SIMAI

e-Lecture Notes

ISSN 1970-4429

Vol.11 - 2013

SIMAI
e-Lecture Notes
ISSN 1970-4429

Nonlinear Evolution Models of Integrable Type

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Vol. 11 - 2013

ISBN-A: 10.978.88905708/03

Published by:
SIMAI - Società Italiana di Matematica Applicata e Industriale
Via dei Taurini, 19 c/o IAC/CNR
00185, ROMA (ITALY)

SIMAI e-Lecture Notes
ISSN: 1970-4429

Volume 11, 2013
ISBN-13: 978-88-905708-0-3
ISBN-A: 10.978.88905708/03

Dedicated to the memory of my parents and my aunt

Preface

Over the past 45 years we have seen a growing interest in integrable linear systems and their applications. These equations include the Korteweg-de Vries (KdV), nonlinear Schrödinger (NLS), sine-Gordon (SG), modified Korteweg-de Vries (mKdV), Toda lattice, integrable discrete nonlinear Schrödinger (IDNLS), Camassa-Holm (CH), and Degasperis-Procesi (DP) equations. They have important applications to surface wave dynamics, fibre optics, Josephson junction transmission lines, Alfvén waves in collisionless plasmas, charge density waves, surfaces of constant Gaussian curvature, traffic congestion, nonlinearly coupled oscillators, and breaking wave dynamics. The mathematics used involves techniques from fields as diverse as functional analysis, Lie groups, differential geometry, numerical linear algebra, and linear control theory. The mathematical problems studied range from unique solvability issues in Sobolev spaces to analytical and numerical solution algorithms to the derivation of conservation laws from hamiltonian principles. The net result of this disparity in applications and mathematical techniques has been the creation of an immense research area where mathematicians, physicists, and engineers, in other words scientists of various pedigrees, can fruitfully work together towards a variety of common goals.

This book is based on a 20 hour minicourse given to graduate students at the University of Cagliari in the early Summer of 2012. The philosophy of this course was to discuss techniques to solve various integrable linear systems by means of the inverse scattering transform (IST) method, where the solution of the integrable nonlinear system is associated with the "potential" in a linear eigenvalue problem. Using the direct and inverse scattering theory of the linear eigenvalue problem the time evolution according to the integrable nonlinear system is converted into the time evolution of the scattering data. The crux of the IST is that the time evolution of the scattering data is so elementary that the IST method in principle yields an explicit method of solving the integrable linear system. In certain cases the exact solvability of the direct and inverse scattering problems allows one to derive extensive families of exact solutions. Discretization of the direct and inverse scattering problems leads to a numerical method to solve the integrable nonlinear system. It is the inverse scattering transform method that we wish to highlight in this monograph. Although the analytical and numerical aspects of the IST are an important part of the research conducted by the Numerical Analysis and Mathematical Modelling Group of the Department of Mathematics and Computer Science of the University of Cagliari, in this monograph we focus on its analytical aspects.

Over the years I have had the pleasure to collaborate with a variety of capable scientists, many of which I have grown to value as friends. In the first place I would like to mention Tuncay Aktosun (University of Texas at Arlington) and Martin Klaus (Virginia Polytechnic Institute and State Uni-

versity) with whom I have worked on inverse scattering problems and later on nonlinear evolution equations. Especially the collaboration with Tuncay Aktosun has continued until the present day. More recently, I have collaborated with Francesco Demontis, Sebastiano Seatzu, Giuseppe Rodriguez, and Luisa Fermo (University of Cagliari), and Antonio Aricò (Second University of Napels) on the analytical (FD) and numerical (SbS, GR, AA, LF) aspects. A still more recent collaboration regards Barbara Prinari (University of Colorado at Colorado Springs, University of Lecce) and Federica Vitale (University of Lecce). In one way or the other these people, plus countless others I have not mentioned, have contributed to this monograph. The research itself has been financed over the years by various funding agencies such as the Italian Ministery of Universities and Research (MIUR), the Autonomous Region of Sardinia (RAS), the University of Cagliari (UNICA), and the University of Texas at Arlington (UTA). Without the assistance of these entities, it would undoubtedly have been much harder to maintain the scientific contacts that have stimulated the research in such a major fashion. I would also like to express my appreciation to the Italian Society of Applied and Industrial Mathematics (SIMAI) for allowing the publication of this monograph in their book series.

The monograph itself contains many well-known results, written up in a different context, as well as new results. The philosophy has been to maximize the use of linear algebra in order to arrive at more concise equations and proofs and to facilitate the development of numerical methods. On the other hand, in spite of my background in functional analysis, the philosophy has also been to minimize the use of functional analysis and to arrive at a treatment that is mathematically as elementary as possible.

Cornelis van der Mee

Contents

Chapter 1

Introduction to Integrable Equations

In this chapter we introduce the principal examples of integrable evolution equations. In particular, we discuss their Lax pairs, AKNS pairs, hybrid Lax-AKNS pairs, and hamiltonian formulations.

1.1 A brief history of integrable equations

The documented sighting, in 1834 by the Scottish engineer John Scott Russell [1808-1882], of the solitary lump-shaped wave travelling along the Union Canal between Edinburgh and Glasgow is generally considered to be the starting point of the study of integrable nonlinear equations. Among Scott Russell's observations was the proportionality of wavespeed and height. The observations were confirmed by conducting water tank experiments [87]. The lack of a theory to explain these phenomena led the scientific community to largely disbelieve Scott Russell's idea of water waves propagating without changing their shape.

A theory to explain Scott Russell's observations was supplied by Joseph Valentin Boussinesq [1842-1929], author of a well-known monograph on fluid dynamics [25]. Describing water waves on an incompressible fluid sustaining irrotational flow in the xz-plane, in 1871-1872 he derived the shallow water equation [23, 24, 25]

$$\frac{\partial^2 h}{\partial t^2} = gH\frac{\partial^2 h}{\partial x^2} + gH\frac{\partial^2}{\partial x^2}\left[\frac{3h^2}{2H} + \frac{1}{3}H^2\frac{\partial^2 h}{\partial x^2}\right],$$

where the height of the water is written as $y = H + h(x,t)$. Assuming propagation in only one direction and writing $\omega(x,t)$ for the wave velocity and using conservation of mass $h_t + (\omega h)_x = 0$, he derived the equation

$$\frac{\partial \omega h}{\partial t} + gH\frac{\partial}{\partial x}\left[h + \tfrac{3}{2}\frac{h^2}{H} + \tfrac{1}{3}H^2\frac{\partial^2 h}{\partial x^2}\right] = 0.$$

Substituting into this equation the expression for $\omega(x,t)$ [39], Boussinesq should have arrived at the PDE[1]

$$\frac{\partial h}{\partial t} + \tfrac{3}{2}\sqrt{\frac{g}{H}}\,\frac{\partial}{\partial x}\left[\tfrac{2}{3}Hh + \tfrac{1}{2}h^2 + \tfrac{1}{9}H^3\frac{\partial^2 h}{\partial x^2}\right] = 0$$

and derived travelling wave solutions of the type $h(x,t) = f(x-ct)$, but in fact he did not. In a more circuitous way, Boussinesq derived such travelling wave solutions, which he coined "ondes solitaires," and thus explained Scott Russell's observations theoretically.

A substantially more transparent derivation of shallow water equations was given by Diederik Johannes Korteweg [1848-1941] and Gustav de Vries [1866-1934] in 1895 [72]. Their equation is the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} - 6uu_x = 0.$$

Their substitution $u(x,t) = f(x-ct)$ led to the ODE

$$-cf'(x) + f'''(x) - 6f(x)f'(x) = 0.$$

Integrating this equation once, they got

$$-cf(x) + f''(x) - 3f(x)^2 = A,$$

where A is a constant of integration. Multiplying by $2f'(x)$ and integrating again, they got for a second integration constant B

$$-cf(x)^2 + f'(x)^2 - 2f(x)^3 = 2Af(x) + B,$$

which led to a separable first order equation. One family of solutions (for $A = B = 0$) is given by

$$u(x,t) = \frac{-\frac{1}{2}c}{\cosh^2\left[\frac{1}{2}\sqrt{c}\,(x-ct-a)\right]},$$

where $c > 0$ and $a \in \mathbb{R}$ are suitable constants. This travelling wave function satisfies the Scott Russell observation that the speed c is proportional to the height $\frac{1}{2}c$. Korteweg and De Vries also derived travelling wave solutions expressed in elliptic functions (by taking $A \neq 0$).

The KdV equation is not the first integrable equation to appear in the literature. In 1862 the French engineer Edmond Bour [1832-1866] derived the sine-Gordon (SG) equation

$$u_{xt} = \frac{1}{\rho^2}\sin(u)$$

[1] Boussinesq's 1877 monograph [25] contains the KdV equation without the simplifications induced by proper rescaling. See Eq. (283bis) on p. 360.

in the study of surfaces of constant negative Gaussian curvature $K = -1/\rho^2$ [22]. This equation was derived by Bour from the Gauss-Mainardi-Codazzi system for pseudospherical surfaces. In 1882 Bäcklund [1845-1922] [15] discovered a transformation that allows one to construct pseudospherical surfaces from other pseudospherical surfaces. In modern terms, these Bäcklund transformations generate solutions of integrable equations from other solutions of integrable equations, allowing one to find a hierarchy of solutions by starting from $u = 0$. These Bäcklund transformations were studied in detail by Luigi Bianchi [1856-1928] [19, 20, 21].

On the KdV front there were few developments in the period 1900-1954. In 1954, at Los Alamos, Fermi, Pasta, and Ulam [56] conducted numerical simulations[2] on coupled oscillations described by the coupled difference equations

$$m\ddot{x}_j = k(x_{j+1} + x_{j-1} - 2x_j)[1 + \alpha(x_{j+1} - x_{j-1})], \qquad j = 0, 1, \ldots, N-1,$$

leading to numerical results which with hindsight can only be interpreted as soliton interactions. The authors did not understand the lack of energy equipartition inherent in the numerical results and therefore left the so-called Fermi-Pasta-Ulam puzzle to a later generation.

In 1960 Gardner and Morikawa [61] rediscovered the KdV equation in an analysis of the transmission of hydromagnetic waves. Another rediscovery involved the pioneering work of Zabusky and Kruskal (1965) [100] on the Fermi-Pasta-Ulam puzzle, where the KdV equation was obtained as the continuum limit of an anharmonic lattice model with cubic nonlinearity. Zabusky and Kruskal supplied numerical evidence of the existence of solitary waves and introduced the term "soliton." To describe soliton interactions, it appeared necessary to study the superposition of two or more travelling wave solutions of the KdV equation. However, the nonlinearity of the KdV equation did not allow one to just add the solitons.

In 1967, Gardner, Greene, Kruskal, and Miura [59] found the *inverse scattering transform* as an elegant and really spectacular way to solve the Cauchy problem of the KdV equation. Starting from an initial condition $u(x, 0)$, they considered the Schrödinger equation

$$-\psi_{xx} + u(x, 0)\psi = k^2\psi, \qquad x \in \mathbb{R},$$

and evaluated the so-called Jost solution $f_r(k, x)$ satisfying

$$f_r(k, x) = \begin{cases} \dfrac{1}{T(k)} e^{-ikx} + \dfrac{R(k)}{T(k)} e^{ikx} + o(1), & x \to +\infty, \\ e^{-ikx}[1 + o(1)], & x \to -\infty. \end{cases}$$

Here $T(k)$ is the *transmission coefficient* (meromorphic in the upper half-plane, with only finitely many, N say, simple poles $i\kappa_j$ which are positive

[2]The actual programming was done by Mary Tsingou Menzel [born 1928] [35].

imaginary) and $R(k)$ the *reflection coefficient* (continuous in $k \in \mathbb{R}$), where $|T(k)|^2 + |R(k)|^2 = 1$ for $k \in \mathbb{R}$. Fortunately, at this point Faddeev (1964) [53] had already developed the direct and inverse scattering theory of the Schrödinger equation on the line. Thus, given the reflection coefficient and N positive so-called norming constants N_j, they wrote down the so-called Marchenko integral equation

$$K(x,y) + \Omega(x+y) + \int_x^\infty dz\, K(x,z)\Omega(z+y) = 0,$$

where the Marchenko integral kernel

$$\Omega(x) = \sum_{j=1}^N N_j e^{-\kappa_j x} + \frac{1}{2\pi}\int_{-\infty}^\infty dk\, e^{ikx} R(k)$$

was constructed from the scattering data $\{R(k), \{\kappa_j, N_j\}_{j=1}^N\}$. The potential $u(x,0)$ then followed from the identity

$$u(x,0) = 2\frac{d}{dx}K(x,x).$$

The crux of the inverse scattering transform (IST) turned out to be that the propagation of the KdV solution $u(x,0) \mapsto u(x,t)$ corresponded exactly with the elementary propagation of the scattering data

$$\{R(k), \{\kappa_j, N_j\}_{j=1}^N\} \mapsto \{R(k)e^{8ik^3t}, \{\kappa_j, N_j e^{8\kappa_j^3 t}\}_{j=1}^N\}.$$

Summarizing, the inverse scattering transform developed by Gardner, Greene, Kruskal and Miura consists of the following three steps:

1. **Direct scattering**: Find the scattering data $\{R(k), \{\kappa_j, N_j\}_{j=1}^N\}$ of the Schrödinger equation with potential $u(x,0)$.

2. **Propagation of scattering data**: Propagate the scattering data as follows:

$$\{R(k), \{\kappa_j, N_j\}_{j=1}^N\} \mapsto \{R(k)e^{8ik^3t}, \{\kappa_j, N_j e^{8\kappa_j^3 t}\}_{j=1}^N\}.$$

3. **Inverse scattering**: Solve the Marchenko integral equation for the time evolved scattering data and obtain the potential $u(x,t)$.

This sequence of three steps can be depicted by the following commutative diagram:

$$\begin{tikzcd}
u(x,0) \arrow[r, "\text{direct scattering}"] \arrow[d, "\text{KdV}"'] & \{R(k), \{\kappa_j, N_j\}_{j=1}^N\} \arrow[d, "\text{time evolution}"] \\
u(x,t) & \{R(k)e^{8ik^3t}, \{\kappa_j, N_j e^{8\kappa_j^3 t}\}_{j=1}^N\} \arrow[l, "\text{inverse scattering}"']
\end{tikzcd}$$

The Gardner-Greene-Kruskal-Miura papers (at least six in all [59, 77, 78, 89, 58, 60]) caught a lot of attention and sparked a search of other nonlinear equations allowing an inverse scattering transform. In 1972, Zakharov and Shabat [101] showed that the Cauchy problem for the nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} \pm 2|u|^2 u = 0$$

can be solved by an inverse scattering transform, irrespective of the choice of the $\pm$ sign. The accompanying linear eigenvalue problem is the so-called Zakharov-Shabat system

$$\psi_x = \begin{pmatrix} -ik & u \\ \mp u^* & ik \end{pmatrix} \psi,$$

where $\psi(x,t)$ is a column vector of length 2 and k is a spectral parameter. The result of the Zakharov-Shabat paper has been an avalanche of papers on examples of inverse scattering transforms for various nonlinear evolution equations. We mention the Manakov system [75] (a 3×3 adaptation of the NLS) and the AKNS system [2] (an $(m+n) \times (m+n)$ matrix generalization of the NLS), where AKNS stands for Mark Ablowitz, Alan Newell, David Kaup, and Harvey Segur. These days there exists a bewildering jungle of very diverse integrable equations. Some of them are discrete in position (such as the Toda lattice, the Kac-Van Moerbeke system, and the integrable discrete nonlinear Schrödinger (IDNLS) equation). Others are bidimensional in position (such as the Kadomtsev-Petviashvili (KPI and KPII) equations).

The nonlinear equations discussed above are called "integrable" for a variety of reasons, even though a precise definition of integrability does not exist. The structure of these equations involves all (or most) of the following features, each of which is considered to be an indication of integrability:

1. There are travelling wave solutions;

2. There are infinitely many conservation laws;

3. The equation can be derived from two independent hamiltonians;

4. There is a Lax pair $\{L, A\}$ of linear operators such that nonlinear equation has the form

$$L_t + LA - AL = 0.$$

5. There is an AKNS pair $\{X, T\}$ of matrices depending on position, time, and a spectral parameter such that nonlinear equation has the form

$$X_t - T_x + XT - TX = 0$$

independent of the spectral parameter.

6. There is an inverse scattering transform to solve the initial-value problem.

In this monograph our ultimate goal is to produce item 6: the inverse scattering transform, and in its wake an analytical method to derive closed form solutions and a numerical method for solving the initial-value problem in general. However, many researchers in the field concentrate on the first three items and/or one of the items 4-5.

The field of integrable equations is very attractive to work in. In the author's opinion, this has the following reasons:

1. There are truckloads of applications in many different fields, often even truckloads of applications of the same equation. Thus researchers from many disciplines can work in it.

2. Integrable equations can be studied with techniques from many fields of mathematics: differential geometry, Lie groups, PDE's, linear algebra, numerical analysis, and functional analysis. Thus mathematicians with very different backgrounds can work in it.

3. There is no exhaustion of the field anywhere in sight. Thus it is possible to remain in this field for one's entire scientific career.

4. The field is nearly uniformly distributed over the important scientific result producing countries. Within each such country, various universities and research institutes are involved, requiring the participation of people from different disciplines. Thus anyone sufficiently competent working in this field will always have co-authors and jobs available.

1.2 Lax pairs

1. General principle. In 1968 Peter Lax [74] has given a general abstract mechanism to explain to some extent why certain nonlinear evolution equations are integrable in the sense that their initial-value problem can be solved by means of an inverse scattering transform. The idea is to depart from a so-called Lax pair $\{L, A\}$ of (possibly unbounded) linear operators L and A which are to satisfy the linear equations

$$\begin{cases} L\Psi = \lambda\Psi, & \text{spatial evolution of } \Psi, \\ \Psi_t = A\Psi, & \text{time evolution of } \Psi, \end{cases}$$

where the "wave function" Ψ depends on position x, time t, and spectral parameter λ. The Lax method amounts to finding the linear operator A from the given linear operator L such that the Lax evolution equations are satisfied. Either operator may depend on t, but the spectral parameter λ does not. By formally writing

$$(L\Psi)_t = (\lambda\Psi)_t = \lambda\Psi_t = \lambda A\Psi = A(\lambda\Psi) = AL\Psi,$$
$$(L\Psi)_t = L_t\Psi + L\Psi_t = L_t\Psi + LA\Psi = (L_t + LA)\Psi,$$

the linear operators L and A are to be related by

$$L_t + LA - AL = 0. \tag{1.1}$$

Equation (1.1) is the integrable nonlinear evolution equation whose Lax pair we are seeking.

Suppose that we can construct the evolution system of linear operators $U(t,s)$ such that

$$\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s), \quad U(s,s) = I, \tag{1.2}$$

where I stands for the identity operator on a suitable complex Hilbert space of (vector) functions. Then $U(t,s)$ can formally be constructed as the unique solution of the integral equation[3]

$$U(t,s) = I + \int_s^t d\tau\, A(\tau)U(\tau,s).$$

Thus for a fixed vector ϕ, $U(t,s)\phi$ is the solution of $\Psi_t = A\Psi$ under the initial condition $\Psi(s) = \phi$. Unique solvability of the usual initial value problem implies the product rule

$$U(t,r)U(r,s) = U(t,s),$$

where $t, r, s \in \mathbb{R}$. Thus $U(t,s)$ is invertible with inverse $U(s,t)$. Therefore

$$\begin{aligned}\frac{\partial}{\partial t}U(s,t) &= \frac{\partial}{\partial t}[U(t,s)^{-1}] \\ &= -U(t,s)^{-1}\left(\frac{\partial}{\partial t}U(t,s)\right)U(t,s)^{-1} \\ &= -U(t,s)^{-1}A(t)U(t,s)U(t,s)^{-1} = -U(s,t)A(t).\end{aligned}$$

By computing

$$\frac{\partial}{\partial t}[U(s,t)L(t)U(t,s)] = U(s,t)[-A(t)L(t) + L_t + L(t)A(t)]U(t,s) = 0,$$

we see that $U(s,t)L(t)U(t,s)$ does not depend on t and hence must coincide with its value at $t = s$, namely with $L(s)$. As a result,

$$U(s,t)L(t)U(t,s) = L(s). \tag{1.3}$$

[3]We shall not discuss the functional analysis of evolution systems [cf. [30]] in detail. Here we shall only assume that everything works out.

Since $U(t,s)$ and $U(s,t)$ are each other's inverses, we see that the operators $L(t)$ and $L(s)$ are similar and hence have the same spectrum. In other words, the abstract Lax equation (1.1) implies *isospectrality*: The spectrum of the linear operator L appearing in the eigenvalue problem associated with the nonlinear evolution equation by means of the inverse scattering transform is time independent.

Let us now discuss in detail a few Lax pairs relevant to well-known integrable evolution systems.

Example 1.1 (KdV) Let us find the Lax pair for the KdV equation. Of course, we take

$$L\Psi = -\tfrac{d^2}{dx^2}\Psi + u(x,t)\Psi,$$

the Schrödinger operator on the line. The problem is to find A. So let us try

$$A = \alpha_3\partial_x^3 + \alpha_2\partial_2^2 + \alpha_1\partial_x + \alpha_0,$$

where the coefficients α_j $(j = 0,1,2,3)$ may depend on x and t but not on λ. Then[4]

$$\begin{aligned}
0 &= L_t + LA - AL \\
&= u_t + u(\alpha_3\partial_x^3 + \alpha_2\partial_2^2 + \alpha_1\partial_x + \alpha_0) - (\alpha_3\partial_x^3 + \alpha_2\partial_2^2 + \alpha_1\partial_x + \alpha_0)u \\
&- \partial_x^2(\alpha_3\partial_x^3 + \alpha_2\partial_x^2 + \alpha_1\partial_x + \alpha_0) + (\alpha_3\partial_x^3 + \alpha_2\partial_2^2 + \alpha_1\partial_x + \alpha_0)\partial_x^2 \\
&= u_t + u(\alpha_3\partial_x^3 + \alpha_2\partial_2^2 + \alpha_1\partial_x + \alpha_0) - \alpha_3(u\partial_x^3 + 3u_x\partial_x^2 + 3u_{xx}\partial_x + u_{xxx}) \\
&- \alpha_2(u\partial_x^2 + 2u_x\partial_x + u_{xx}) - \alpha_1(u\partial_x + u_x) - \alpha_0 u \\
&= -(\alpha_3\partial_x^5 + 2[\alpha_3]_x\partial_x^4 + [\alpha_3]_{xx}\partial_x^3 + \alpha_2\partial_x^4 + 2[\alpha_2]_x\partial_x^3 + [\alpha_2]_{xx}\partial_x^2 \\
&\quad + \alpha_1\partial_x^3 + 2[\alpha_1]_x\partial_x^2 + [\alpha_1]_{xx}\partial_x + \alpha_0\partial_x^2 + 2[\alpha_0]_x\partial_x + [\alpha_0]_{xx}) \\
&+ (\alpha_3\partial_x^3 + \alpha_2\partial_2^2 + \alpha_1\partial_x + \alpha_0)\partial_x^2 \\
&+ (u\alpha_3 - \alpha_3 u)\partial_x^3 + (u\alpha_2 - 3\alpha_3 u_x - \alpha_2 u)\partial_x^2 \\
&+ (u\alpha_1 - 3\alpha_3 u_{xx} - 2\alpha_2 u_x - \alpha_1 u)\partial_x \\
&+ (u_t + u\alpha_0 - \alpha_3 u_{xxx} - \alpha_2 u_{xx} - \alpha_1 u_x - \alpha_0 u) \\
&= -(2[\alpha_3]_x\partial_x^4 + [\alpha_3]_{xx}\partial_x^3 + 2[\alpha_2]_x\partial_x^3 + [\alpha_2]_{xx}\partial_x^2 \\
&\quad + 2[\alpha_1]_x\partial_x^2 + [\alpha_1]_{xx}\partial_x + 2[\alpha_0]_x\partial_x + [\alpha_0]_{xx}) \\
&- 3\alpha_3 u_x\partial_x^2 + (-3\alpha_3 u_{xx} - 2\alpha_2 u_x)\partial_x + (u_t - \alpha_3 u_{xxx} - \alpha_2 u_{xx} - \alpha_1 u_x) \\
&= -2[\alpha_3]_x\partial_x^4 - ([\alpha_3]_{xx} + 2[\alpha_2]_x)\partial_x^3 - ([\alpha_2]_{xx} + 2[\alpha_1]_x + 3\alpha_3 u_x)\partial_x^2 \\
&- ([\alpha_1]_{xx} + 2[\alpha_0]_x + 3\alpha_3 u_{xx} + 2\alpha_2 u_x)\partial_x \\
&+ (-[\alpha_0]_{xx} + u_t - \alpha_3 u_{xxx} - \alpha_2 u_{xx} - \alpha_1 u_x).
\end{aligned}$$

[4] In the calculation below, functions "are" premultiplication operators by functions. As a result, $\partial_x u = u\partial_x + u_x$, $\partial_x^2 u = u\partial_x^2 + 2u_x\partial_x + u_{xx}$, and $\partial_x^3 u = u\partial_x^3 + 3u_x\partial_x^2 + 3u_{xx}\partial_x + u_{xxx}$.

Thus the coefficients of the various powers of ∂_x should vanish. This leads to the coupled system of differential equations

$$\begin{aligned}
[\alpha_3]_x &= 0,\\
[\alpha_3]_{xx} + 2[\alpha_2]_x &= 0,\\
[\alpha_2]_{xx} + 2[\alpha_1]_x + 3\alpha_3 u_x &= 0,\\
[\alpha_1]_{xx} + 2[\alpha_0]_x + 3\alpha_3 u_{xx} + 2\alpha_2 u_x &= 0,\\
-[\alpha_0]_{xx} + u_t - \alpha_3 u_{xxx} - \alpha_2 u_{xx} - \alpha_1 u_x &= 0.
\end{aligned}$$

The first two equations imply that α_3 and α_2 do not depend on x and hence that $[\alpha_3]_{xx} = [\alpha_2]_{xx} = 0$. Thus there remain the three equations

$$\begin{aligned}
2[\alpha_1]_x + 3\alpha_3 u_x &= 0,\\
[\alpha_1]_{xx} + 2[\alpha_0]_x + 3\alpha_3 u_{xx} + 2\alpha_2 u_x &= 0,\\
-[\alpha_0]_{xx} + u_t - \alpha_3 u_{xxx} - \alpha_2 u_{xx} - \alpha_1 u_x &= 0.
\end{aligned}$$

Now the first equation implies that $\gamma_1 = 2\alpha_1 + 3\alpha_3 u$ does not depend on x, leading to $\alpha_1 = \frac{1}{2}\gamma_1 - \frac{3}{2}\alpha_3 u$. Now the last two equations become

$$\begin{aligned}
-\tfrac{3}{2}\alpha_3 u_{xx} + 2[\alpha_0]_x + 3\alpha_3 u_{xx} + 2\alpha_2 u_x &= 0,\\
-[\alpha_0]_{xx} + u_t - \alpha_3 u_{xxx} - \alpha_2 u_{xx} - \tfrac{1}{2}\gamma_1 u_x + \tfrac{3}{2}\alpha_3 u u_x &= 0.
\end{aligned}$$

Now the first equation can be integrated to yield

$$\tfrac{3}{2}\alpha_3 u_x + 2\alpha_0 + 2\alpha_2 u = \gamma_2,$$

where γ_2 does not depend on x. Thus

$$\alpha_0 = \tfrac{1}{2}\gamma_2 - \tfrac{3}{4}\alpha_3 u_x - \alpha_2 u.$$

Now substitution in the final equation yields

$$\tfrac{3}{4}\alpha_3 u_{xxx} + \alpha_2 u_{xx} + u_t - \alpha_3 u_{xxx} - \alpha_2 u_{xx} - \tfrac{1}{2}\gamma_1 u_x + \tfrac{3}{2}\alpha_3 u u_x = 0.$$

Finally, we have obtained the PDE

$$u_t - \tfrac{1}{2}\gamma_1 u_x - \tfrac{1}{4}\alpha_3 u_{xxx} + \tfrac{3}{2}\alpha_3 u u_x = 0.$$

A direct comparison with the KdV equation

$$u_t + u_{xxx} - 6uu_x = 0$$

yields $\gamma_1 = 0$ and $\alpha_3 = -4$. In other words,

$$A = -4\partial_x^3 + \alpha_2\partial_x^2 + 6u\partial_x + [\tfrac{1}{2}\gamma_2 + 3u_x - \alpha_2 u],$$

where α_2 and γ_2 are parameters which we can choose to vanish.

Example 1.2 (NLS) Let us now try to find a Lax pair for the following nonsymmetric generalization of the nonlinear Schrödinger equation:

$$\begin{aligned} iq_t + q_{xx} + 2q^2 r &= 0, \\ -ir_t + r_{xx} + 2qr^2 &= 0. \end{aligned}$$

Putting $q = u$ and $r = \pm u^*$, we get the NLS equation

$$iu_t + u_{xx} \pm 2|u|^2 u = 0.$$

The equation with the plus sign is called the *focusing* NLS, whereas the equation with the minus sign is called the *defocusing* NLS.

By means of the inverse scattering transform, the nonsymmetric NLS system is associated with the nonsymmetric Zakharov-Shabat system

$$\begin{aligned} \xi_x &= -i\lambda\xi + q\eta, \\ \eta_x &= r\xi + i\lambda\eta. \end{aligned}$$

This system can also be written as

$$\begin{pmatrix} i\partial_x & -iq \\ ir & -i\partial_x \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

We therefore define L as follows:

$$L = \begin{pmatrix} i\partial_x & -iq \\ ir & -i\partial_x \end{pmatrix},$$

so that the nonsymmetric Zakharov-Shabat system has the form $L\Psi = \lambda\Psi$ for $\Psi = \left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right)$. For A we choose

$$A = \begin{pmatrix} a_2\partial_x^2 + a_1\partial_x + a_0 & b_2\partial_x^2 + b_1\partial_x + b_0 \\ c_2\partial_x^2 + c_1\partial_x + c_0 & d_2\partial_x^2 + d_1\partial_x + d_0 \end{pmatrix},$$

where a_j, b_j, c_j, and d_j $(j = 0, 1, 2)$ depend on x and t but not on λ. We get for the entries of the 2×2 matrix representing the left-hand side of (1.1)

$$\begin{aligned} 0 &= L_t^{11} + L^{11}A^{11} - A^{11}L^{11} + L^{12}A^{21} - A^{12}L^{21} \\ &= i\partial_x(a_2\partial_x^2 + a_1\partial_x + a_0) - i(a_2\partial_x^2 + a_1\partial_x + a_0)\partial_x \\ &\quad - iq(c_2\partial_x^2 + c_1\partial_x + c_0) + i(c_2\partial_x^2 + c_1\partial_x + c_0)q \\ &= i[a_2]_x\partial_x^2 + i[a_1]_x\partial_x + i[a_0]_x + ic_2(2q_x\partial_x + q_{xx}) + ic_1q_x \\ &= i[a_2]_x\partial_x^2 + i([a_1]_x + 2c_2q_x)\partial_x + i([a_0]_x + c_1q_x); \end{aligned}$$

$$\begin{aligned} 0 &= L_t^{22} + L^{22}A^{22} - A^{22}L^{22} + L^{21}A^{12} - A^{21}L^{12} \\ &= -i\partial_x(d_2\partial_x^2 + d_1\partial_x + d_0) + i(d_2\partial_x^2 + a_1\partial_x + a_0)\partial_x \\ &\quad + ir(b_2\partial_x^2 + b_1\partial_x + b_0) - i(b_2\partial_x^2 + b_1\partial_x + b_0)r \\ &= -i[d_2]_x\partial_x^2 - i[d_1]_x\partial_x - i[d_0]_x - ib_2(2r_x\partial_x + r_{xx}) - ib_1r_x \\ &= -i[d_2]_x\partial_x^2 - i([d_1]_x + 2b_2r_x)\partial_x - i([d_0]_x + b_1r_x); \end{aligned}$$

$$
\begin{aligned}
0 &= L_t^{12} + L^{11}A^{12} - A^{11}L^{12} + L^{12}A^{22} - A^{12}L^{22} \\
&= -iq_t + i\partial_x(b_2\partial_x^2 + b_1\partial_x + b_0) + i(a_2\partial_x^2 + a_1\partial_x + a_0)q \\
&- iq(d_2\partial_x^2 + d_1\partial_x + d_0) + i(b_2\partial_x^2 + b_1\partial_x + b_0)\partial_x \\
&= -iq_t + 2i(b_2\partial_x^2 + b_1\partial_x + b_0)\partial_x \\
&+ iq(a_2\partial_x^2 + a_1\partial_x + a_0) + ia_2(2q_x\partial_x + q_{xx}) + ia_1q_x \\
&- iq(d_2\partial_x^2 + d_1\partial_x + d_0) + i([b_2]_x\partial_x^2 + [b_1]_x\partial_x + [b_0]_x) \\
&= 2ib_2\partial_x^3 + i(2b_1 + qa_2 - qd_2 + [b_2]_x)\partial_x^2 \\
&+ i(2b_0 + qa_1 + 2a_2q_x - qd_1 + [b_1]_x)\partial_x \\
&+ i(-q_t + qa_0 + a_2q_{xx} + a_1q_x - qd_0 + [b_0]_x);
\end{aligned}
$$

$$
\begin{aligned}
0 &= L_t^{21} + L^{22}A^{21} - A^{22}L^{21} + L^{21}A^{11} - A^{21}L^{11} \\
&= ir_t - i\partial_x(c_2\partial_x^2 + c_1\partial_x + c_0) - i(d_2\partial_x^2 + d_1\partial_x + d_0)r \\
&+ ir(a_2\partial_x^2 + a_1\partial_x + a_0) - i(c_2\partial_x^2 + c_1\partial_x + c_0)\partial_x \\
&= ir_t - 2i(c_2\partial_x^2 + c_1\partial_x + c_0)\partial_x \\
&- ir(d_2\partial_x^2 + d_1\partial_x + d_0) - id_2(2r_x\partial_x + r_{xx}) - id_1r_x \\
&+ ir(a_2\partial_x^2 + a_1\partial_x + a_0) - i([c_2]_x\partial_x^2 + [c_1]_x\partial_x + [c_0]_x) \\
&= -2ic_2\partial_x^3 - i(2c_1 + rd_2 - ra_2 + [c_2]_x)\partial_x^2 \\
&- i(2c_0 + rd_1 + 2d_2r_x - ra_1 + [c_1]_x)\partial_x \\
&- i(-r_t + rd_0 + d_2r_{xx} + d_1r_x - ra_0 + [c_0]_x).
\end{aligned}
$$

Equating coefficients of the same powers of ∂_x, we get the 14 equations

$$[a_2]_x = 0, \tag{1.4a}$$
$$[a_1]_x + 2c_2q_x = 0, \tag{1.4b}$$
$$[a_0]_x + c_1q_x = 0, \tag{1.4c}$$
$$[d_2]_x = 0, \tag{1.4d}$$
$$[d_1]_x + 2b_2r_x = 0, \tag{1.4e}$$
$$[d_0]_x + b_1r_x = 0, \tag{1.4f}$$
$$b_2 = 0, \tag{1.4g}$$
$$2b_1 + qa_2 - qd_2 + [b_2]_x = 0, \tag{1.4h}$$
$$2b_0 + qa_1 + 2a_2q_x - qd_1 + [b_1]_x = 0, \tag{1.4i}$$
$$-q_t + qa_0 + a_2q_{xx} + a_1q_x - qd_0 + [b_0]_x = 0, \tag{1.4j}$$
$$c_2 = 0, \tag{1.4k}$$
$$2c_1 + rd_2 - ra_2 + [c_2]_x = 0, \tag{1.4l}$$
$$2c_0 + rd_1 + 2d_2r_x - ra_1 + [c_1]_x = 0, \tag{1.4m}$$
$$-r_t + rd_0 + d_2r_{xx} + d_1r_x - ra_0 + [c_0]_x = 0. \tag{1.4n}$$

Equations (1.4g) and (1.4k) yield $b_2 = c_2 = 0$, while (1.4a) and (1.4d) imply that a_2 and d_2 do not depend on x. We now get the following 10 equations

$$[a_1]_x = 0, \tag{1.5a}$$
$$[a_0]_x + c_1 q_x = 0, \tag{1.5b}$$
$$[d_1]_x = 0, \tag{1.5c}$$
$$[d_0]_x + b_1 r_x = 0, \tag{1.5d}$$
$$2b_1 + (a_2 - d_2)q = 0, \tag{1.5e}$$
$$2b_0 + qa_1 + 2a_2 q_x - qd_1 + [b_1]_x = 0, \tag{1.5f}$$
$$-q_t + qa_0 + a_2 q_{xx} + a_1 q_x - qd_0 + [b_0]_x = 0, \tag{1.5g}$$
$$2c_1 + (d_2 - a_2)r = 0, \tag{1.5h}$$
$$2c_0 + rd_1 + 2d_2 r_x - ra_1 + [c_1]_x = 0, \tag{1.5i}$$
$$-r_t + rd_0 + d_2 r_{xx} + d_1 r_x - ra_0 + [c_0]_x = 0. \tag{1.5j}$$

Thus a_1 and d_1 do not depend on x, while

$$b_1 = \tfrac{1}{2}(d_2 - a_2)q, \qquad c_1 = \tfrac{1}{2}(a_2 - d_2)r.$$

Thus we now get the six equations

$$[a_0]_x + \tfrac{1}{2}(a_2 - d_2)rq_x = 0, \tag{1.6a}$$
$$[d_0]_x + \tfrac{1}{2}(d_2 - a_2)qr_x = 0, \tag{1.6b}$$
$$2b_0 + (a_1 - d_1)q + \tfrac{1}{2}(d_2 + 3a_2)q_x = 0, \tag{1.6c}$$
$$-q_t + q(a_0 - d_0) + a_2 q_{xx} + a_1 q_x + [b_0]_x = 0, \tag{1.6d}$$
$$2c_0 + (d_1 - a_1)r + \tfrac{1}{2}(a_2 + 3d_2)r_x = 0, \tag{1.6e}$$
$$-r_t + d_2 r_{xx} + d_1 r_x - r(a_0 - d_0) + [c_0]_x = 0. \tag{1.6f}$$

Thus (1.6c) and (1.6e) imply

$$b_0 = \tfrac{1}{2}(d_1 - a_1)q - \tfrac{1}{4}(d_2 + 3a_2)q_x,$$
$$c_0 = \tfrac{1}{2}(a_1 - d_1)r - \tfrac{1}{4}(a_2 + 3d_2)r_x.$$

Further, subtracting (1.6a) and (1.6b) we see that

$$a_0 - d_0 = \gamma_1 + \tfrac{1}{2}(a_2 - d_2)qr,$$

where γ_1 does not depend on x. Substituting the last three identities into (1.6d) and (1.6f) we obtain

$$iq_t - \tfrac{1}{4}i(a_2 - d_2)q_{xx} - \tfrac{1}{2}i(a_2 - d_2)q^2 r - \tfrac{1}{2}i(a_1 + d_1)q_x - i\gamma_1 q = 0,$$
$$-ir_t - \tfrac{1}{4}i(a_2 - d_2)r_{xx} - \tfrac{1}{2}i(a_2 - d_2)qr^2 + \tfrac{1}{2}i(a_1 + d_1)r_x - i\gamma_1 r = 0.$$

Finally, choosing $-\frac{1}{4}i(a_2 - d_2) = 1$, $\frac{1}{2}i(a_1 + d_1) = 0$, and $i\gamma_1 = 0$, we get the unsymmetric NLS system

$$iq_t + q_{xx} + 2q^2 r = 0,$$
$$-ir_t + r_{xx} + 2qr^2 = 0.$$

Example 1.3 (sine-Gordon) Let us derive the following Lax pair for the sine-Gordon equation $u_{xt} = \sin(u)$:

$$
\begin{aligned}
L &= \begin{pmatrix} i\partial_x & \frac{1}{2}iu_x \\ \frac{1}{2}iu_x & -i\partial_x \end{pmatrix}, \\
A &= \tfrac{1}{4}\begin{pmatrix} C\partial_x^{-1}C - S\partial_x^{-1}S & -S\partial_x^{-1}C - C\partial_x^{-1}S \\ S\partial_x^{-1}C + C\partial_x^{-1}S & C\partial_x^{-1}C - S\partial_x^{-1}S \end{pmatrix},
\end{aligned}
$$

where

$$
Cf = \cos(\tfrac{1}{2}u)f, \qquad Sf = \sin(\tfrac{1}{2}u)f,
$$

and

$$
[\partial_x^{-1}f](x) = \tfrac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right) ds\, f(s).
$$

Using that

$$
\begin{aligned}
\partial_x C &= C\partial_x - \tfrac{1}{2}u_x S, \\
\partial_x S &= S\partial_x + \tfrac{1}{2}u_x C,
\end{aligned}
$$

and hence that

$$
\begin{aligned}
\partial_x^{-1}C\partial_x &= \partial_x^{-1}[\partial_x C + \tfrac{1}{2}u_x S] = C + \partial_x^{-1}\tfrac{1}{2}u_x S, \\
\partial_x^{-1}S\partial_x &= \partial_x^{-1}[\partial_x S - \tfrac{1}{2}u_x C] = S - \partial_x^{-1}\tfrac{1}{2}u_x C,
\end{aligned}
$$

we get for the entries of the matrix representing the left-hand side of (1.1)

$$
\begin{aligned}
&L_t^{11} + L^{11}A^{11} - A^{11}L^{11} + L^{12}A^{21} - A^{12}L^{21} \\
&\quad = \tfrac{1}{4}i\left\{\partial_x[C\partial_x^{-1}C - S\partial_x^{-1}S] - [C\partial_x^{-1}C - S\partial_x^{-1}S]\partial_x\right. \\
&\quad + \left.\tfrac{1}{2}u_x[S\partial_x^{-1}C + C\partial_x^{-1}S] + [S\partial_x^{-1}C + C\partial_x^{-1}S]\tfrac{1}{2}u_x\right\} \\
&\quad = \tfrac{1}{4}i\left\{C\partial_x\partial_x^{-1}C - \tfrac{1}{2}u_x S\partial_x^{-1}C - S\partial_x\partial_x^{-1}S - \tfrac{1}{2}u_x C\partial_x^{-1}S\right. \\
&\quad - C\partial_x^{-1}\tfrac{1}{2}u_x S - S\partial_x^{-1}\tfrac{1}{2}u_x C - C^2 + S^2 \\
&\quad + \left.\tfrac{1}{2}u_x S\partial_x^{-1}C + \tfrac{1}{2}u_x C\partial_x^{-1}S + S\partial_x^{-1}C\tfrac{1}{2}u_x + C\partial_x^{-1}S\tfrac{1}{2}u_x\right\} = 0;
\end{aligned}
$$

$$
\begin{aligned}
&L_t^{22} + L^{22}A^{22} - A^{22}L^{22} + L^{21}A^{12} - A^{21}L^{12} \\
&\quad = \tfrac{1}{4}i\left\{-\partial_x[C\partial_x^{-1}C - S\partial_x^{-1}S] + [C\partial_x^{-1}C - S\partial_x^{-1}S]\partial_x\right. \\
&\quad - \left.\tfrac{1}{2}u_x[S\partial_x^{-1}C + C\partial_x^{-1}S] - [S\partial_x^{-1}C + C\partial_x^{-1}S]\tfrac{1}{2}u_x\right\} \\
&\quad = \tfrac{1}{4}i\left\{-C\partial_x\partial_x^{-1}C + \tfrac{1}{2}u_x S\partial_x^{-1}C + S\partial_x\partial_x^{-1}S + \tfrac{1}{2}u_x C\partial_x^{-1}S\right. \\
&\quad + C\partial_x^{-1}\tfrac{1}{2}u_x S + S\partial_x^{-1}\tfrac{1}{2}u_x C + C^2 - S^2 \\
&\quad - \left.\tfrac{1}{2}u_x S\partial_x^{-1}C - \tfrac{1}{2}u_x C\partial_x^{-1}S - S\partial_x^{-1}C\tfrac{1}{2}u_x - C\partial_x^{-1}S\tfrac{1}{2}u_x\right\} = 0;
\end{aligned}
$$

$$
\begin{aligned}
&L_t^{12} + L^{11}A^{12} - A^{11}L^{12} + L^{12}A^{22} - A^{12}L^{22} \\
&\quad = \tfrac{1}{4}i\left\{2u_{xt} - \partial_x[S\partial_x^{-1}C + C\partial_x^{-1}S] - [C\partial_x^{-1}C - S\partial_x^{-1}S]u_x\right. \\
&\quad \left. + \tfrac{1}{2}u_x[C\partial_x^{-1}C - S\partial_x^{-1}S] - [S\partial_x^{-1}C + C\partial_x^{-1}S]\partial_x\right\} \\
&\quad = \tfrac{1}{4}i\left\{2u_{xt} - S\partial_x\partial_x^{-1}C - \tfrac{1}{2}u_xC\partial_x^{-1}C - C\partial_x\partial_x^{-1}S + \tfrac{1}{2}u_xS\partial_x^{-1}S\right. \\
&\quad - C\partial_x^{-1}C\tfrac{1}{2}u_x + S\partial_x^{-1}S\tfrac{1}{2}u_x + \tfrac{1}{2}u_xC\partial_x^{-1}C - \tfrac{1}{2}u_xS\partial_x^{-1}S \\
&\quad \left. - S\partial_x^{-1}\tfrac{1}{2}u_xS + C\partial_x^{-1}\tfrac{1}{2}u_xC - 2SC - 2CS\right\} \\
&\quad = \tfrac{1}{4}i[2u_{xt} - 4CS] = \tfrac{1}{2}i[u_{xt} - 2\cos(\tfrac{1}{2}u)\sin(\tfrac{1}{2}u)] = \tfrac{1}{2}i[u_{xt} - \sin(u)];
\end{aligned}
$$

$$
\begin{aligned}
&L_t^{21} + L^{22}A^{21} - A^{22}L^{21} + L^{21}A^{11} - A^{21}L^{11} \\
&\quad = \tfrac{1}{4}i\left\{2u_{xt} - \partial_x[S\partial_x^{-1}C + C\partial_x^{-1}S] - [C\partial_x^{-1}C - S\partial_x^{-1}S]\tfrac{1}{2}u_x\right. \\
&\quad \left. + \tfrac{1}{2}u_x[C\partial_x^{-1}C - S\partial_x^{-1}S] - [S\partial_x^{-1}C + C\partial_x^{-1}S]\partial_x\right\} \\
&\quad = \tfrac{1}{4}i\left\{2u_{xt} - S\partial_x\partial_x^{-1}C - \tfrac{1}{2}u_xC\partial_x^{-1}C - C\partial_x\partial_x^{-1}S + \tfrac{1}{2}u_xS\partial_x^{-1}S\right. \\
&\quad + C\partial_x^{-1}C\tfrac{1}{2}u_x - S\partial_x^{-1}S\tfrac{1}{2}u_x + \tfrac{1}{2}u_xC\partial_x^{-1}C - \tfrac{1}{2}u_xS\partial_x^{-1}S \\
&\quad \left. + S\partial_x^{-1}\tfrac{1}{2}u_xS - C\partial_x^{-1}\tfrac{1}{2}u_xC - 2SC - 2CS\right\} \\
&\quad = \tfrac{1}{4}i[2u_{xt} - 4CS] = \tfrac{1}{2}i[u_{xt} - 2\cos(\tfrac{1}{2}u)\sin(\tfrac{1}{2}u)] = \tfrac{1}{2}i[u_{xt} - \sin(u)].
\end{aligned}
$$

implying $u_{xt} = \sin(u)$ in order for the Lax equation (1.1) to be true.

Example 1.4 (Toda lattice) Let us derive a Lax pair for the Toda lattice equations. These equations were first formulated in 1967 by Morikazu Toda [1917-2010] [91, 92] to model an infinite sequence of nonlinearly coupled oscillators converging to the KdV equation as the distance between consecutive oscillators vanishes. In this model the position variable is an integer n. Thus instead of functions $u(x,t)$ we now study sequences $\{u_n(t)\}_{n=-\infty}^{\infty}$. The linear eigenvalue problem associated with the Toda lattice equations is the biinfinite Jacobi system

$$a_{n+1}u_{n+1} + a_nu_{n-1} + b_nu_n = \lambda u_n, \tag{1.7}$$

where $\{a_n\}_{n=-\infty}^{\infty}$ is a sequence of positive numbers tending to $\frac{1}{2}$, $\{b_n\}_{n=-\infty}^{\infty}$ is a sequence of real numbers tending to zero as $n \to \pm\infty$, and λ is a spectral parameter. Defining the forward shift operator S^+ and the backward shift operator S^- by

$$(S^+\boldsymbol{x})_n = x_{n+1}, \quad (S^-\boldsymbol{x})_n = x_{n-1}, \qquad \boldsymbol{x} = \{x_n\}_{n=-\infty}^{\infty},$$

we define the Lax pair $\{L, A\}$ as follows [57]:

$$(L\boldsymbol{x})_n = a_{n+1}x_{n+1} + a_nx_{n-1} + b_nx_n, \tag{1.8a}$$

$$(A\boldsymbol{x})_n = a_{n+1}x_{n+1} - a_{n-1}x_{n-1}, \tag{1.8b}$$

where $\boldsymbol{x} = \{x_n\}_{n=-\infty}^{\infty}$. Thus

$$L = a_{n+1}S^+ + a_n S^- + b_n, \qquad A = a_{n+1}S^+ - a_n S^-,$$

where a_n etc. are premultipliers. Thus $S^+ d_n = d_{n+1}S^+$ and $S^- d_n = d_{n-1}S^-$. Writing $S^{++} = [S^+]^2$ and $S^{--} = [S^-]^2$, we now compute

$$\begin{aligned}
L_t + LA - AL &= [a_{n+1}]_t S^+ + [a_n]_t S^- + [b_n]_t \\
&\quad + (a_{n+1}S^+ + a_n S^- + b_n)(a_{n+1}S^+ - a_n S^-) \\
&\quad - (a_{n+1}S^+ - a_n S^-)(a_{n+1}S^+ + a_n S^- + b_n) \\
&= [a_{n+1}]_t S^+ + [a_n]_t S^- + [b_n]_t \\
&\quad + a_{n+1}a_{n+2}S^{++} - a_{n+1}^2 + a_n^2 - a_n a_{n-1}S^{--} + b_n a_{n+1}S^+ - b_n a_n S^- \\
&\quad - a_{n+1}a_{n+2}S^{++} - a_{n+1}^2 - a_{n+1}b_{n+1}S^+ + a_n^2 + a_n a_{n-1}S^{--} + a_n b_{n-1}S^- \\
&= ([a_{n+1}]_t + a_{n+1}(b_n - b_{n+1}))S^+ + ([a_n]_t + a_n(b_{n-1} - b_n))S^- \\
&\quad + [b_n]_t + 2(a_n^2 - a_{n+1}^2).
\end{aligned}$$

Consequently,

$$\begin{aligned}
[a_{n+1}]_t &= a_{n+1}(b_{n+1} - b_n), \\
[a_n]_t &= a_n(b_n - b_{n-1}), \\
[b_n]_t &= 2(a_{n+1}^2 - a_n^2).
\end{aligned}$$

Putting

$$a_n = \tfrac{1}{2}e^{-\frac{1}{2}(q_n - q_{n-1})}, \qquad b_n = -\frac{1}{2}p_n, \tag{1.9}$$

we obtain the Toda lattice equations

$$\begin{aligned}
[p_n]_t &= e^{-(q_n - q_{n-1})} - e^{-(q_{n+1} - q_n)}, \\
[q_n]_t &= p_n.
\end{aligned}$$

1.3 AKNS pairs

1. General principle. An alternative to Lax pairs has been developed by Mark Ablowitz, David Kaup, Alan Newell, and Harvey Segur [2]. Instead of constructing a pair of linear operators $\{L, A\}$ on a suitable complex Hilbert space of vector functions, they constructed two matrix functions X and T which depend on position, time, and the spectral variable λ. This is in contrast to the Lax method, where the operators L and A do not depend on the spectral variable. This time the treatment of the continuous and discrete cases is different.

In the continuous case, the nonlinear evolution equation is derived from the pair of evolution equations

$$\Psi_x = X\Psi, \qquad \Psi_t = T\Psi, \tag{1.10}$$

where Ψ is a column vector and X and T are square matrices, all of them depending on (x, t, λ). Then the obvious identity

$$(\Psi_x)_t = (\Psi_t)_x$$

implies that $(X\Psi)_t = (T\Psi)_x$, or that

$$X_t - T_x + XT - TX = 0. \tag{1.11}$$

In a geometrical context, the identities (1.10) are sometimes called the Gauss-Mainardi-Codazzi equations, while the compatibility condition (1.11) is called the zero curvature condition.

In the discrete case, the nonlinear evolution equation is derived from the pair of difference-differential equations

$$\Psi_{n+1} = X_n \Psi_n, \qquad [\Psi_n]_t = T_n \Psi_n.$$

Differentiating the first equation with respect to t, we get

$$\begin{aligned} [\Psi_{n+1}]_t &= [X_n]_t \Psi_n + X_n T_n \Psi_n, \\ [\Psi_{n+1}]_t &= T_{n+1} \Psi_{n+1} = T_{n+1} X_n \Psi_n. \end{aligned}$$

As a result, we get

$$[X_n]_t + X_n T_n - T_{n+1} X_n = 0. \tag{1.12}$$

Let us now discuss in detail AKNS pairs for a few important nonlinear evolution equations [2, 6, 1]. In most cases we present matrix generalizations of these nonlinear evolution equations (NLS, mKdV, KdV) and derive a Lax pair from each AKNS pair, thus generalizing some of the Lax pairs derived before.

Example 1.5 (matrix KdV) Consider the AKNS pair

$$X = \begin{pmatrix} 0_{n\times n} & u - \lambda I_n \\ I_n & 0_{n\times n} \end{pmatrix}, \quad T = \begin{pmatrix} u_x & -4\lambda^2 I_n + 2\lambda u + 2u^2 - u_{xx} \\ 4\lambda I_n + 2u & -u_x \end{pmatrix},$$

where $\Psi = \left(\begin{smallmatrix} \psi_x \\ \psi \end{smallmatrix}\right)$. Substitution into the contingency equation $X_t - T_x + XT - TX = 0_{2n\times 2n}$ yields zero, as far as the $(1,1)$-, $(2,1)$-, and $(2,2)$-blocks are concerned. Also, $\Psi_x = X\Psi$. The remaining block can be computed as follows:

$$\begin{aligned} &[X^{12}]_t - [T^{12}]_x + X^{11}T^{12} + X^{21}T^{22} - T^{11}X^{12} - T^{22}X^{22} \\ &= u_t - 2\lambda u_x - 2(u^2)_x + u_{xxx} - (u - \lambda I_n)u_x - u_x(u - \lambda I_n) \\ &= u_t - 3(u^2)_x + u_{xxx}. \end{aligned}$$

Thus $\{X, T\}$ is an AKNS pair for the matrix KdV equation

$$u_t - 3(u^2)_x + u_{xxx} = 0_{n\times n}.$$

Moreover, the equation $\psi_{xx} = (u - \lambda)\psi$ implies that $L\psi \stackrel{\text{def}}{=} [-\partial_x^2 I_n + u]\psi$ is the first operator in a Lax pair. Since

$$\psi_t = 4[-\psi_{xx} + u\psi]_x + 2u\psi_x - u_x\psi = -4\psi_{xxx} + 6u\psi_x + 3u_x\psi,$$

we see that $A\psi = [-4\partial_x^3 I_n + 6u\partial_x + 3u_x]\psi$ is the second operator.

Example 1.6 (matrix NLS) Consider the matrix NLS system

$$\begin{aligned} iq_t + q_{xx} - 2qrq &= 0_{m\times n}, \\ -ir_t + r_{xx} - 2rqr &= 0_{n\times m}, \end{aligned}$$

where q is $m \times n$ and r is $n \times m$. Introducing the megamatrix

$$\boldsymbol{Q} = \begin{pmatrix} 0_{m\times m} & q \\ r & 0_{n\times n} \end{pmatrix},$$

we can write the matrix NLS system in the concise form

$$iJ\boldsymbol{Q}_t + \boldsymbol{Q}_{xx} - 2\boldsymbol{Q}^3 = 0_{(m+n)\times(m+n)}, \qquad \boldsymbol{Q}J = -J\boldsymbol{Q},$$

where $J = I_m \oplus (-I_n)$. Following [5], we now define

$$X = -i\lambda J + \boldsymbol{Q}, \qquad T = -2i\lambda^2 J - iJ\boldsymbol{Q}^2 + 2\lambda\boldsymbol{Q} + iJ\boldsymbol{Q}_x.$$

Here I_m denotes the $m \times m$ identity matrix. Then

$$\begin{aligned} &iJ[X_t - T_x + XT - TX] = iJ\boldsymbol{Q}_t - 2i\lambda J\boldsymbol{Q}_x - (\boldsymbol{Q}^2)_x + \boldsymbol{Q}_{xx} \\ &- 2i\lambda^3 J - i\lambda J\boldsymbol{Q}^2 + 2\lambda^2\boldsymbol{Q} + i\lambda J\boldsymbol{Q}_x - 2\lambda^2\boldsymbol{Q} - \boldsymbol{Q}^3 + 2i\lambda J\boldsymbol{Q}^2 + \boldsymbol{Q}\boldsymbol{Q}_x \\ &+ 2i\lambda^3 J + i\lambda J\boldsymbol{Q}^2 + 2\lambda^2\boldsymbol{Q} + i\lambda J\boldsymbol{Q}_x - 2\lambda^2\boldsymbol{Q} - \boldsymbol{Q}^3 - 2i\lambda J\boldsymbol{Q}^2 + \boldsymbol{Q}_x\boldsymbol{Q} \\ &= iJ\boldsymbol{Q}_t + \boldsymbol{Q}_{xx} - 2\boldsymbol{Q}^3. \end{aligned}$$

Thus $\{X, T\}$ is an AKNS pair if and only the matrix NLS system

$$iJ\boldsymbol{Q}_t + \boldsymbol{Q}_{xx} - 2\boldsymbol{Q}^3 = 0_{(m+n)\times(m+n)}$$

is satisfied.

Let us now derive a Lax pair $\{L, A\}$ from the AKNS pair $\{X, T\}$. The identity $\Psi_x = X\Psi = -i\lambda J\Psi + \boldsymbol{Q}\Psi$ can be written in the form

$$L\Psi \stackrel{\text{def}}{=} (iJ\partial_x - iJ\boldsymbol{Q})\Psi = \lambda\Psi.$$

We can then write $\Psi_t = -2iJ(\lambda^2\Psi) + 2\boldsymbol{Q}(\lambda\Psi) + [-iJ\boldsymbol{Q}^2 + iJ\boldsymbol{Q}_x]\Psi$ in the form $\Psi_t = A\Psi$, where

$$A\Psi \stackrel{\text{def}}{=} -2iJL^2 + 2\boldsymbol{Q}L - iJ\boldsymbol{Q}^2 + iJ\boldsymbol{Q}_x.$$

Thus we have derived a Lax pair $\{L, A\}$ for the matrix NLS system from an AKNS pair. To simplify A, we write

$$L = iJ(\partial_x - \boldsymbol{Q}), \qquad L^2 = (iJ)^2(\partial_x + \boldsymbol{Q})(\partial_x - \boldsymbol{Q}) = -\partial_x^2 + \boldsymbol{Q}^2 + \boldsymbol{Q}_x.$$

Then

$$\begin{aligned} A &= -2iJ(-\partial_x^2 + \boldsymbol{Q}^2 + \boldsymbol{Q}_x) - 2iJ\boldsymbol{Q}(\partial_x - \boldsymbol{Q}) - iJ\boldsymbol{Q}^2 + iJ\boldsymbol{Q}_x \\ &= -2IJ\left\{-\partial_x^2 + \boldsymbol{Q}^2 + \boldsymbol{Q}_x + \boldsymbol{Q}(\partial_x - \boldsymbol{Q}) + \tfrac{1}{2}\boldsymbol{Q}^2 - \tfrac{1}{2}\boldsymbol{Q}_x\right\} \\ &= 2iJ\left\{\partial_x^2 - \boldsymbol{Q}\partial_x - \tfrac{1}{2}\boldsymbol{Q}_x - \tfrac{1}{2}\boldsymbol{Q}^2\right\} \\ &= \begin{pmatrix} 2i\partial_x^2 - iqr & -2iq\partial_x - iq_x \\ 2ir\partial_x + ir_x & -2i\partial_x^2 + irq \end{pmatrix}. \end{aligned}$$

Example 1.7 (matrix mKdV) Let us consider the AKNS pair

$$X = -i\lambda J + J\boldsymbol{Q}, \quad T = -4i\lambda^3 J + 2i\lambda J\boldsymbol{Q}^2 + 4\lambda^2 J\boldsymbol{Q} + 2i\lambda\boldsymbol{Q}_x - J\boldsymbol{Q}_{xx} - 2J\boldsymbol{Q}^3,$$

where $J = I_m \oplus (-I_n)$ and $J\boldsymbol{Q} = -\boldsymbol{Q}J$. Then

$$\begin{aligned} J[X_t - T_x + XT - TX] = \boldsymbol{Q}_t & \\ &- 2i\lambda(\boldsymbol{Q}^2)_x - 4\lambda^2\boldsymbol{Q}_x - 2i\lambda J\boldsymbol{Q}_{xx} + \boldsymbol{Q}_{xxx} + 2(\boldsymbol{Q}^3)_x \\ &- 4\lambda^4 J + 2\lambda^2 J\boldsymbol{Q}^2 - 4i\lambda^3 J\boldsymbol{Q} + 2\lambda^2\boldsymbol{Q}_x + i\lambda J\boldsymbol{Q}_{xx} + 2i\lambda J\boldsymbol{Q}^3 \\ &+ 4i\lambda^3 J\boldsymbol{Q} - 2i\lambda J\boldsymbol{Q}^3 - 4\lambda^2 J\boldsymbol{Q}^2 + 2i\lambda\boldsymbol{Q}\boldsymbol{Q}_x + J\boldsymbol{Q}\boldsymbol{Q}_{xx} + 2J\boldsymbol{Q}^4 \\ &+ 4\lambda^4 J - 2\lambda^2 J\boldsymbol{Q}^2 - 4i\lambda^3 J\boldsymbol{Q} + 2\lambda^2\boldsymbol{Q}_x + i\lambda J\boldsymbol{Q}_{xx} + 2i\lambda J\boldsymbol{Q}^3 \\ &+ 4i\lambda^3 J\boldsymbol{Q} - 2i\lambda J\boldsymbol{Q}^3 + 4\lambda^2 J\boldsymbol{Q}^2 + 2i\lambda\boldsymbol{Q}_x\boldsymbol{Q} - J\boldsymbol{Q}_{xx}\boldsymbol{Q} - 2J\boldsymbol{Q}^4 \\ &= \boldsymbol{Q}_t - 2i\lambda(\boldsymbol{Q}^2)_x - 4\lambda^2\boldsymbol{Q}_x - 2i\lambda J\boldsymbol{Q}_{xx} + \boldsymbol{Q}_{xxx} + 2(\boldsymbol{Q}^3)_x \\ &- 4\lambda^4 J + 2\lambda^2 J\boldsymbol{Q}^2 - 4i\lambda^3 J\boldsymbol{Q} + 2\lambda^2\boldsymbol{Q}_x + i\lambda J\boldsymbol{Q}_{xx} + 2i\lambda J\boldsymbol{Q}^3 \\ &+ 4i\lambda^3 J\boldsymbol{Q} - 2i\lambda J\boldsymbol{Q}^3 - 4\lambda^2 J\boldsymbol{Q}^2 + 2i\lambda\boldsymbol{Q}\boldsymbol{Q}_x + J\boldsymbol{Q}\boldsymbol{Q}_{xx} + 2J\boldsymbol{Q}^4 \\ &+ 4\lambda^4 J - 2\lambda^2 J\boldsymbol{Q}^2 - 4i\lambda^3 J\boldsymbol{Q} + 2\lambda^2\boldsymbol{Q}_x + i\lambda J\boldsymbol{Q}_{xx} + 2i\lambda J\boldsymbol{Q}^3 \\ &+ 4i\lambda^3 J\boldsymbol{Q} - 2i\lambda J\boldsymbol{Q}^3 + 4\lambda^2 J\boldsymbol{Q}^2 + 2i\lambda\boldsymbol{Q}_x\boldsymbol{Q} - J\boldsymbol{Q}_{xx}\boldsymbol{Q} - 2J\boldsymbol{Q}^4 \\ &= \boldsymbol{Q}_t + 2(\boldsymbol{Q}^3)_x + \boldsymbol{Q}_{xxx}. \end{aligned}$$

Thus the contingency condition (1.11) leads to the nonsymmetric matrix mKdV equation

$$\boldsymbol{Q}_t + 2(\boldsymbol{Q}^3)_x + \boldsymbol{Q}_{xxx} = 0_{(m+n)\times(m+n)}, \qquad J\boldsymbol{Q} = -\boldsymbol{Q}J.$$

Assuming $m = n = 1$ and $\boldsymbol{Q} = \left(\begin{smallmatrix} 0 & u \\ u & 0 \end{smallmatrix}\right)$ with u real-valued, we obtain the modified Korteweg-de Vries equation

$$u_t + 6u^2 u_x + u_{xxx} = 0. \tag{1.13}$$

Let us now derive a Lax pair $\{L, A\}$ from the AKNS pair $\{X, T\}$. The eigenvalue equation $\Psi_x = X\Psi$ can be written in the form $L\Psi = \lambda\Psi$, where

$$L = iJ\partial_x - i\boldsymbol{Q}.$$

We then write

$$\begin{aligned} \Psi_t = T\Psi &= -4iJ(\lambda^3\Psi) + 4J\boldsymbol{Q}(\lambda^2\Psi) + 2i(J\boldsymbol{Q}^2 + \boldsymbol{Q}_x)\lambda\Psi - J(\boldsymbol{Q}_{xx} + 2\boldsymbol{Q}^3)\Psi \\ &= \left[-4iJL^3 + 4J\boldsymbol{Q}L^2 + 2i(J\boldsymbol{Q}^2 + \boldsymbol{Q}_x)L - J(\boldsymbol{Q}_{xx} + 2\boldsymbol{Q}^3)\right]\Psi = A\Psi, \end{aligned}$$

Thus $\{L, A\}$ is a Lax pair for the nonsymmetric matrix mKdV equation.

Example 1.8 (sine-Gordon) Let us define the AKNS pair as follows:

$$X = \begin{pmatrix} -i\lambda & -\frac{1}{2}u_x \\ \frac{1}{2}u_x & i\lambda \end{pmatrix}, \qquad T = \frac{i}{4\lambda}\begin{pmatrix} \cos(u) & \sin(u) \\ \sin(u) & -\cos(u) \end{pmatrix}.$$

It is easily verified that

$$X_t - T_x + XT - TX = \begin{pmatrix} 0 & -\frac{1}{2}u_{xt} + \frac{1}{2}\sin(u) \\ \frac{1}{2}u_{xt} - \frac{1}{2}\sin(u) & 0 \end{pmatrix},$$

so that $\{X, T\}$ is indeed an AKNS for the sine-Gordon equation

$$u_{xt} = \sin(u). \tag{1.14}$$

Let us now derive a Lax pair for the sine-Gordon equation from an AKNS pair. Starting from $\Psi_x = X\Psi = [-i\lambda J - \frac{1}{2}J\boldsymbol{Q}_x]\Psi$, where $\boldsymbol{Q} = \left(\begin{smallmatrix} 0 & u \\ u & 0 \end{smallmatrix}\right)$, we obtain

$$L\Psi \overset{\text{def}}{=} (iJ\partial_x + \tfrac{1}{2}i\boldsymbol{Q}_x)\Psi = \lambda\Psi.$$

Solving $L\Psi = \Phi$, we get

$$\Psi = -iW^{-1}\partial_x^{-1}WJ\Phi,$$

where $W_x = \frac{1}{2}WJ\boldsymbol{Q}_x$. We may thus choose

$$W = \begin{pmatrix} \cos(\frac{u}{2}) & \sin(\frac{u}{2}) \\ -\sin(\frac{u}{2}) & \cos(\frac{u}{2}) \end{pmatrix}.$$

Consequently,

$$\begin{aligned} \Psi_t &= \tfrac{i}{4}\begin{pmatrix} \cos(u) & \sin(u) \\ \sin(u) & -\cos(u) \end{pmatrix}(-i)W^{-1}\partial_x^{-1}WJ\Psi \\ &= \tfrac{1}{4}\begin{pmatrix} \cos(u) & \sin(u) \\ \sin(u) & -\cos(u) \end{pmatrix}\begin{pmatrix} \cos(\frac{u}{2}) & -\sin(\frac{u}{2}) \\ \sin(\frac{u}{2}) & \cos(\frac{u}{2}) \end{pmatrix}\partial_x^{-1}\begin{pmatrix} \cos(\frac{u}{2}) & -\sin(\frac{u}{2}) \\ -\sin(\frac{u}{2}) & -\cos(\frac{u}{2}) \end{pmatrix}\Psi \\ &= \tfrac{1}{4}\begin{pmatrix} \cos(\frac{u}{2}) & \sin(\frac{u}{2}) \\ \sin(\frac{u}{2}) & -\cos(\frac{u}{2}) \end{pmatrix}\partial_x^{-1}\begin{pmatrix} \cos(\frac{u}{2}) & -\sin(\frac{u}{2}) \\ -\sin(\frac{u}{2}) & -\cos(\frac{u}{2}) \end{pmatrix} = A\Psi, \end{aligned}$$

where A is defined as in Example 1.3. As a result, $\{L, A\}$ is the Lax pair of the sine-Gordon equation (1.14).

Example 1.9 (matrix IDNLS) Let us consider the matrix IDNLS system

$$\begin{aligned} i[Q_n]_t &= Q_{n+1} - 2Q_n + Q_{n-1} - Q_{n+1}R_nQ_n - Q_nR_nQ_{n-1}, \\ -i[R_n]_t &= R_{n+1} - 2R_n + R_{n-1} - R_{n+1}Q_nR_n - R_nQ_nR_{n-1}, \end{aligned}$$

where Q_n is an $N \times M$ matrix and R_n is an $M \times N$ matrix. Writing

$$\boldsymbol{Q}_n = \begin{pmatrix} 0_{N\times N} & Q_n \\ R_n & 0_{M\times M} \end{pmatrix}, \qquad J = I_N \oplus (-I_M),$$

we can write the matrix IDNLS system in a concise way as follows:

$$iJ[\boldsymbol{Q}_n]_t = \boldsymbol{Q}_{n+1} - 2\boldsymbol{Q}_n + \boldsymbol{Q}_{n-1} - \boldsymbol{Q}_{n+1}\boldsymbol{Q}_n^2 - \boldsymbol{Q}_n^2\boldsymbol{Q}_{n-1}, \quad \boldsymbol{Q}_nJ = -J\boldsymbol{Q}_n.$$

Now put ([5, Eq. (5.1.2)];[93, Eqs. (2.3)-(2.4)])

$$X_n = \boldsymbol{Z}+\boldsymbol{Q}_n, \quad T_n = -\tfrac{1}{2}iJ(\boldsymbol{Z}-\boldsymbol{Z}^{-1})^2+iJ\boldsymbol{Q}_n\boldsymbol{Q}_{n-1}-iJ\boldsymbol{Z}\boldsymbol{Q}_n+iJ\boldsymbol{Z}^{-1}\boldsymbol{Q}_{n-1},$$

where $\boldsymbol{Z} = zI_N \oplus z^{-1}I_M$ for some spectral variable $0 \neq z \in \mathbb{C}$. Then $\boldsymbol{Z}$ and J commute, while

$$\boldsymbol{Z}\boldsymbol{Q}_n = \boldsymbol{Q}_n\boldsymbol{Z}^{-1}, \quad \boldsymbol{Z}^{-1}\boldsymbol{Q}_n = \boldsymbol{Q}_n\boldsymbol{Z}, \quad (\boldsymbol{Z}-\boldsymbol{Z}^{-1})^2\boldsymbol{Q}_n = \boldsymbol{Q}_n(\boldsymbol{Z}-\boldsymbol{Z}^{-1})^2.$$

We now compute

$$\begin{aligned}
&-iJ([X_n]_t + X_nT_n - T_{n+1}X_n) \\
&= -iJ[\boldsymbol{Q}_n]_t + (\boldsymbol{Z}-\boldsymbol{Q}_n)[-\tfrac{1}{2}(\boldsymbol{Z}-\boldsymbol{Z}^{-1})^2 + \boldsymbol{Q}_n\boldsymbol{Q}_{n-1} - \boldsymbol{Z}\boldsymbol{Q}_n + \boldsymbol{Z}^{-1}\boldsymbol{Q}_{n-1}] \\
&- [-\tfrac{1}{2}(\boldsymbol{Z}-\boldsymbol{Z}^{-1})^2 + \boldsymbol{Q}_{n+1}\boldsymbol{Q}_n - \boldsymbol{Z}\boldsymbol{Q}_{n+1} + \boldsymbol{Z}^{-1}\boldsymbol{Q}_n](\boldsymbol{Z}+\boldsymbol{Q}_n) \\
&= -iJ[\boldsymbol{Q}_n]_t + \boldsymbol{Z}[\boldsymbol{Q}_n\boldsymbol{Q}_{n-1} - \boldsymbol{Z}\boldsymbol{Q}_n + \boldsymbol{Z}^{-1}\boldsymbol{Q}_{n-1}] \\
&- [\boldsymbol{Q}_{n+1}\boldsymbol{Q}_n - \boldsymbol{Z}\boldsymbol{Q}_{n+1} + \boldsymbol{Z}^{-1}\boldsymbol{Q}_n]\boldsymbol{Z} \\
&- \boldsymbol{Q}_n[-\tfrac{1}{2}(\boldsymbol{Z}-\boldsymbol{Z}^{-1})^2 + \boldsymbol{Q}_n\boldsymbol{Q}_{n-1} - \boldsymbol{Z}\boldsymbol{Q}_n + \boldsymbol{Z}^{-1}\boldsymbol{Q}_{n-1}] \\
&- [-\tfrac{1}{2}(\boldsymbol{Z}-\boldsymbol{Z}^{-1})^2 + \boldsymbol{Q}_{n+1}\boldsymbol{Q}_n - \boldsymbol{Z}\boldsymbol{Q}_{n+1} + \boldsymbol{Z}^{-1}\boldsymbol{Q}_n]\boldsymbol{Q}_n \\
&= -iJ[\boldsymbol{Q}_n]_t + \boldsymbol{Z}\boldsymbol{Q}_{n+1}\boldsymbol{Z} + \boldsymbol{Q}_{n-1} + (-\boldsymbol{Z}^2\boldsymbol{Q}_n - \boldsymbol{Z}^{-1}\boldsymbol{Q}_n\boldsymbol{Z} + \tfrac{1}{2}(\boldsymbol{Z}-\boldsymbol{Z}^{-1})^2\boldsymbol{Q}_n \\
&+ \boldsymbol{Q}_n\tfrac{1}{2}(\boldsymbol{Z}-\boldsymbol{Z}^{-1})^2) - \boldsymbol{Q}_{n+1}\boldsymbol{Q}_n^2 - \boldsymbol{Q}_n^2\boldsymbol{Q}_{n-1} \\
&+ [\boldsymbol{Z}\boldsymbol{Q}_n\boldsymbol{Q}_{n-1} - \boldsymbol{Q}_n\boldsymbol{Z}^{-1}\boldsymbol{Q}_{n-1}] + [-\boldsymbol{Q}_{n+1}\boldsymbol{Q}_n\boldsymbol{Z} + \boldsymbol{Z}\boldsymbol{Q}_{n+1}\boldsymbol{Q}_n] \\
&+ [\boldsymbol{Q}_n\boldsymbol{Z}\boldsymbol{Q}_n - \boldsymbol{Z}^{-1}\boldsymbol{Q}_n^2] \\
&= -iJ[\boldsymbol{Q}_n]_t + \boldsymbol{Q}_{n+1} + \boldsymbol{Q}_{n-1} - 2\boldsymbol{Q}_n - \boldsymbol{Q}_{n+1}\boldsymbol{Q}_n^2 - \boldsymbol{Q}_n^2\boldsymbol{Q}_{n-1}.
\end{aligned}$$

Thus $\{X_n, T_n\}$ is an AKNS pair for the matrix IDNLS system.

Since the spectral variable z does not appear in X in a linear way, there is no straightforward method to convert the AKNS pair into a Lax pair (as we did for the matrix NLS).

1.4 Hybrid Lax-AKNS pairs

When using this method, the nonlinear evolution equation follows by applying the contingency condition that higher order partial derivatives do not depend on the order of partial differentiation for sufficiently smooth functions [Schwarz's theorem], where the differential expressions may depend on λ. For instance, from the hybrid Lax-AKNS pair equations

$$\begin{aligned}\psi_{xx} &= (u-\lambda)\psi, \\ \psi_t &= (4\lambda+2u)\psi_x - u_x\psi,\end{aligned}$$

we obtain from the contingency condition $(\psi_{xx})_t = (\psi_t)_{xx}$

$$\begin{aligned}0 &= (\psi_{xx})_t - (\psi_t)_{xx} = [(u-\lambda)\psi]_t - [(4\lambda+2u)\psi_x - u_x\psi]_{xx} \\ &= (u-\lambda)\psi_t + u_t\psi - (4\lambda+2u)(\psi_{xx})_x - 4u_x\psi_{xx} \\ &- 2u_{xx}\psi_x + u_x\psi_{xx} + 2u_{xx}\psi_x + u_{xxx}\psi \\ &= (u-\lambda)\{(4\lambda+2u)\psi_x - u_x\psi\} + u_t\psi - (4\lambda+2u)\{(u-\lambda)\psi_x + u_x\psi\} \\ &- 3u_x(u-\lambda)\psi + u_{xxx}\psi \\ &= \lambda^2\{-4\psi_x + 4\psi_x\} \\ &+ \lambda\{-2u\psi_x + u_x\psi + 4u\psi_x - 4u\psi_x - 4u_x\psi + 2u\psi_x + 3u_x\psi\} \\ &+ 2u^2\psi_x - uu_x\psi + u_t\psi - 2u^2\psi_x - 2uu_x\psi - 3uu_x\psi + u_{xxx}\psi \\ &= [u_t + u_{xxx} - 6uu_x]\psi,\end{aligned}$$

which implies the KdV equation. Since the first hybrid pair equation can be written as $L\psi = (-\partial_x^2 + u)\psi = \lambda\psi$ and the second as

$$\begin{aligned}\psi_t &= 4(\lambda\psi)_x + 2u\psi_x - u_x\psi = 4(-\partial_x^2\psi + u\psi)_x + 2u\psi_x - u_x\psi \\ &= [-4\partial_x^3 + 6u\partial_x + 3u_x]\psi = A\psi,\end{aligned}$$

we obtain the Lax pair $\{L, A\}$ given by Example 1.1.

We illustrate the hybrid Lax-AKNS pairs by means of the Camassa-Holm (CH) and Degasperis-Procesi (DP) equations.

Example 1.10 (Camassa-Holm) Consider the Camassa-Holm equation

$$u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0, \tag{1.15}$$

where ω is a nonnegative constant. This equation was formulated by Camassa and Holm [27] in 1993 to take into account the phenomenon of breaking waves not incorporated in the KdV model. The CH equation describes the unidirectional propagation of shallow water waves over a flat bottom [27, 66, 67] as well as that of axially symmetric waves in a hyperelastic rod

[32, 34]. The inverse scattering transform (IST) relates the CH equation for $\omega > 0$ to the Schrödinger equation on the line

$$-\psi_{xx} = -\left(\frac{1}{4} - \frac{m+\omega}{2\lambda}\right)\psi = \left(\frac{m+\omega}{\omega}k^2 + \frac{m}{4\omega}\right)\psi,$$

where $m = u - u_{xx}$ and $\lambda(k) = \frac{\omega}{2}\frac{1}{k^2+\frac{1}{4}}$. We have the following hybrid Lax-AKNS equations [27, 28, 31]:

$$\psi_{xx} = \left(\frac{1}{4} - \frac{m(x,t)+\omega}{2\lambda}\right)\psi, \tag{1.16a}$$

$$\psi_t = (-u-\lambda)\psi_x + \tfrac{1}{2}u_x\psi. \tag{1.16b}$$

The contingency condition $(\psi_{xx})_t = (\psi_t)_{xx}$ leads to the CH equation. Indeed, substitution into the contingency condition implies that

$$\begin{aligned}\left(\frac{1}{4} - \frac{m+\omega}{2\lambda}\right)&[(-u-\lambda)\psi_x + \tfrac{1}{2}u_x\psi] - \frac{m_t}{2\lambda}\psi = -u_{xx}\psi_x - 2u_x\psi_{xx}\\ &+ (-u-\lambda)\psi_{xxx} + \tfrac{1}{2}u_{xxx}\psi + u_{xx}\psi_x + \tfrac{1}{2}u_x\psi_{xx}\\ &= -u_{xx}\psi_x - 2u_x\left(\frac{1}{4} - \frac{m+\omega}{2\lambda}\right)\psi + (-u-\lambda)\left(\frac{1}{4} - \frac{m+\omega}{2\lambda}\right)\psi_x\\ &+ (-u-\lambda)\frac{-m_x}{2\lambda}\psi + \tfrac{1}{2}u_{xxx}\psi + u_{xx}\psi_x + \tfrac{1}{2}u_x\left(\frac{1}{4} - \frac{m+\omega}{2\lambda}\right)\psi.\end{aligned}$$

Observing that the ψ_x terms in the first and third members cancel out, we get

$$\begin{aligned}\left(\frac{1}{4} - \frac{m+\omega}{2\lambda}\right)\tfrac{1}{2}u_x\psi - \frac{m_t}{2\lambda}\psi = -2u_x&\left(\frac{1}{4} - \frac{m+\omega}{2\lambda}\right)\psi + (-u-\lambda)\frac{-m_x}{2\lambda}\psi\\ &+ \tfrac{1}{2}u_{xxx}\psi + \tfrac{1}{2}u_x\left(\frac{1}{4} - \frac{m+\omega}{2\lambda}\right)\psi.\end{aligned}$$

The terms not containing 2λ in the denominator cancel out, because $m = u - u_{xx}$. When multiplying by 2λ, the other terms yield the equation

$$-(m+\omega)\tfrac{1}{2}u_x\psi - m_t\psi = 2u_x(m+\omega) + um_x\psi - \tfrac{1}{2}u_x(m+\omega)\psi.$$

This equation can also be written in the form

$$[m_t + 2u_x(m+\omega) + um_x]\psi = 0,$$

which is equivalent to the Camassa-Holm equation (1.15) applied to ψ.

Writing (1.16a) in the form

$$L\psi = \frac{1}{m+\omega}(-\partial_x^2 + \tfrac{1}{4})\psi = \frac{1}{2\lambda}\psi,$$

we obtain $\psi_t = A\psi$, where

$$A\psi = (-u + \tfrac{1}{2}u_x)\psi - \tfrac{1}{2}(-\partial_x^2 + \tfrac{1}{4})^{-1}(m+\omega)\psi.$$

As a result, $\{L, A\}$ is a Lax pair for the CH equation.

Example 1.11 (Degasperis-Procesi) A similar, integrable, equation is the Degasperis-Procesi equation [37]

$$u_t - u_{xxt} + 2\omega u_x + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0.$$

We have the following hybrid Lax-AKNS equations [37]:

$$\psi_{xxx} = \psi_x + \lambda(u - u_{xx} + \tfrac{2}{3}\omega)\psi, \tag{1.17a}$$

$$\psi_t = \frac{1}{\lambda}\psi_{xx} - u\psi_x + u_x\psi. \tag{1.17b}$$

The contingency condition $(\psi_{xxx})_t = (\psi_t)_{xxx}$ leads to the DP equation. Indeed,

$$\begin{aligned}
0 &= (\psi_{xxx})_t - (\psi_t)_{xxx} = (\psi_t)_x + \lambda[u_t - u_{xxt}]\psi + \lambda[u - u_{xx} + \tfrac{2}{3}\omega]\psi_t \\
&- \frac{1}{\lambda}(\psi_{xxx})_{xx} + u(\psi_{xxx})_x + 3u_x\psi_{xxx} + 3u_{xx}\psi_{xx} + u_{xxx}\psi_x \\
&- u_x\psi_{xxx} - 3u_{xx}\psi_{xx} - 3u_{xxx}\psi_x - u_{xxxx}\psi \\
&= \left[\frac{1}{\lambda}\psi_{xx} - u\psi_x + u_x\psi\right]_x + \lambda[u_t - u_{xxt}]\psi \\
&+ \lambda(u - u_{xx} + \tfrac{2}{3}\omega)(\frac{1}{\lambda}\psi_{xx} - u\psi_x + u_x\psi) - \frac{1}{\lambda}\left[\psi_x + \lambda(u - u_{xx} + \tfrac{2}{3}\omega)\psi\right]_{xx} \\
&+ u\left[\psi_x + \lambda(u - u_{xx} + \tfrac{2}{3}\omega)\psi\right]_x + 2u_x\left[\psi_x + \lambda(u - u_{xx} + \tfrac{2}{3}\omega)\psi\right] \\
&- 2u_{xxx}\psi_x - u_{xxxx}\psi \\
&= \lambda\left\{u_t - u_{xxt} + 4uu_x - 3u_xu_{xx} - uu_{xxx} + 2\omega u_x\right\}\psi,
\end{aligned}$$

which implies the DP equation.

Writing (1.17a) in the form

$$L\psi = (m + \tfrac{2}{3}\omega)^{-1}(\partial_x^3 - \partial_x)\psi = \lambda\psi,$$

we obtain $\psi_t = A\psi$, where

$$A\psi = \left[\partial_x^2(\partial_x^3 - \partial_x)^{-1}(m + \tfrac{2}{3}\omega) - u\partial_x + u_x\right]\psi.$$

As a result, $\{L, A\}$ is a Lax pair for the DP equation.

1.5 Hamiltonian formulation

In this section we derive the nonlinear evolution equations in one of the following two ways: 1) as the Hamilton equations from a hamiltonian, or 2) as the Euler-Lagrange equations from a lagrangian. In the case of a discrete system, the worst that can happen is to have infinitely many variables. In the case of a continuous system, we deal with hamiltonian or lagrangian densities and the corresponding Hamilton equations or Euler-Lagrange equations. Our approach is quite distinct from that in the standard source on the hamiltonian formulation of integrable nonlinear evolution equations, the Faddeev-Takhtajan book [54].

In a discrete integrable system the hamiltonian $\boldsymbol{H}$ depends on often infinitely many independent variables:

$$\boldsymbol{H} = \boldsymbol{H}\left(\{p_n\}_{n=-\infty}^{\infty}, \{q_n\}_{n=-\infty}^{\infty}, t\right).$$

Then Hamilton equations have the following form:

$$\frac{\partial \boldsymbol{H}}{\partial p_n} = [q_n]_t, \qquad \frac{\partial \boldsymbol{H}}{\partial q_n} = -[p_n]_t.$$

Furthermore, if the hamiltonian $\boldsymbol{H}$ does not depend on t explicitly, then $\boldsymbol{H}$ is a conserved quantity [64]. Indeed,

$$\begin{aligned}\frac{d\boldsymbol{H}}{dt} &= \sum_{j=-\infty}^{\infty}\left(\frac{\partial \boldsymbol{H}}{\partial p_j}[p_j]_t + \frac{\partial \boldsymbol{H}}{\partial q_j}[q_j]_t\right) + \frac{\partial \boldsymbol{H}}{\partial t}\\ &= \sum_{j=-\infty}^{\infty}\left([q_j]_t[p_j]_t - [p_j]_t[q_j]_t\right) + \frac{\partial \boldsymbol{H}}{\partial t} = \frac{\partial \boldsymbol{H}}{\partial t},\end{aligned}$$

which vanishes identically if and only $\boldsymbol{H}$ does not depend explicitly on t.

Example 1.12 (Toda lattice) As an example, we consider the Toda lattice hamiltonian [91]

$$\boldsymbol{H} = \sum_{n=-\infty}^{\infty}\left(\tfrac{1}{2}p_n^2 + V(q_{n+1}-q_n)\right), \tag{1.18a}$$

where the potential is given by

$$V(r) = e^{-r} + r - 1 = \tfrac{1}{2}r^2 + O(r^3), \qquad r \to 0. \tag{1.18b}$$

Then the Hamilton equations of the Toda lattice system are as follows:

$$\begin{aligned}[q_n]_t &= \frac{\partial \boldsymbol{H}}{\partial p_n} = p_n,\\ [p_n]_t &= -\frac{\partial \boldsymbol{H}}{\partial q_n} = V'(q_{n+1}-q_n) - V'(q_n - q_{n-1})\\ &= e^{-(q_n-q_{n-1})} - e^{-(q_{n+1}-q_n)}.\end{aligned}$$

Here we have used that $V'(r) = 1 - e^{-r}$. We thus get the nonlinear Toda lattice equation

$$[q_n]_{tt} = e^{-(q_n - q_{n-1})} - e^{-(q_{n+1} - q_n)}. \tag{1.19}$$

which concludes the example.

Let us recall the definition of the Poisson bracket [64]

$$\{f, g\} = \sum_{j=-\infty}^{\infty} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right).$$

We therefore get

$$\{p_n, p_m\} = \sum_{j=-\infty}^{\infty} \left(\frac{\partial p_n}{\partial p_j} \frac{\partial p_m}{\partial q_j} - \frac{\partial p_n}{\partial q_j} \frac{\partial p_m}{\partial p_j} \right) = \sum_{j=-\infty}^{\infty} (\delta_{n,j} \cdot 0 - 0 \cdot \delta_{m,j}) = 0,$$

$$\{q_n, q_m\} = \sum_{j=-\infty}^{\infty} \left(\frac{\partial q_n}{\partial p_j} \frac{\partial q_m}{\partial q_j} - \frac{\partial q_n}{\partial q_j} \frac{\partial q_m}{\partial p_j} \right) = \sum_{j=-\infty}^{\infty} (0 \cdot \delta_{m,j} - \delta_{n,j} \cdot 0) = 0,$$

$$\{p_n, q_m\} = \sum_{j=-\infty}^{\infty} \left(\frac{\partial p_n}{\partial p_j} \frac{\partial q_m}{\partial q_j} - \frac{\partial p_n}{\partial q_j} \frac{\partial q_m}{\partial p_j} \right) = \sum_{j=-\infty}^{\infty} (\delta_{n,j}\delta_{m,j} - 0 \cdot 0) = \delta_{n,m}.$$

As a result,

$$[q_n]_t = \{q_n, \boldsymbol{H}\}, \qquad [p_n]_t = \{p_n, \boldsymbol{H}\}.$$

Since the hamiltonian $\boldsymbol{H}$ does not depend explicitly on time t, it is a conserved quantity:

$$\frac{d\boldsymbol{H}}{dt} = 0.$$

Introducing the lagrangian $\boldsymbol{L}$ by

$$\boldsymbol{L} = \sum_{n=-\infty}^{\infty} p_n [q_n]_t - \boldsymbol{H} = \sum_{n=-\infty}^{\infty} p_n^2 - \boldsymbol{H},$$

we obtain for the Toda lattice example

$$\boldsymbol{L} = \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}[q_n]_t^2 - V(q_{n+1} - q_n) \right).$$

The Euler-Lagrange equations

$$\frac{\partial \boldsymbol{L}}{\partial q_n} = \frac{d}{dt} \frac{\partial \boldsymbol{L}}{\partial [q_n]_t}$$

then lead to the equations of motion (1.19) derived before from the hamiltonian.

In a continuous system, where the position variable $x \in \mathbb{R}$ (KdV, NLS, SG, etc.) or $(x, y) \in \mathbb{R}^2$ (KPI, KPII), the lagrangian and hamiltonian are to be replaced by a lagrangian density or a hamiltonian density, respectively (cf. [64, Ch. 13], [55, Ch. 20]). Restricting ourselves to just one position variable x, the Euler-Lagrange equation for the lagrangian density

$$\mathcal{L} = \mathcal{L}(q, [q]_x, [q]_t, t)$$

reads

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial [q]_t} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial [q]_x} = 0,$$

as a result of the variational principle

$$\begin{aligned}\frac{dS}{d\alpha} &= \int_{-\infty}^{\infty} dx \int_{t_1}^{t_2} dt \left(\frac{\partial \mathcal{L}}{\partial q}\frac{\partial q}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial q_t}\frac{\partial q_t}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial q_x}\frac{\partial q_x}{\partial \alpha} \right) \\ &= \int_{-\infty}^{\infty} dx \int_{t_1}^{t_2} dt \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial q_t} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial q_x} \right) \frac{\partial q}{\partial \alpha} = 0.\end{aligned}$$

The hamiltonian density $\mathcal{H}$ can be expressed in the lagrangian density as follows:

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial [q]_t}[q]_t - \mathcal{L}.$$

Introducing $p = \frac{\partial \mathcal{L}}{\partial [q]_t}$, $\mathcal{H}$ is a function of q, p, q_x, p_x, and t. We then obtain the Hamilton equations

$$\begin{aligned}[q]_t &= \frac{\partial \mathcal{H}}{\partial p} - \frac{d}{dx}\frac{\partial \mathcal{H}}{\partial [p]_x}, \\ -[p]_t &= \frac{\partial \mathcal{H}}{\partial q} - \frac{d}{dx}\frac{\partial \mathcal{H}}{\partial [q]_x}.\end{aligned}$$

The Euler-Lagrange and Hamilton equations extend in a natural way to more than one spatial variable and to several generalized coordinates.

As an example, we consider the (scalar) nonsymmetric NLS system. Starting from the hamiltonian density

$$\mathcal{H} = -i[q_x r_x + (qr)^2],$$

where q is the generalized coordinate and r is the corresponding momentum, we get the Hamilton equations

$$\begin{aligned}iq_t + q_{xx} - 2q^2 r &= 0, \\ -ir_t + r_{xx} - 2qr^2 &= 0.\end{aligned}$$

It is easily verified that $\{q, q\} = \{r, r\} = 0$ and $\{r, q\} = 1$.

Let us extend this example to q as a row vector and r as a column vector of the same length. Let us define the hamiltonian density

$$\mathcal{H} = -i\left(q_x r_x - (qr)^2\right) = -i\sum_j [q_j]_x [r_j]_x - i\left(\sum_j q_j r_j\right)^2,$$

which is a scalar. Then the Hamilton equations are as follows:

$$0 = i[q_n]_t - i\frac{\partial\mathcal{H}}{\partial r_n} + i\frac{\partial}{\partial x}\frac{\partial\mathcal{H}}{\partial[r_n]_x} = i[q_n]_t - 2\left(\sum_j q_j r_j\right) q_n + [q_n]_{xx},$$

$$0 = -i[r_n]_t - i\frac{\partial\mathcal{H}}{\partial q_n} + i\frac{\partial}{\partial x}\frac{\partial\mathcal{H}}{\partial[q_n]_x} = -i[r_n]_t - 2r_n\left(\sum_j q_j r_j\right) + [r_n]_{xx}.$$

It is easily seen that

$$\{q_n, q_m\} = \{r_n, r_m\} = 0, \qquad \{r_n, q_m\} = \delta_{n,m}.$$

Let us consider two examples involving the lagrangian formalism.

Example 1.13 In the KdV case, we deal with a lagrangian density $\mathcal{L}$ of the type

$$\mathcal{L} = \mathcal{L}(q, q_t, q_x, q_{xx}, t).$$

The variational principle

$$\begin{aligned}\frac{dS}{d\alpha} &= \int_{-\infty}^{\infty} dx \int_{t_1}^{t_2} dt\left(\frac{\partial\mathcal{L}}{\partial q}\frac{\partial q}{\partial\alpha} + \frac{\partial\mathcal{L}}{\partial q_t}\frac{\partial q_t}{\partial\alpha} + \frac{\partial\mathcal{L}}{\partial q_x}\frac{\partial q_x}{\partial\alpha} + \frac{\partial\mathcal{L}}{\partial q_{xx}}\frac{\partial q_{xx}}{\partial\alpha}\right)\\ &= \int_{-\infty}^{\infty} dx \int_{t_1}^{t_2} dt\left(\frac{\partial\mathcal{L}}{\partial q} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial q_t} - \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial q_x} + \frac{\partial^2}{\partial x^2}\frac{\partial\mathcal{L}}{\partial q_{xx}}\right)\frac{\partial q}{\partial\alpha} = 0\end{aligned}$$

leads to the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial[q]_t} - \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial[q]_x} + \frac{\partial^2}{\partial x^2}\frac{\partial\mathcal{L}}{\partial[q]_{xx}} = 0. \qquad (1.20)$$

Taking [58]

$$\mathcal{L} = -\tfrac{1}{2}q_x q_t + (q_x)^3 + \tfrac{1}{2}(q_{xx})^2, \qquad u = q_x,$$

we obtain

$$0 = \tfrac{1}{2}(q_x)_t - \left(-\tfrac{1}{2}q_t + 3(q_x)^2\right)_x + (q_{xx})_{xx},$$

and hence

$$u_t + u_{xxx} - 6uu_x = 0.$$

The modified KdV equation (1.13) follows from the lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}q_x q_t - \tfrac{1}{2}(q_x)^4 + \tfrac{1}{2}(q_{xx})^2, \qquad u = q_x.$$

In this case, (1.20) implies that

$$0 = \tfrac{1}{2}(q_x)_t - (-\tfrac{1}{2}q_t - 2(q_x)^3)_x + (q_{xx})_{xx},$$

and therefore

$$u_t + 6u^2u_x + u_{xxx} = 0.$$

Chapter 2

Study of the NLS through the Zakharov-Shabat system

In this chapter we write up the direct and inverse scattering theory of the matrix Zakharov-Shabat system

$$iJ\frac{\partial X}{\partial x}(\lambda,x) - V(x)X(\lambda,x) = \lambda X(\lambda,x), \tag{2.1}$$

where

$$J = \begin{pmatrix} I_m & 0_{m\times n} \\ 0_{n\times m} & -I_n \end{pmatrix}, \qquad V(x) = \begin{pmatrix} 0_{m\times m} & iq(x) \\ ir(x) & 0_{n\times n} \end{pmatrix}, \tag{2.2}$$

the potentials $q(x)$ and $r(x)$ have their entries in $L^1(\mathbb{R})$, and λ is a spectral parameter. In the *defocusing case* we have[1] $r(x) = -q(x)^\dagger$ and hence $V(x)^\dagger = V(x)$; in the *focusing case* $r(x) = q(x)^\dagger$ and hence $V(x)^\dagger = -V(x)$.

We mention various textbooks in which Zakharov-Shabat systems are discussed [2, 76, 79, 54, 1, 5], although in not all of them the matrix Zakharov-Shabat system is treated our way. With respect to [10, 94, 41, 44], in the matrix Zakharov-Shabat system $-iJ$ gets replaced by iJ. The resulting replacement of $e^{i\lambda Jx}$ by $e^{-i\lambda Jx}$ implies an interchange of the roles of the upper and lower half-planes when analyticity properties come up. Apart from that, we shall adopt the same definitions of Jost solutions, transition coefficients, and reflection coefficients. The custom of having the transmission coefficients analytic in the upper half-plane forces us to define them in a different way.

Many of the techniques of proving the basic results have been developed in various publications [2, 10, 94, 5, 54]. For this reason we omit most of the proofs. We shall make extensive use of matrix notations.

[1] We write daggers to denote the complex conjugate transpose of a matrix or, less often, the adjoint operator on a complex Hilbert space. We denote complex conjugates of scalars by the asterisk.

2.1 Jost solutions and transition matrices

In this section we define the Jost solutions and transition coefficients and derive their analyticity properties. We conclude with their Wronskian relations.

1. Jost functions and transition coefficients. Let us define the $(m+n)\times m$ and $(m+n)\times n$ *Jost functions from the right* $\overline{\psi}(\lambda,x)$ and $\psi(\lambda,x)$, the $(m+n)\times m$ and $(m+n)\times n$ *Jost functions from the left* $\phi(\lambda,x)$ and $\overline{\phi}(\lambda,x)$, and the $(m+n)\times(m+n)$ *Jost matrices* $\Psi(\lambda,x)$ and $\Phi(\lambda,x)$ from the right and the left as those solutions to the matrix Zakharov-Shabat system (2.1) satisfying the asymptotic conditions

$$\Psi(\lambda,x)=\begin{pmatrix}\overline{\psi}(\lambda,x) & \psi(\lambda,x)\end{pmatrix}=\begin{cases}e^{-i\lambda Jx}[I_{m+n}+o(1)], & x\to+\infty,\\ e^{-i\lambda Jx}a_l(\lambda)+o(1), & x\to-\infty,\end{cases} \tag{2.3a}$$

$$\Phi(\lambda,x)=\begin{pmatrix}\phi(\lambda,x) & \overline{\phi}(\lambda,x)\end{pmatrix}=\begin{cases}e^{-i\lambda Jx}[I_{m+n}+o(1)], & x\to-\infty,\\ e^{-i\lambda Jx}a_r(\lambda)+o(1), & x\to+\infty.\end{cases} \tag{2.3b}$$

Then the system of equations (2.1) being first order implies

$$\Phi(\lambda,x)=\Psi(\lambda,x)a_r(\lambda),\qquad \Psi(\lambda,x)=\Phi(\lambda,x)a_l(\lambda). \tag{2.4}$$

We shall call $a_l(\lambda)$ and $a_r(\lambda)$ *transition matrices* from the left and the right, respectively.

2. Volterra integral equations and analyticity. Writing the matrix Zakharov-Shabat system (2.1) in the form

$$\frac{\partial}{\partial y}\left(e^{-i\lambda J(x-y)}X(\lambda,y)\right)=-iJe^{-i\lambda J(x-y)}V(y)X(\lambda,y),$$

we get

$$\Psi(\lambda,x)=e^{-i\lambda Jx}+iJ\int_x^{\infty}dy\,e^{i\lambda J(y-x)}V(y)\Psi(\lambda,y), \tag{2.5a}$$

$$\Phi(\lambda,x)=e^{-i\lambda Jx}-iJ\int_{-\infty}^{x}dy\,e^{-i\lambda J(x-y)}V(y)\Phi(\lambda,y). \tag{2.5b}$$

The Volterra integral equations (2.5) can be used to prove the existence and uniqueness of the solutions $\Psi(\lambda,x)$ and $\Phi(\lambda,x)$ to (2.1) that satisfy the asymptotic conditions

$$\begin{aligned}\Psi(\lambda,x)&=e^{-i\lambda Jx}[I_{m+n}+o(1)], & x&\to+\infty,\\ \Phi(\lambda,x)&=e^{-i\lambda Jx}[I_{m+n}+o(1)], & x&\to-\infty.\end{aligned}$$

In the proofs it is used in an essential way that the spectral parameter λ is real and the entries of the potential $V(x)$ belong to $L^1(\mathbb{R})$. The existence of and the expressions for the transition coefficients as well as all continuity and analyticity properties then follow as corollaries. These proofs have been detailed in many places (e.g., [10, 5, 41]).

Rewriting (2.5) we get

$$\Psi(\lambda,x)e^{i\lambda Jx} = I_{m+n}+iJ\int_x^\infty dy\, e^{i\lambda J(y-x)}V(y)\Big[\Psi(\lambda,y)e^{i\lambda Jy}\Big]\,e^{i\lambda J(x-y)}, \tag{2.6a}$$

$$\Phi(\lambda,x)e^{i\lambda Jx} = I_{m+n}-iJ\int_{-\infty}^x dy\, e^{-i\lambda J(x-y)}V(y)\Big[\Phi(\lambda,y)e^{i\lambda Jy}\Big]e^{i\lambda J(x-y)}. \tag{2.6b}$$

For any $(m+n)\times p$ matrix G we now define the $m\times p$ and $n\times p$ matrices

$$G^{\rm up} = \begin{pmatrix} I_m & 0_{m\times n}\end{pmatrix} G, \qquad G^{\rm dn} = \begin{pmatrix} 0_{m\times n} & I_n\end{pmatrix} G,$$

respectively. Splitting up (2.5) according to its four blocks we get

$$e^{i\lambda x}\overline{\psi}^{\rm up}(\lambda,x) = I_m - \int_x^\infty dy\, q(y)\left[\overline{\psi}^{\rm dn}(\lambda,y)e^{i\lambda y}\right], \tag{2.7a}$$

$$e^{-i\lambda x}\psi^{\rm up}(\lambda,x) = -\int_x^\infty dy\, e^{2i\lambda(y-x)}q(y)\left[\psi^{\rm dn}(\lambda,y)e^{-i\lambda y}\right], \tag{2.7b}$$

$$e^{i\lambda x}\overline{\psi}^{\rm dn}(\lambda,x) = \int_x^\infty dy\, e^{-2i\lambda(y-x)}r(y)\left[\overline{\psi}^{\rm up}(\lambda,y)e^{i\lambda y}\right], \tag{2.7c}$$

$$e^{-i\lambda x}\psi^{\rm dn}(\lambda,x) = I_n + \int_x^\infty dy\, r(y)\left[\psi^{\rm up}(\lambda,y)e^{-i\lambda y}\right], \tag{2.7d}$$

as well as

$$e^{i\lambda x}\phi^{\rm up}(\lambda,x) = I_m + \int_{-\infty}^x dy\, q(y)\left[\phi^{\rm dn}(\lambda,y)e^{i\lambda y}\right], \tag{2.8a}$$

$$e^{-i\lambda x}\overline{\phi}^{\rm up}(\lambda,x) = \int_{-\infty}^x dy\, e^{-2i\lambda(x-y)}q(y)\left[\overline{\phi}^{\rm dn}(\lambda,y)e^{-i\lambda y}\right], \tag{2.8b}$$

$$e^{i\lambda x}\phi^{\rm dn}(\lambda,x) = -\int_{-\infty}^x dy\, e^{2i\lambda(x-y)}r(y)\left[\phi^{\rm up}(\lambda,y)e^{i\lambda y}\right], \tag{2.8c}$$

$$e^{-i\lambda x}\overline{\phi}^{\rm dn}(\lambda,x) = I_n - \int_{-\infty}^x dy\, r(y)\left[\overline{\phi}^{\rm up}(\lambda,y)e^{-i\lambda y}\right]. \tag{2.8d}$$

Therefore, $e^{-i\lambda x}\psi^{\rm up}(\lambda,x)$, $e^{-i\lambda x}\psi^{\rm dn}(\lambda,x)$, $e^{i\lambda x}\phi^{\rm up}(\lambda,x)$, and $e^{i\lambda x}\phi^{\rm dn}(\lambda,x)$ are continuous in $\lambda\in\overline{\mathbb{C}^+}$, are analytic in $\lambda\in\mathbb{C}^+$, and approach $0_{m\times n}$, I_n, I_m, and $0_{n\times m}$ as $|\lambda|\to+\infty$ from within $\overline{\mathbb{C}^+}$. Analogously, $e^{i\lambda x}\overline{\psi}^{\rm up}(\lambda,x)$, $e^{i\lambda x}\overline{\psi}^{\rm dn}(\lambda,x)$, $e^{-i\lambda x}\overline{\phi}^{\rm up}(\lambda,x)$, and $e^{-i\lambda x}\overline{\phi}^{\rm dn}(\lambda,x)$ are continuous in $\lambda\in\overline{\mathbb{C}^-}$, are analytic in $\lambda\in\mathbb{C}^-$, and approach I_m, $0_{n\times m}$, $0_{m\times n}$, and I_n as $|\lambda|\to+\infty$ from within $\overline{\mathbb{C}^-}$.

Let us take the appropriate limits of the expressions in (2.7) and (2.8) as $x \to \pm\infty$. We get

$$a_l(\lambda) = \begin{pmatrix} a_{l1}(\lambda) & a_{l2}(\lambda) \\ a_{l3}(\lambda) & a_{l4}(\lambda) \end{pmatrix}, \qquad a_r(\lambda) = \begin{pmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda) \end{pmatrix},$$

where the various blocks are called *transition coefficients* and

$$a_{l1}(\lambda) = I_m - \int_{-\infty}^{\infty} dy\, q(y) \left[e^{i\lambda y} \overline{\psi}^{\mathrm{dn}}(\lambda, y) \right], \tag{2.9a}$$

$$a_{l4}(\lambda) = I_n + \int_{-\infty}^{\infty} dy\, r(y) \left[e^{-i\lambda y} \psi^{\mathrm{up}}(\lambda, y) \right], \tag{2.9b}$$

$$a_{r1}(\lambda) = I_m + \int_{-\infty}^{\infty} dy\, q(y) \left[e^{i\lambda y} \phi^{\mathrm{dn}}(\lambda, y) \right], \tag{2.9c}$$

$$a_{r4}(\lambda) = I_n - \int_{-\infty}^{\infty} dy\, r(y) \left[e^{-i\lambda y} \overline{\phi}^{\mathrm{up}}(\lambda, y) \right]. \tag{2.9d}$$

Thus, $a_{r1}(\lambda)$ and $a_{l4}(\lambda)$ are continuous in $\lambda \in \overline{\mathbb{C}^+}$, are analytic in $\lambda \in \mathbb{C}^+$, and tend to the identity matrix as $|\lambda| \to +\infty$ from within $\overline{\mathbb{C}^+}$. Analogously, $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are continuous in $\lambda \in \overline{\mathbb{C}^-}$, are analytic in $\lambda \in \mathbb{C}^-$, and tend to the identity matrix as $|\lambda| \to +\infty$ from within $\overline{\mathbb{C}^-}$. Similarly, we get

$$a_{l2}(\lambda) = -\int_{\infty}^{\infty} dy\, e^{2i\lambda y} q(y) \left[\psi^{\mathrm{dn}}(\lambda, y) e^{-i\lambda y} \right], \tag{2.10a}$$

$$a_{l3}(\lambda) = \int_{-\infty}^{\infty} dy\, e^{-2i\lambda y} r(y) \left[\overline{\psi}^{\mathrm{up}}(\lambda, y) e^{i\lambda y} \right], \tag{2.10b}$$

$$a_{r2}(\lambda) = \int_{-\infty}^{\infty} dy\, e^{2i\lambda y} q(y) \left[\overline{\phi}^{\mathrm{dn}}(\lambda, y) e^{-i\lambda y} \right], \tag{2.10c}$$

$$a_{r3}(\lambda) = -\int_{-\infty}^{\infty} dy\, e^{-2i\lambda y} r(y) \left[\phi^{\mathrm{up}}(\lambda, y) e^{i\lambda y} \right]. \tag{2.10d}$$

As a result, $a_{l2}(\lambda)$, $a_{l3}(\lambda)$, $a_{r2}(\lambda)$, and $a_{r3}(\lambda)$ are continuous in $\lambda \in \mathbb{R}$ and vanish as $\lambda \to \pm\infty$.

3. Wronskian relations. We conclude this section with the so-called *Wronskian relations*. Let $X(\lambda, x)$ be an $(m+n) \times q$ matrix solution of the matrix Zakharov-Shabat system (2.1) and $Y(\lambda, x)$ a $p \times (m+n)$ solution of the dual matrix Zakharov-Shabat system

$$-i \frac{\partial Y}{\partial x}(\lambda, x) J + Y(\lambda, x) V(x) = \lambda Y(\lambda, x). \tag{2.11}$$

Then

$$\begin{aligned} \frac{\partial}{\partial x} \left(Y(\lambda, x) X(\lambda, x) \right) &= iY(\lambda, x)[\lambda I_{m+n} - V(x)] J X(\lambda, x) \\ &\quad - iY(\lambda, x) J [\lambda I_{m+n} + V(x)] X(\lambda, x) \\ &= i\lambda [Y(\lambda, x) J X(\lambda, x) - Y(\lambda, x) J X(\lambda, x)] \\ &\quad - iY(\lambda, x)[V(x) J + J V(x)] X(\lambda, x) = 0_{p \times q}, \end{aligned}$$

because $V(x)$ and J anticommute. For $\lambda \in \overline{\mathbb{C}^+}$ we then get four Wronskian relations for $X \in \{\psi, \phi\}$ and $Y \in \{\breve{\psi}, \breve{\phi}\}$ and for $\lambda \in \overline{\mathbb{C}^-}$ four Wronskian relations for $X \in \{\overline{\psi}, \overline{\phi}\}$ and $Y \in \{\breve{\overline{\psi}}, \breve{\overline{\phi}}\}$. Other Wronskian relations are only valid for $\lambda \in \mathbb{R}$, because they involve solutions X and Y that are analytic in different half-planes.

Using that $\mathrm{Tr}[\lambda I_{m+n} + V(x)] = \lambda(m-n)$, for $\lambda \in \mathbb{R}$ the functions $\det \Psi(\lambda, x)$ and $\det \Phi(\lambda, x)$ satisfy the ordinary differential equation[2]

$$\psi'(x) = -i\lambda(m-n)\psi(x).$$

As a result of (2.3), for $\lambda \in \mathbb{R}$ we have

$$\det \Psi(\lambda, x) = \det \Phi(\lambda, x) = e^{-i\lambda(m-n)}. \tag{2.12}$$

In the *defocusing* case we have $V(x)^\dagger = V(x)$. Thus if $X(\lambda, x)$ satisfies the matrix Zakharov-Shabat system, $Y(\lambda, x) \stackrel{\text{def}}{=} JX(\lambda^*, x)^\dagger J$ satisfies the dual system (2.11). As a result, in the defocusing case the Jost matrices $\Psi(\lambda, x)$ and $\Phi(\lambda, x)$ and the transition matrices $a_l(\lambda)$ and $a_r(\lambda)$ are J-unitary in the sense that

$$\Psi(\lambda, x)^{-1} = J\Psi(\lambda, x)^\dagger J, \qquad \Phi(\lambda, x)^{-1} = J\Phi(\lambda, x)^\dagger J, \tag{2.13a}$$
$$a_r(\lambda) = a_l(\lambda)^{-1} = Ja_l(\lambda)^\dagger J, \qquad a_l(\lambda) = a_r(\lambda)^{-1} = Ja_r(\lambda)^\dagger J, \tag{2.13b}$$

where $\lambda \in \mathbb{R}$.

In the *focusing* case we have $V(x)^\dagger = -V(x)$. Thus if $X(\lambda, x)$ satisfies the matrix Zakharov-Shabat system, $Y(\lambda, x) \stackrel{\text{def}}{=} X(\lambda^*, x)^\dagger$ satisfies the dual system (2.11). As a result, in the focusing case the Jost matrices $\Psi(\lambda, x)$ and $\Phi(\lambda, x)$ and the transition matrices $a_l(\lambda)$ and $a_r(\lambda)$ are unitary in the sense that

$$\Psi(\lambda, x)^{-1} = \Psi(\lambda, x)^\dagger, \qquad \Phi(\lambda, x)^{-1} = \Phi(\lambda, x)^\dagger, \tag{2.14a}$$
$$a_r(\lambda) = a_l(\lambda)^{-1} = a_l(\lambda)^\dagger, \qquad a_l(\lambda) = a_r(\lambda)^{-1} = a_r(\lambda)^\dagger, \tag{2.14b}$$

where $\lambda \in \mathbb{R}$.

2.2 Jost Solutions as Fourier Transforms

In this section we write the Jost solutions as Fourier transforms of so-called kernel functions $\overline{K}(x, y)$, $K(x, y)$, $M(x, y)$, and $\overline{M}(x, y)$ which in turn will turn out to be integrable with respect to y for fixed x.

[2] We easily prove that the matrix differential equation $\Psi'(x) = A(x)\Psi(x)$ implies the scalar differential equation $(\det \Psi)'(x) = [\mathrm{Tr} A(x)] \det \Psi(x)$.

Write

$$\overline{\psi}(\lambda,x)=e^{-i\lambda x}\begin{pmatrix}I_m\\0_{n\times m}\end{pmatrix}+\int_x^\infty dy\,e^{-i\lambda y}\overline{K}(x,y),\tag{2.15a}$$
$$\psi(\lambda,x)=e^{i\lambda x}\begin{pmatrix}0_{m\times n}\\I_n\end{pmatrix}+\int_x^\infty dy\,e^{i\lambda y}K(x,y),\tag{2.15b}$$
$$\phi(\lambda,x)=e^{-i\lambda x}\begin{pmatrix}I_m\\0_{n\times m}\end{pmatrix}+\int_{-\infty}^x dy\,e^{-i\lambda y}M(x,y),\tag{2.15c}$$
$$\overline{\phi}(\lambda,x)=e^{i\lambda x}\begin{pmatrix}0_{m\times n}\\I_n\end{pmatrix}+\int_{-\infty}^x dy\,e^{i\lambda y}\overline{M}(x,y).\tag{2.15d}$$

Inverting the Fourier transforms we get for the *kernel functions*

$$\overline{K}(x,y)=\frac{1}{2\pi}\int_{-\infty}^\infty d\lambda\,e^{i\lambda y}\left\{\overline{\psi}(\lambda,x)-e^{-i\lambda x}\begin{pmatrix}I_m\\0_{n\times m}\end{pmatrix}\right\},\tag{2.16a}$$
$$K(x,y)=\frac{1}{2\pi}\int_{-\infty}^\infty d\lambda\,e^{-i\lambda y}\left\{\psi(\lambda,x)-e^{i\lambda x}\begin{pmatrix}0_{m\times n}\\I_n\end{pmatrix}\right\},\tag{2.16b}$$
$$M(x,y)=\frac{1}{2\pi}\int_{-\infty}^\infty d\lambda\,e^{i\lambda y}\left\{\phi(\lambda,x)-e^{-i\lambda x}\begin{pmatrix}I_m\\0_{n\times m}\end{pmatrix}\right\},\tag{2.16c}$$
$$\overline{M}(x,y)=\frac{1}{2\pi}\int_{-\infty}^\infty d\lambda\,e^{-i\lambda y}\left\{\overline{\phi}(\lambda,x)-e^{i\lambda x}\begin{pmatrix}0_{m\times n}\\I_n\end{pmatrix}\right\}.\tag{2.16d}$$

The half-line $y\geq x$ on which $\overline{K}(x,y)$ and $K(x,y)$ are supported, and the half-line $y\leq x$ on which $M(x,y)$ and $\overline{M}(x,y)$ are supported, are intimately related to the analyticity properties of the Jost functions.

Equations (2.5a), (2.15a), and (2.15b) imply that

$$\overline{K}^{\mathrm{up}}(x,y)=-\int_x^\infty dz\,q(z)\overline{K}^{\mathrm{dn}}(z,z+y-x),\tag{2.17a}$$
$$\overline{K}^{\mathrm{dn}}(x,y)=\tfrac{1}{2}r(\tfrac{1}{2}(x+y))+\int_x^{\frac{1}{2}(x+y)}dz\,r(z)\overline{K}^{\mathrm{up}}(z,x+y-z),\tag{2.17b}$$
$$K^{\mathrm{up}}(x,y)=-\tfrac{1}{2}q(\tfrac{1}{2}(x+y))-\int_x^{\frac{1}{2}(x+y)}dz\,q(z)K^{\mathrm{dn}}(z,x+y-z),\tag{2.17c}$$
$$K^{\mathrm{dn}}(x,y)=\int_x^\infty dz\,r(z)K^{\mathrm{up}}(z,z+y-x).\tag{2.17d}$$

As a result, we can express the potentials and their (partial) energy integrals in terms of the kernel functions from the right as follows:

$$q(x)=-2K^{\mathrm{up}}(x,x),\tag{2.18a}$$
$$r(x)=2\overline{K}^{\mathrm{dn}}(x,x),\tag{2.18b}$$

$$\int_x^\infty dz\, r(z)q(z) = -2K^{\rm dn}(x,x), \tag{2.18c}$$
$$\int_x^\infty dz\, q(z)r(z) = -2\overline{K}^{\rm up}(x,x). \tag{2.18d}$$

On the other hand, Eqs. (2.5b), (2.15c), and (2.15d) imply that

$$M^{\rm up}(x,y) = \int_{-\infty}^x dz\, q(z)M^{\rm dn}(z,z+y-x), \tag{2.19a}$$
$$M^{\rm dn}(x,y) = -\tfrac{1}{2}r(\tfrac{1}{2}(x+y)) - \int_{\frac{1}{2}(x+y)}^x dz\, r(z)M^{\rm up}(z,x+y-z), \tag{2.19b}$$
$$\overline{M}^{\rm up}(x,y) = \tfrac{1}{2}q(\tfrac{1}{2}(x+y)) + \int_{\frac{1}{2}(x+y)}^x dz\, q(z)\overline{M}^{\rm dn}(z,x+y-z), \tag{2.19c}$$
$$\overline{M}^{\rm dn}(x,y) = -\int_{-\infty}^x dz\, r(z)\overline{M}^{\rm up}(z,z+y-x). \tag{2.19d}$$

As a result, we can express the potentials and their (partial) energy integrals in terms of the kernel functions from the left as follows:

$$q(x) = 2\overline{M}^{\rm up}(x,x), \tag{2.20a}$$
$$r(x) = -2M^{\rm dn}(x,x), \tag{2.20b}$$
$$\int_{-\infty}^x dz\, r(z)q(z) = -2\overline{M}^{\rm dn}(x,x), \tag{2.20c}$$
$$\int_{-\infty}^x dz\, q(z)r(z) = -2M^{\rm up}(x,x). \tag{2.20d}$$

Using Gronwall's inequality[3] we easily prove the following estimates:

Theorem 2.1 *For potentials $q(x)$ and $r(x)$ with entries in $L^1(\mathbb{R})$ we have*

$$\int_x^\infty dy\,\|\overline{K}^{up}(x,y)\| \le \frac{C}{2}\nu_+(x)^2, \quad \int_x^\infty dy\,\|\overline{K}^{dn}(x,y)\| \le C\,\nu_+(x), \tag{2.21a}$$
$$\int_x^\infty dy\,\|K^{dn}(x,y)\| \le \frac{C}{2}\nu_+(x)^2, \quad \int_x^\infty dy\,\|K^{up}(x,y)\| \le C\,\nu_+(x), \tag{2.21b}$$
$$\int_x^\infty dy\,\|M^{up}(x,y)\| \le \frac{C}{2}\nu_-(x)^2, \quad \int_x^\infty dy\,\|M^{dn}(x,y)\| \le C\,\nu_-(x), \tag{2.21c}$$
$$\int_x^\infty dy\,\|\overline{M}^{dn}(x,y)\| \le \frac{C}{2}\nu_-(x)^2, \quad \int_x^\infty dy\,\|\overline{M}^{up}(x,y)\| \le C\,\nu_-(x), \tag{2.21d}$$

[3] Suppose $h(x)$ is a nonnegative function such that $|F(x)| \le 1+\int_x^\infty dy\, h(y)|F(y)|$. Then

$$|F(x)| \le \sum_{n=0}^\infty \frac{1}{n!}\left(\int_x^\infty dy\, h(y)\right)^n = \exp\left(\int_x^\infty dy\, h(y)\right).$$

The proof proceeds by iteration.

where $\nu_\pm(x)$ *are given by*

$$\nu_+(x) = \max\left(\int_x^\infty dy\,\|q(y)\|, \int_x^\infty dy\,\|r(y)\|\right), \tag{2.22a}$$

$$\nu_-(x) = \max\left(\int_{-\infty}^x dy\,\|q(y)\|, \int_{-\infty}^x dy\,\|r(y)\|\right), \tag{2.22b}$$

and $C = \exp(\frac{1}{2}[\max\{\|q\|_1, \|r\|_1\}]^2)$.

At this point we introduce the so-called Wiener algebra. Let $\mathcal{W}_{p\times q}$ denote the complex Banach space of all $p \times q$ matrix-valued functions $F(\lambda)$ of the form

$$F(\lambda) = F_\infty + \int_{-\infty}^\infty dy\,\hat{F}(y)e^{i\lambda y}, \qquad \lambda \in \mathbb{R},$$

where the entries of $\hat{F}(y)$ belong to $L^1(\mathbb{R})$. Similarly, let $\mathcal{W}^+_{p\times q}$ denote the complex Banach space of all $p \times q$ matrix-valued functions $F(\lambda)$ of the form

$$F(\lambda) = F_\infty + \int_0^\infty dy\,\hat{F}(y)e^{i\lambda y}, \qquad \lambda \in \mathbb{R},$$

where the entries of $\hat{F}(y)$ belong to $L^1(\mathbb{R}^+)$. Finally, let $\mathcal{W}^-_{p\times q}$ denote the complex Banach space of all $p \times q$ matrix-valued functions $F(\lambda)$ of the form

$$F(\lambda) = F_\infty + \int_{-\infty}^0 dy\,\hat{F}(y)e^{i\lambda y}, \qquad \lambda \in \mathbb{R},$$

where the entries of $\hat{F}(y)$ belong to $L^1(\mathbb{R}^-)$. Then the estimates (2.21) imply that for each $x \in \mathbb{R}$ the matrix functions $\Psi(\lambda, x)e^{i\lambda Jx}$ and $\Psi(\lambda, x)e^{i\lambda Jx}$ belong to $\mathcal{W}_{(m+n)\times(m+n)}$.

According to *Wiener's theorem* [cf. [97] in the scalar case], the matrix function $F \in \mathcal{W}_{p\times p}$ has its inverse $F(\cdot)^{-1}$ in $\mathcal{W}_{p\times p}$ whenever F_∞ and, for each $\lambda \in \mathbb{R}$, $F(\lambda)$ are nonsingular matrices. By the same token, the matrix function $F \in \mathcal{W}^\pm_{p\times p}$ has its inverse $F(\cdot)^{-1}$ in $\mathcal{W}^\pm_{p\times p}$ whenever F_∞ and, for each $\lambda \in \overline{\mathbb{C}^\pm}$, $F(\lambda)$ are nonsingular matrices. In fact, the matrix case follows from the scalar case by using the algebra property on the determinant and the entries of the cofactor matrix.

Using (2.17b) we estimate for $\delta > 0$

$$\begin{aligned}\int_0^\infty dx\,\|\overline{K}^{\mathrm{dn}}(x,x+\delta)\| &\leq \tfrac{1}{2}\int_0^\infty dx\,\|r(x+\tfrac{1}{2}\delta)\| \\ &+ \int_0^\infty dx \int_x^{x+\frac{1}{2}\delta} dz\,\|r(z)\|\,\|\overline{K}^{\mathrm{up}}(z, x+\tfrac{1}{2}-z)\| \\ &= \tfrac{1}{2}\int_{\frac{1}{2}\delta}^\infty dz\,\|r(z)\| + \int_0^\infty dz\,\|r(z)\| \int_0^{\frac{1}{2}\delta} dw\,\|\overline{K}^{\mathrm{up}}(z,w)\|.\end{aligned}$$

We derive similar estimates K^{up}, M^{dn}, and $\overline{M}^{\mathrm{up}}$. Now (2.9) and (2.15) imply that

$$a_{l1}(\lambda) = I_m - \int_{-\infty}^{\infty} dy\, q(y) \int_0^{\infty} dz\, e^{-i\lambda z} \overline{K}^{\mathrm{dn}}(y, y+z), \tag{2.23a}$$

$$a_{l4}(\lambda) = I_n + \int_{-\infty}^{\infty} dy\, r(y) \int_0^{\infty} dz\, e^{i\lambda z} K^{\mathrm{up}}(y, y+z), \tag{2.23b}$$

$$a_{r1}(\lambda) = I_m + \int_{-\infty}^{\infty} dy\, q(y) \int_0^{\infty} dz\, e^{i\lambda z} M^{\mathrm{dn}}(y, y+z), \tag{2.23c}$$

$$a_{r4}(\lambda) = I_n - \int_{-\infty}^{\infty} dy\, r(y) \int_0^{\infty} dz\, e^{-i\lambda z} \overline{M}^{\mathrm{up}}(y, y+z), \tag{2.23d}$$

The above estimates imply that $a_{l1}(\lambda)$, $a_{l4}(\lambda)$, $a_{r1}(\lambda)$, and $a_{r4}(\lambda)$ belong to $\mathcal{W}^-_{m\times m}$, $\mathcal{W}^+_{n\times n}$, $\mathcal{W}^+_{m\times m}$, and $\mathcal{W}^-_{n\times n}$, respectively.

Equations (2.10) and (2.15) imply that

$$\begin{aligned} a_{l2}(\lambda) &= -\int_{-\infty}^{\infty} dy\, e^{2i\lambda y} q(y) \left(I_n + \int_0^{\infty} dz\, e^{i\lambda z} K^{\mathrm{dn}}(y, y+z) \right) \\ &= -\int_{-\infty}^{\infty} dy\, e^{2i\lambda y} q(y) - \int_{-\infty}^{\infty} dy\, q(y) \int_{2y}^{\infty} dz\, e^{i\lambda z} K^{\mathrm{dn}}(y, z-y), \end{aligned} \tag{2.24a}$$

$$\begin{aligned} a_{l3}(\lambda) &= \int_{-\infty}^{\infty} dy\, e^{-2i\lambda y} r(y) \left(I_m + \int_0^{\infty} dz\, e^{-i\lambda z} \overline{K}^{\mathrm{up}}(y, y+z) \right) \\ &= \int_{-\infty}^{\infty} dy\, e^{-2i\lambda y} r(y) + \int_{-\infty}^{\infty} dy\, r(y) \int_{2y}^{\infty} dz\, e^{-i\lambda z} \overline{K}^{\mathrm{up}}(y, z-y), \end{aligned} \tag{2.24b}$$

$$\begin{aligned} a_{r2}(\lambda) &= \int_{-\infty}^{\infty} dy\, e^{2i\lambda y} q(y) \left(I_n + \int_0^{\infty} dz\, e^{-i\lambda z} \overline{M}^{\mathrm{dn}}(y, y-z) \right) \\ &= \int_{-\infty}^{\infty} dy\, e^{2i\lambda y} q(y) + \int_{-\infty}^{\infty} dy\, q(y) \int_{-2y}^{\infty} dz\, e^{-i\lambda z} \overline{M}^{\mathrm{dn}}(y, -y-z), \end{aligned} \tag{2.24c}$$

$$\begin{aligned} a_{r3}(\lambda) &= -\int_{-\infty}^{\infty} dy\, e^{-2i\lambda y} r(y) \left(I_m + \int_0^{\infty} dz\, e^{i\lambda z} M^{\mathrm{up}}(y, y-z) \right) \\ &= -\int_{-\infty}^{\infty} dy\, e^{-2i\lambda y} r(y) - \int_{-\infty}^{\infty} dy\, r(y) \int_{-2y}^{\infty} dz\, e^{i\lambda z} M^{\mathrm{up}}(y, -y-z), \end{aligned} \tag{2.24d}$$

respectively. Now observe that $\int_{-\infty}^{\infty} dy \int_{2y}^{\infty} dz$ and $\int_{-\infty}^{\infty} dy \int_{-2y}^{\infty} dz$ can be transformed into $\int_{-\infty}^{\infty} dz \int_{-\infty}^{\frac{1}{2}z} dy$ and $\int_{-\infty}^{\infty} dz \int_{-\frac{1}{2}z}^{\infty} dy$, respectively. So let us rearrange the following double integral:

$$\begin{aligned} &\int_{-\infty}^{\infty} dz \int_{-\infty}^{\frac{1}{2}z} dy\, \|q(y)\|\, \|K^{\mathrm{dn}}(y, z-y)\| = \int_{-\infty}^{\infty} dy\, \|q(y)\| \int_{2y}^{\infty} dz \|K^{\mathrm{dn}}(y, z-y)\| \\ &= \int_{-\infty}^{\infty} dy\, \|q(y)\| \int_y^{\infty} dw\, \|K^{\mathrm{dn}}(y, w)\| = \int_{-\infty}^{\infty} dy\, \|q(y)\|\, \mu(K^{\mathrm{dn}}; y), \end{aligned}$$

which is finite. As a result, $a_{l2}(\lambda)$ belongs to $\mathcal{W}_{m\times n}$. In the same way we prove that $a_{l3}(\lambda)$, $a_{r2}(\lambda)$, and $a_{r3}(\lambda)$ belong to $\mathcal{W}_{n\times m}$, $\mathcal{W}_{m\times n}$, and $\mathcal{W}_{n\times m}$, respectively. Thus $a_l(\lambda)$ and $a_r(\lambda)$ belong to $\mathcal{W}_{(m+n)\times(m+n)}$.

2.3 Scattering coefficients

The scattering data used as input to the inverse scattering problem for the matrix NLS equation do not involve the transition coefficients but rather the scattering coefficients. These can be derived from the transition coefficients by writing the Jost solutions as the columns of modified Jost matrices which are analytic in either the upper or the lower half-plane.

1. Scattering coefficients. Let us reshuffle the columns of the Jost matrices in such a way that the resulting *modified Jost matrices* are analytic in either the upper or the lower complex half-plane. In other words, put

$$F_+(\lambda,x) = \begin{pmatrix}\phi(\lambda,x) & \psi(\lambda,x)\end{pmatrix}, \tag{2.25a}$$

$$F_-(\lambda,x) = \begin{pmatrix}\overline{\psi}(\lambda,x) & \overline{\phi}(\lambda,x)\end{pmatrix}. \tag{2.25b}$$

Then $F_+(\lambda,x)$ is continuous in $\lambda \in \overline{\mathbb{C}^+}$, is analytic in $\lambda \in \mathbb{C}^+$, and tends to I_{m+n} as $|\lambda| \to +\infty$ from within $\overline{\mathbb{C}^+}$. Also, $F_-(\lambda,x)$ is continuous in $\lambda \in \overline{\mathbb{C}^-}$, is analytic in $\lambda \in \mathbb{C}^-$, and tends to I_{m+n} as $|\lambda| \to +\infty$ from within $\overline{\mathbb{C}^-}$. Moreover, $F_+(\lambda,x)$ belongs to $\mathcal{W}^+_{(m+n)\times(m+n)}$ and $F_-(\lambda,x)$ to $\mathcal{W}^-_{(m+n)\times(m+n)}$. The modified Jost matrices satisfy

$$F_+(\lambda,x) = \begin{cases} e^{-i\lambda Jx}\begin{pmatrix} a_{r1}(\lambda) & 0_{m\times n} \\ a_{r3}(\lambda) & I_n \end{pmatrix} + o(1), & x \to +\infty, \\ e^{-i\lambda Jx}\begin{pmatrix} I_m & a_{l2}(\lambda) \\ 0_{n\times m} & a_{l4}(\lambda) \end{pmatrix} + o(1), & x \to -\infty, \end{cases} \tag{2.26a}$$

$$F_-(\lambda,x) = \begin{cases} e^{-i\lambda Jx}\begin{pmatrix} I_m & a_{r2}(\lambda) \\ 0_{n\times m} & a_{r4}(\lambda) \end{pmatrix} + o(1), & x \to +\infty, \\ e^{-i\lambda Jx}\begin{pmatrix} a_{l1}(\lambda) & 0_{m\times n} \\ a_{l3}(\lambda) & I_n \end{pmatrix} + o(1), & x \to -\infty. \end{cases} \tag{2.26b}$$

As a result, we obtain

$$F_+(\lambda,x) = \Psi(\lambda,x)\begin{pmatrix} a_{r1}(\lambda) & 0_{m\times n} \\ a_{r3}(\lambda) & I_n \end{pmatrix} = \Phi(\lambda,x)\begin{pmatrix} I_m & a_{l2}(\lambda) \\ 0_{n\times m} & a_{l4}(\lambda) \end{pmatrix}, \tag{2.27a}$$

$$F_-(\lambda,x) = \Psi(\lambda,x)\begin{pmatrix} I_m & a_{r2}(\lambda) \\ 0_{n\times m} & a_{r4}(\lambda) \end{pmatrix} = \Phi(\lambda,x)\begin{pmatrix} a_{l1}(\lambda) & 0_{m\times n} \\ a_{l3}(\lambda) & I_n \end{pmatrix}. \tag{2.27b}$$

Lemma 2.2 *For $\lambda \in \overline{\mathbb{C}^+}$, $a_{l4}(\lambda)$ is invertible iff $a_{r1}(\lambda)$ is invertible, while, for $\lambda \in \overline{\mathbb{C}^-}$, $a_{r4}(\lambda)$ is invertible iff $a_{l1}(\lambda)$ is invertible. Moreover,*

$$\det a_{l4}(\lambda) = \det a_{r1}(\lambda), \qquad \lambda \in \overline{\mathbb{C}^+}, \tag{2.28a}$$

$$\det a_{r4}(\lambda) = \det a_{l1}(\lambda), \qquad \lambda \in \overline{\mathbb{C}^-}. \tag{2.28b}$$

Moreover, in the defocusing case the matrices $a_{l1}(\lambda)$, $a_{l4}(\lambda)$, $a_{r1}(\lambda)$, and $a_{r4}(\lambda)$ are nonsingular for $\lambda \in \mathbb{R}$.

Proof. Taking determinants of (2.27a) and by analytic continuation we get

$$\det F_+(\lambda, x) = e^{-i\lambda(m-n)} \det a_{r1}(\lambda) = e^{-i\lambda(m-n)} \det a_{l4}(\lambda), \qquad \lambda \in \overline{\mathbb{C}^+},$$
$$\det F_-(\lambda, x) = e^{-i\lambda(m-n)} \det a_{r4}(\lambda) = e^{-i\lambda(m-n)} \det a_{l1}(\lambda), \qquad \lambda \in \overline{\mathbb{C}^-}.$$

Thus, for $\lambda \in \overline{\mathbb{C}^+}$, $a_{r1}(\lambda)$ is invertible iff $a_{l4}(\lambda)$ is invertible. On the other hand, for $\lambda \in \overline{\mathbb{C}^-}$, $a_{l1}(\lambda)$ is invertible iff $a_{r4}(\lambda)$ is invertible.

Next, in the defocusing case the transition matrices $a_l(\lambda)$ and $a_r(\lambda)$ are J-unitary. Then for $\lambda \in \mathbb{R}$ the identities

$$a_l(\lambda)^\dagger J a_l(\lambda) J = a_r(\lambda)^\dagger J a_r(\lambda) J = I_{m+n}$$

imply that for $\lambda \in \mathbb{R}$

$$a_{l1}(\lambda)^\dagger a_{l1}(\lambda) = I_m + a_{l3}(\lambda)^\dagger a_{l3}(\lambda),$$
$$a_{l4}(\lambda)^\dagger a_{l4}(\lambda) = I_n + a_{l2}(\lambda)^\dagger a_{l2}(\lambda),$$

so that $\|a_{l1}(\lambda)\boldsymbol{x}\| \geq \|\boldsymbol{x}\|$ for each $\boldsymbol{x} \in \mathbb{C}^m$ and $\|a_{l4}(\lambda)\boldsymbol{y}\| \geq \|\boldsymbol{y}\|$ for each $\boldsymbol{y} \in \mathbb{C}^n$. Hence, $a_{l1}(\lambda)$ and $a_{l4}(\lambda)$ are nonsingular for $\lambda \in \mathbb{R}$. The nonsingularity of the other two diagonal transition coefficients follows from the first part of this lemma. ∎

By a *spectral singularity* we mean a value of $\lambda \in \mathbb{R}$ for which at least one of the numbers $\det a_{l4}(\lambda) = \det a_{r1}(\lambda)$ and $\det a_{r4}(\lambda) = \det a_{l1}(\lambda)$ vanishes. If there are no spectral singularities, the number of zeros of $\det a_{l4}(\lambda) = \det a_{r1}(\lambda)$ in $\mathbb{C}^+$ and the number of zeros of $\det a_{r4}(\lambda) = \det a_{l1}(\lambda)$ in $\mathbb{C}^-$ are finite. In the defocusing case there are no spectral singularities, because of the second part of Lemma 2.2. Spectral singularities are often considered unwarranted complications to scattering theory and are therefore usually assumed away.

Let us now seek a *scattering matrix* $S(\lambda)$ such that

$$F_-(\lambda, x) = F_+(\lambda, x) J S(\lambda) J, \tag{2.29}$$

where $x, \lambda \in \mathbb{R}$ (except in points $\lambda \in \mathbb{R}$ where $S(\lambda)$ is discontinuous). Writing

$$S(\lambda) = \begin{pmatrix} T_r(\lambda) & L(\lambda) \\ R(\lambda) & T_l(\lambda) \end{pmatrix},$$

we get the pair of matrix equations

$$\begin{aligned} I_m &= a_{r1}(\lambda)T_r(\lambda), & a_{r2}(\lambda) &= -a_{r1}(\lambda)L(\lambda), \\ a_{r4}(\lambda) &= T_l(\lambda) - a_{r3}(\lambda)L(\lambda), & 0_{n\times m} &= a_{r3}(\lambda)T_r(\lambda) - R(\lambda), \\ I_n &= a_{l4}(\lambda)T_l(\lambda), & a_{l3}(\lambda) &= -a_{l4}(\lambda)R(\lambda), \\ a_{l1}(\lambda) &= T_r(\lambda) - a_{l2}(\lambda)R(\lambda), & 0_{m\times n} &= a_{l2}(\lambda)T_l(\lambda) - L(\lambda). \end{aligned}$$

Equation (2.29) is the Riemann-Hilbert problem used by most researchers to solve the inverse scattering problem [1, 5, 6, 54]. Observe that the existence of $S(\lambda)$ satisfying (2.29) implies (and is in fact equivalent to) the invertibility of $a_{r1}(\lambda)$ and $a_{l4}(\lambda)$. In this case we define the *transmission coefficients* by

$$T_r(\lambda) = a_{r1}(\lambda)^{-1}, \qquad T_l(\lambda) = a_{l4}(\lambda)^{-1}. \tag{2.30a}$$

Thus in the scalar case ($m = n = 1$) the two transmission coefficients coincide [cf. (2.28)]. We now easily find the *reflection coefficients* to be given by

$$L(\lambda) = -a_{r1}(\lambda)^{-1}a_{r2}(\lambda) = a_{l2}(\lambda)a_{l4}(\lambda)^{-1}, \tag{2.30b}$$

$$R(\lambda) = -a_{l4}(\lambda)^{-1}a_{l3}(\lambda) = a_{r3}(\lambda)a_{r1}(\lambda)^{-1}. \tag{2.30c}$$

We easily check that

$$\begin{aligned} S(\lambda) &= \begin{pmatrix} a_{r1}(\lambda)^{-1} & 0_{m\times n} \\ 0_{n\times m} & a_{l4}(\lambda)^{-1} \end{pmatrix} \begin{pmatrix} I_m & -a_{r2}(\lambda) \\ -a_{l3}(\lambda) & I_n \end{pmatrix} \\ &= \begin{pmatrix} I_m & a_{l2}(\lambda) \\ a_{r3}(\lambda) & I_n \end{pmatrix} \begin{pmatrix} a_{r1}(\lambda)^{-1} & 0_{m\times n} \\ 0_{n\times m} & a_{l4}(\lambda)^{-1} \end{pmatrix}. \end{aligned} \tag{2.31}$$

Since $a_l(\lambda)$ and $a_r(\lambda)$ are each other's inverses, we have

$$T_l(\lambda) = a_{l4}(\lambda)^{-1} = a_{r4}(\lambda) - a_{r3}(\lambda)a_{r1}(\lambda)^{-1}a_{r2}(\lambda), \tag{2.32a}$$

$$T_r(\lambda) = a_{r1}(\lambda)^{-1} = a_{l1}(\lambda) - a_{l2}(\lambda)a_{l4}(\lambda)^{-1}a_{l3}(\lambda), \tag{2.32b}$$

provided one (and hence both of) $a_{r1}(\lambda)$ and $a_{l4}(\lambda)$ is invertible. We now easily verify that

$$F_+(\lambda, x)^{-1} = \begin{cases} \begin{pmatrix} T_r(\lambda) & 0_{m\times n} \\ -R(\lambda) & I_n \end{pmatrix} e^{i\lambda Jx} + o(1), & x \to +\infty, \\ \begin{pmatrix} I_m & -L(\lambda) \\ 0_{n\times m} & T_l(\lambda) \end{pmatrix} e^{i\lambda Jx} + o(1), & x \to -\infty, \end{cases} \tag{2.33}$$

Therefore,

$$F_+(\lambda, x)^{-1} = \begin{pmatrix} T_r(\lambda) & 0_{m\times n} \\ -R(\lambda) & I_n \end{pmatrix} \Psi(\lambda, x)^{-1} = \begin{pmatrix} I_m & -L(\lambda) \\ 0_{n\times m} & T_l(\lambda) \end{pmatrix} \Phi(\lambda, x)^{-1}. \tag{2.34}$$

It is now clear from (2.31) that $S(\lambda)$ belongs to $\mathcal{W}_{(m+n)\times(m+n)}$ whenever $a_{r1}(\lambda)$ and $a_{l4}(\lambda)$ are nonsingular for $\lambda \in \mathbb{R}$.

In the defocusing case it follows directly from (2.13) that $S(\lambda)$ is continuous in $\lambda \in \mathbb{R}$ and unitary. By the same token, in the focusing case it follows from (2.14) that $S(\lambda)$ is J-unitary, but it need not be continuous in $\lambda \in \mathbb{R}$. These discontinuities in the reflection and transmission matrices can only occur if there exist diagonal transition coefficients that are not invertible, i.e., if there exist spectral singularities.

2. Dual scattering coefficients. If there are no adjoint symmetries on the potential [i.e., if we are not in the focusing or the defocusing case], then we need to introduce the duals of the matrix functions introduced so far. In the focusing and the defocusing cases, a simple adjoint (plus an occasional $\pm$ sign) would suffice.

The invertibility of the Jost matrices $\Psi(\lambda, x)$ and $\Phi(\lambda, x)$ for $\lambda \in \mathbb{R}$ implies

$$\Psi(\lambda,x)^{-1} = \begin{pmatrix} \breve{\psi}(\lambda,x) \\ \overline{\breve{\psi}}(\lambda,x) \end{pmatrix} = \begin{cases} [I_{m+n} + o(1)]e^{i\lambda Jx}, & x \to +\infty, \\ a_r(\lambda)e^{i\lambda Jx} + o(1), & x \to -\infty, \end{cases} \tag{2.35a}$$

$$\Phi(\lambda,x)^{-1} = \begin{pmatrix} \overline{\breve{\phi}}(\lambda,x) \\ \breve{\phi}(\lambda,x) \end{pmatrix} = \begin{cases} [I_{m+n} + o(1)]e^{i\lambda Jx}, & x \to -\infty, \\ a_l(\lambda)e^{i\lambda Jx} + o(1), & x \to +\infty, \end{cases} \tag{2.35b}$$

where we have employed (2.3) and $a_r(\lambda) = a_l(\lambda)^{-1}$. Rewriting the matrix Zakharov-Shabat system (2.1) and using that $V(x)$ and J anticommute, we see that $\Psi(\lambda,x)^{-1}$ and $\Phi(\lambda,x)^{-1}$ satisfy the dual matrix Zakharov-Shabat system

$$-i\frac{\partial Y}{\partial x}(\lambda,x)J + Y(\lambda,x)V(x) = \lambda Y(\lambda,x). \tag{2.36}$$

We can thus define the *dual Jost matrices* $\breve{\Psi}(\lambda,x)$ and $\breve{\Phi}(\lambda,x)$ as the inverses of the Jost matrices $\Psi(\lambda,x)$ and $\Phi(\lambda,x)$, respectively. In principle, the direct and inverse scattering theory for the matrix Zakharov-Shabat system (2.1) could just as easily be developed for the dual matrix Zakharov-Shabat system (2.36), where we rely on the Volterra integral equations

$$\Psi(\lambda,x)^{-1} = I_{m+n} + i\int_x^\infty dy\, \Psi(\lambda,y)^{-1}V(y)Je^{i\lambda J(x-y)}, \tag{2.37a}$$

$$\Phi(\lambda,x)^{-1} = I_{m+n} - i\int_{-\infty}^x dy\, \Phi(\lambda,y)^{-1}V(y)Je^{i\lambda J(x-y)}. \tag{2.37b}$$

However, in the defocusing and focusing cases the symmetries present make it unnecessary to do so.

Let us now define the *dual scattering matrix* $\breve{S}(\lambda)$ as follows:

$$F_+(\lambda,x) = F_-(\lambda,x)J\breve{S}(\lambda)J = F_-(\lambda,x)JS(\lambda)^{-1}J, \tag{2.38}$$

where $\lambda, x \in \mathbb{R}$ (except in points $\lambda \in \mathbb{R}$ where $\breve{S}(\lambda)$ is not continuous in $\lambda \in \mathbb{R}$) and the rightmost part of (2.38) follows straight from (2.29). Put

$$\breve{S}(\lambda) = S(\lambda)^{-1} = \begin{pmatrix} \breve{T}_l(\lambda) & \breve{R}(\lambda) \\ \breve{L}(\lambda) & \breve{T}_r(\lambda) \end{pmatrix}.$$

Then the dual transmission coefficients $\breve{T}_l(\lambda)$ and $\breve{T}_r(\lambda)$ are meromorphic in $\lambda \in \mathbb{C}^-$. From $J\breve{S}(\lambda)J = F_-(\lambda, x)^{-1}F_+(\lambda, x)$ we easily obtain

$$\breve{T}_l(\lambda) = a_{l1}(\lambda)^{-1} = a_{r1}(\lambda) - a_{r2}(\lambda)a_{r4}(\lambda)^{-1}a_{r3}(\lambda), \tag{2.39a}$$

$$\breve{T}_r(\lambda) = a_{r4}(\lambda)^{-1} = a_{l4}(\lambda) - a_{l3}(\lambda)a_{l1}(\lambda)^{-1}a_{l2}(\lambda). \tag{2.39b}$$

$$\breve{R}(\lambda) = a_{r2}(\lambda)a_{r4}(\lambda)^{-1} = -a_{l1}(\lambda)^{-1}a_{l2}(\lambda), \tag{2.39c}$$

$$\breve{L}(\lambda) = a_{l3}(\lambda)a_{l1}(\lambda)^{-1} = -a_{r4}(\lambda)^{-1}a_{r3}(\lambda). \tag{2.39d}$$

As a result, we get

$$F_-(\lambda, x)^{-1} = \begin{cases} \begin{pmatrix} I_m & -\breve{R}(\lambda) \\ 0_{n\times m} & \breve{T}_r(\lambda) \end{pmatrix} e^{i\lambda Jx} + o(1), & x \to +\infty, \\ \begin{pmatrix} \breve{T}_l(\lambda) & 0_{m\times n} \\ -\breve{L}(\lambda) & I_n \end{pmatrix} e^{i\lambda Jx} + o(1), & x \to -\infty. \end{cases} \tag{2.40}$$

Therefore,

$$F_-(\lambda, x)^{-1} = \begin{pmatrix} I_m & -\breve{R}(\lambda) \\ 0_{n\times m} & \breve{T}_r(\lambda) \end{pmatrix} \Psi(\lambda, x)^{-1} = \begin{pmatrix} \breve{T}_l(\lambda) & 0_{m\times n} \\ -\breve{L}(\lambda) & I_n \end{pmatrix} \Phi(\lambda, x)^{-1}. \tag{2.41}$$

It is now clear that $\breve{S}(\lambda)$ and, for each $x \in \mathbb{R}$, $\Psi(\lambda, x)^{-1}$ and $\Phi(\lambda, x)^{-1}$ belong to $\mathcal{W}_{(m+n)\times(m+n)}$ whenever $a_{l1}(\lambda)$ and $a_{r4}(\lambda)$ are nonsingular for $\lambda \in \mathbb{R}$.

2.4 Marchenko equations

In this section we introduce the right and left Marchenko equations. In the absence of bound states, we give a full derivation. Even though we present their explicit forms in general, a full derivation of the exact form of the Marchenko integral kernels is complicated if the transmission matrices have multiple poles [41, 44]. Thus, in most of the literature one deals only with situations, where the poles of the transmission matrices are simple.

1. Formulation of the Marchenko equations. Starting from (2.29) and (2.38) we get for $\lambda \in \mathbb{R}$

$$\overline{\psi}(\lambda, x) = \phi(\lambda, x)T_r(\lambda) - \psi(\lambda, x)R(\lambda), \tag{2.42a}$$

$$\psi(\lambda, x) = -\overline{\psi}(\lambda, x)\breve{R}(\lambda) + \overline{\phi}(\lambda, x)\breve{T}_r(\lambda), \tag{2.42b}$$

$$\phi(\lambda, x) = \overline{\psi}(\lambda, x)\breve{T}_l(\lambda) - \overline{\phi}(\lambda, x)\breve{L}(\lambda), \tag{2.42c}$$

$$\overline{\phi}(\lambda, x) = -\phi(\lambda, x)L(\lambda) + \psi(\lambda, x)T_l(\lambda). \tag{2.42d}$$

Let us now write for $\lambda \in \mathbb{R}$

$$L(\lambda) = \int_{-\infty}^{\infty} dy\, e^{i\lambda y}\ell(y), \qquad \breve{R}(\lambda) = \int_{-\infty}^{\infty} dy\, e^{i\lambda y}\breve{\rho}(y), \tag{2.43a}$$

$$R(\lambda) = \int_{-\infty}^{\infty} dy\, e^{-i\lambda y}\rho(y), \qquad \breve{L}(\lambda) = \int_{-\infty}^{\infty} dy\, e^{-i\lambda y}\breve{\ell}(y), \tag{2.43b}$$

where $L(\lambda)$ and $\breve{R}(\lambda)$ belong to $\mathcal{W}_{m\times n}$ and $R(\lambda)$ and $\breve{L}(\lambda)$ belong to $\mathcal{W}_{n\times m}$. In other words, the entries of $\ell(y)$, $\breve{\rho}(y)$, $\rho(y)$, and $\breve{\ell}(y)$ belong to $L^1(\mathbb{R})$.

In the *defocusing case* we have by the unitarity of the scattering matrix

$$\breve{R}(\lambda) = R(\lambda)^\dagger, \qquad \breve{L}(\lambda) = L(\lambda)^\dagger, \tag{2.44a}$$

where $\lambda \in \mathbb{R}$. In this case $\breve{\ell}(y) = \ell(y)^\dagger$ and $\breve{\rho}(y) = \rho(y)^\dagger$ for a.e. $y \in \mathbb{R}$. In the *focusing case* we have instead by the J-unitarity of the scattering matrix

$$\breve{R}(\lambda) = -R(\lambda)^\dagger, \qquad \breve{L}(\lambda) = -L(\lambda)^\dagger, \tag{2.44b}$$

where $\lambda \in \mathbb{R}$. In this case $\breve{\ell}(y) = -\ell(y)^\dagger$ and $\breve{\rho}(y) = -\rho(y)^\dagger$ for a.e. $y \in \mathbb{R}$.

If there are *no bound states* and hence $T_r(\lambda)$ and $T_l(\lambda)$ are analytic in $\lambda \in \mathbb{C}^+$, we obtain from (2.42) and (2.15)

$$\overline{K}(x,y) + \begin{pmatrix} 0_{m\times n} \\ I_n \end{pmatrix} \rho(x+y) + \int_x^\infty dz\, K(x,z)\rho(z+y) = 0_{(m+n)\times m}, \tag{2.45a}$$

$$K(x,y) + \begin{pmatrix} I_m \\ 0_{n\times m} \end{pmatrix} \breve{\rho}(x+y) + \int_x^\infty dz\, \overline{K}(x,z)\breve{\rho}(z+y) = 0_{(m+n)\times n}, \tag{2.45b}$$

$$M(x,y) + \begin{pmatrix} 0_{m\times n} \\ I_n \end{pmatrix} \breve{\ell}(x+y) + \int_{-\infty}^x dz\, \overline{M}(x,z)\breve{\ell}(z+y) = 0_{(m+n)\times m}, \tag{2.45c}$$

$$\overline{M}(x,y) + \begin{pmatrix} I_m \\ 0_{n\times m} \end{pmatrix} \ell(x+y) + \int_{-\infty}^x dz\, M(x,z)\ell(y+z) = 0_{(m+n)\times n}. \tag{2.45d}$$

In this case Eqs. (2.45) can be summarized as

$$\alpha_l(x,y) + \omega_l(x,y) + \int_x^\infty dz\, \alpha_l(x,z)\omega_l(z,y) = 0_{(m+n)\times(m+n)}, \tag{2.46a}$$

$$\alpha_r(x,y) + \omega_r(x,y) + \int_{-\infty}^x dz\, \alpha_r(x,z)\omega_r(z,y) = 0_{(m+n)\times(m+n)}, \tag{2.46b}$$

where

$$\omega_l(x,y) = \begin{pmatrix} 0_{m\times m} & \breve{\rho}(x+y) \\ \rho(x+y) & 0_{n\times n} \end{pmatrix}, \qquad \alpha_l(x,y) = \begin{pmatrix} \overline{K}(x,y) & K(x,y) \end{pmatrix},$$

$$\omega_r(x,y) = \begin{pmatrix} 0_{m\times m} & \ell(x+y) \\ \breve{\ell}(x+y) & 0_{n\times n} \end{pmatrix}, \qquad \alpha_r(x,y) = \begin{pmatrix} M(x,y) & \overline{M}(x,y) \end{pmatrix}.$$

In the *defocusing case* there are no bound states and we get (2.46), where

$$\omega_l(x,y)=\omega_l(y,x)^\dagger,\qquad \omega_r(x,y)=\omega_r(y,x)^\dagger, \tag{2.47a}$$

which means that $\breve{\rho}(z)=\rho(z)^\dagger$ and $\breve{\ell}(z)=\ell(z)^\dagger$. In the *focusing case without bound states* we get (2.46), where

$$\omega_l(x,y)=J\omega_l(y,x)^\dagger J,\qquad \omega_r(x,y)=J\omega_r(y,x)^\dagger J, \tag{2.47b}$$

which means that $\breve{\rho}(z)=-\rho(z)^\dagger$ and $\breve{\ell}(z)=-\ell(z)^\dagger$.

If the transmission coefficients $T_r(\lambda)$ and $T_l(\lambda)$ are continuous in $\lambda\in\mathbb{R}$ and hence the number of bound states is finite, then the Marchenko equations are given by (2.46), where

$$\omega_l(x,y)=\begin{pmatrix}0_{m\times m} & \breve{\rho}(x+y)+\breve{\mathbf{\Gamma}}_l(x+y)\\ \rho(x+y)+\mathbf{\Gamma}_l(x+y) & 0_{n\times n}\end{pmatrix}, \tag{2.48a}$$

$$\omega_r(x,y)=\begin{pmatrix}0_{m\times m} & \ell(x+y)+\mathbf{\Gamma}_r(x+y)\\ \breve{\ell}(x+y)+\breve{\mathbf{\Gamma}}_r(x+y) & 0_{n\times n}\end{pmatrix}, \tag{2.48b}$$

where

$$\mathbf{\Gamma}_l(x+y)=\sum_{j=1}^{N}e^{-(x+y)a_j}\sum_{s=0}^{N_j-1}\frac{(x+y)^s}{s!}\Gamma_{ljs}, \tag{2.49a}$$

$$\breve{\mathbf{\Gamma}}_l(x+y)=\sum_{j=1}^{\breve{N}}e^{-(x+y)\breve{a}_j}\sum_{s=0}^{\breve{N}_j-1}\frac{(x+y)^s}{s!}\breve{\Gamma}_{ljs}, \tag{2.49b}$$

$$\mathbf{\Gamma}_r(x+y)=\sum_{j=1}^{N}e^{(x+y)a_j}\sum_{s=0}^{N_j-1}\frac{(x+y)^s}{s!}\Gamma_{rjs}, \tag{2.49c}$$

$$\breve{\mathbf{\Gamma}}_r(x+y)=\sum_{j=1}^{\breve{N}}e^{(x+y)\breve{a}_j}\sum_{s=0}^{\breve{N}_j-1}\frac{(x+y)^s}{s!}\breve{\Gamma}_{rjs}. \tag{2.49d}$$

Here $ia_1,\ldots,ia_N$ are the poles of the transmission coefficients $T_l(\lambda)$ and $T_r(\lambda)$ in $\mathbb{C}^+$ and $-i\breve{a}_1,\ldots,-i\breve{a}_{\breve{N}}$ are the poles of the transmission coefficients $\breve{T}_l(\lambda)$ and $\breve{T}_r(\lambda)$ in $\mathbb{C}^-$. The matrices Γ_{ljs}, Γ_{rjs}, $\breve{\Gamma}_{ljs}$, and $\breve{\Gamma}_{rjs}$ are called norming constants.[4] In other words,

$$\overline{K}(x,y)+\begin{pmatrix}0_{m\times n}\\ I_n\end{pmatrix}\Omega_l(x+y)+\int_x^\infty dz\,K(x,z)\Omega_l(z+y)=0_{(m+n)\times m}, \tag{2.50a}$$

$$K(x,y)+\begin{pmatrix}I_m\\ 0_{n\times m}\end{pmatrix}\breve{\Omega}_l(x+y)+\int_x^\infty dz\,\overline{K}(x,z)\breve{\Omega}_l(z+y)=0_{(m+n)\times n}, \tag{2.50b}$$

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[4] Almost everyone assumes the poles to be simple. Then the expressions for $\mathbf{\Gamma}_l(x+y)$ etc. simplify to $\mathbf{\Gamma}_l(x+y)=\sum_{j=1}^N e^{-(x+y)a_j}\Gamma_{lj0}$, etc. The derivation of these kernels for multiple poles can be found in [45, 26].

$$M(x,y)+\begin{pmatrix}0_{m\times n}\\ I_n\end{pmatrix}\breve{\Omega}_r(x+y)+\int_{-\infty}^{x}dz\,\overline{M}(x,z)\breve{\Omega}_r(z+y)=0_{(m+n)\times m}, \tag{2.50c}$$

$$\overline{M}(x,y)+\begin{pmatrix}I_m\\ 0_{n\times m}\end{pmatrix}\Omega_r(x+y)+\int_{-\infty}^{x}dz\,M(x,z)\Omega_r(z+y)=0_{(m+n)\times n}, \tag{2.50d}$$

where

$$\omega_l(x,y)=\begin{pmatrix}0_{n\times n} & \breve{\Omega}_l(x+y)\\ \Omega_l(x+y) & 0_{m\times m}\end{pmatrix},\quad \omega_r(x,y)=\begin{pmatrix}0_{n\times n} & \Omega_r(x+y)\\ \breve{\Omega}_r(x+y) & 0_{n\times n}\end{pmatrix}.$$

In the *focusing case* with transmission coefficients continuous in $\lambda\in\mathbb{R}$ we get instead of (2.47b) [cf. [44]]

$$\omega_l(x,y)=J\omega_l(y,x)^\dagger J,\qquad \omega_r(x,y)=J\omega_r(y,x)^\dagger J. \tag{2.51}$$

2. Unique solvability. The Marchenko equations (2.46a) and (2.46b) are systems of integral equations having a structured integral kernel. The structure consists of having the integral kernel depend on the sum of its arguments rather than on the separate arguments themselves. Another feature is that there is a Marchenko equation, either (2.46a) or (2.46b), for each $x\in\mathbb{R}$, where the kernel does not really depend on $x\in\mathbb{R}$. In fact, when replacing x in (2.46a) by a larger $\hat{x}$, we actually "compress" the integral kernel into a right lower corner. If the system were to be discrete, it is as if the system matrix were to be replaced by a right lower square corner. By the same token, when replacing x in (2.46b) by a smaller $\hat{x}$, we "compress" the integral kernel into a left upper corner.

Theorem 2.3 *Assume that*

$$\int_{2x_0}^{\infty}dz\,\|\omega_l(z)\|<+\infty.$$

Suppose that, for every $x\geq x_0$, (2.46a) *has at least solution $\alpha_l(x,y)$ satisfying*

$$\sup_{x\geq x_0}\int_x^{\infty}dy\,\|\alpha_l(x,y)\|<+\infty.$$

Then for each $x\geq x_0$ and for an arbitrary inhomogeneous term $g_l(y)$ belonging to a suitable vector function space $E(x,+\infty)^{m+n}$, the integral equation

$$f_l(y)+\int_x^{\infty}dz\,f_l(z)\omega_l(z,y)=g_l(y)$$

has a unique solution in $E(x,+\infty)^{m+n}$.

A proof of Theorem 2.3 (though under a superfluous symmetry condition on the Marchenko kernel) can be found in [12]. The allowed function spaces E include L^p $(1 \leq p \leq +\infty)$ and BC. There is an analogous result for (2.46b).

In the defocusing case the Marchenko equations (2.46a) and (2.46b) are uniquely solvable and the integral operator is selfadjoint with norm strictly less than one [10, 41]. This is a direct consequence of the fact that in the defocusing case the reflection coefficients are all matrices of norm strictly less than one. In fact, in the defocusing case the norm of the Marchenko operator on $L^2(x,+\infty)^{n\times 1}$ is bounded above by $\sup_{\lambda\in\mathbb{R}} \|R(\lambda)\|$. Thus, in principle, these Marchenko equations can be solved by iteration.

In the focusing case either Marchenko equation (2.46b) can be split in an upper and a lower portion which are coupled [cf. (2.50)]. Formally,

$$\begin{pmatrix} I & K \\ -K^\dagger & I \end{pmatrix} \begin{pmatrix} \alpha^{\mathrm{up}} \\ \alpha^{\mathrm{dn}} \end{pmatrix} = - \begin{pmatrix} \omega^{\mathrm{up}} \\ \omega^{\mathrm{dn}} \end{pmatrix}.$$

The adjoint symmetry of this system is a result of the symmetry relation (2.51). Since $I + KK^\dagger$ and $I + K^\dagger K$ are invertible linear operators, we get as a unique solution

$$\begin{pmatrix} \alpha^{\mathrm{up}} \\ \alpha^{\mathrm{dn}} \end{pmatrix} = - \begin{pmatrix} (I+KK^\dagger)^{-1} & -K(I+K^\dagger K)^{-1} \\ K^\dagger(I+KK^\dagger)^{-1} & (I+K^\dagger K)^{-1} \end{pmatrix} \begin{pmatrix} \omega^{\mathrm{up}} \\ \omega^{\mathrm{dn}} \end{pmatrix}.$$

Thus the Marchenko integral equations are uniquely solvable [94, 41].[5]

It has been shown [70, 69] that the scalar (1 + 1) focusing Zakharov-Shabat system does not have isolated nonreal eigenvalues if $\int_{-\infty}^{\infty} dx\,|q(x)| < \frac{\pi}{2}$, where the bound is optimal. For single-lobe potentials, where $q(x)$ has a single maximum, the eigenvalues were shown to be imaginary and their exact number was determined. These results were extended in various steps. In [71] it was proven that there do not exist isolated nonreal eigenvalues nor spectral singularities of the matrix Zakharov-Shabat system (3.1) whenever

$$\int_{-\infty}^{\infty} dx\, \max(\|q(x)\|, \|r(x)\|) < \frac{\pi}{2}.$$

Thus under this condition the Marchenko theory is particularly simple.

3. Retrieving the potential from the Marchenko solution. To compute the potentials $q(x)$ and $r(x)$ from the solution of the Marchenko equation (2.46a), we need to use (2.18a)-(2.18b). Analogously, (2.20a)-(2.20b) are required to derive the potentials from the solution of the Marchenko equation (2.46b).

[5] In principle, the unique solvability has only been established for $E = L^2$. A compactness argument [41] is required to extend the result to the other allowed vector function spaces.

Finding a $1,1$-correspondence between potentials (without spectral singularities) and Marchenko kernels is known as the *characterization problem.* It has recently been solved completely [49]: Assuming that

a. the Marchenko equations (2.46a) whose kernels are given by (2.48a), (2.49a), and (2.49b) and where $\rho(x)$ and $\breve{\rho}(x)$ have their entries in $L^1(\mathbb{R})$, are uniquely solvable for $x \geq x_0$, and

b. the Marchenko equations (2.46b) whose kernels are given by (2.48b), (2.49c), and(2.49d) and where $\ell(x)$ and $\breve{\ell}(x)$ have their entries in $L^1(\mathbb{R})$, aree uniquely solvable for $x \leq x_0$,

we get unique potentials $q(x)$ and $r(x)$ with entries belonging to $L^1(\mathbb{R})$. In the focusing case the assumptions on the unique solvability of the Marchenko equations are superfluous. In the defocusing case there is a $1,1$-correspondence between potentials $q(x)$ and reflection coefficients from the right $R(\lambda)$ satisfying

$$\sup_{\lambda\in\mathbb{R}} \|R(\lambda)\| < 1, \qquad R(\lambda) = \int_{-\infty}^{\infty} dy\, e^{-i\lambda y}\rho(y),$$

where $\rho(x)$ has its entries in $L^1(\mathbb{R})$. In the defocusing case there is also a $1,1$-correspondence between potentials $q(x)$ and reflection coefficients from the left $L(\lambda)$ satisfying

$$\sup_{\lambda\in\mathbb{R}} \|L(\lambda)\| < 1, \qquad L(\lambda) = \int_{-\infty}^{\infty} dy\, e^{i\lambda y}\ell(y),$$

where $\ell(x)$ has its entries in $L^1(\mathbb{R})$.

2.5 Propagation of scattering data

1. What are the scattering data? The Marchenko kernel contains all of the information comprising the usual scattering data in a faithful way. In particular, it is the sum of two contributions: 1) the Fourier transform of a reflection coefficient, and 2) a bound state contribution encoding the discrete Zakharov-Shabat eigenvalues and the norming constants. More precisely,

$$\Omega_l(x,y) = \rho(x+y) + \sum_{j=1}^{N} e^{-(x+y)a_j} \sum_{s=0}^{N_j-1} \frac{(x+y)^s}{s!}\Gamma_{ljs},$$

$$\breve{\Omega}_l(x,y) = \breve{\rho}(x+y) + \sum_{j=1}^{\breve{N}} e^{-(x+y)\breve{a}_j} \sum_{s=0}^{\breve{N}_j-1} \frac{(x+y)^s}{s!}\breve{\Gamma}_{ljs},$$

$$\Omega_r(x,y) = \ell(x+y) + \sum_{j=1}^{N} e^{(x+y)a_j} \sum_{s=0}^{N_j-1} \frac{(x+y)^s}{s!}\Gamma_{rjs},$$

$$\breve{\Omega}_r(x,y) = \breve{\ell}(x+y) + \sum_{j=1}^{\breve{N}} e^{(x+y)\breve{a}_j} \sum_{s=0}^{\breve{N}_j-1} \frac{(x+y)^s}{s!}\breve{\Gamma}_{rjs},$$

where $ia_1, \ldots, ia_N$ are the discrete eigenvalues of the matrix Zakharov-Shabat system in $\mathbb{C}^+$ and $-i\breve{a}_1, \ldots, -i\breve{a}_{\breve{N}}$ are the discrete eigenvalues of the matrix Zakharov-Shabat system in $\mathbb{C}^-$. The coefficients Γ_j and $\breve{\Gamma}_j$ are called *norming constants.* In the focusing case, symmetry relations imply that

$$\breve{\Omega}_l(x+y) = -\Omega_l(y+x)^\dagger, \qquad \breve{\Omega}_r(x+y) = -\Omega_r(y+x)^\dagger.$$

In the defocusing case, there are no eigenvalue terms and the symmetry relations are given by

$$\breve{\Omega}_l(x+y) = \Omega_l(y+x)^\dagger, \qquad \breve{\Omega}_r(x+y) = \Omega_r(y+x)^\dagger.$$

Since the Marchenko kernels encode the usual scattering data

$$\{R(\lambda), \{ia_j, \Gamma_{ljs}\}_{s=0}^{N_j-1}{}_{j=1}^{N}; \breve{R}(\lambda), \{-i\breve{a}_j, \breve{\Gamma}_{ljs}\}_{s=0}^{\breve{N}_j-1}{}_{j=1}^{\breve{N}}\}$$

faithfully, the Marchenko kernels $\Omega_l(x+y)$ and $\breve{\Omega}_l(x+y)$ themselves can be used as the scattering data. In the focusing case it is sufficient to use either the "classical" scattering data

$$\{R(\lambda), \{ia_j, \Gamma_{ljs}\}_{s=0}^{N_j-1}{}_{j=1}^{N}\}$$

or the Marchenko kernel $\Omega_l(x+y)$ as scattering data. In the defocusing case, where there are no bound states, we can either take the reflection coefficient $R(\lambda)$ or the Marchenko kernel $\rho(x+y)$ as scattering data.

2. Propagation of scattering data. It is important to understand how the Marchenko kernels propagate in time if the potentials evolve according to the matrix NLS system. We have previously derived the Lax pair

$$L = iJ(\partial_x - \boldsymbol{Q}), \qquad A = 2iJ\{\partial_x^2 - \boldsymbol{Q}\partial_x + \tfrac{1}{2}\boldsymbol{Q}_x - \tfrac{1}{2}\boldsymbol{Q}^2\},$$

of the matrix NLS system

$$iJ\boldsymbol{Q}_t + \boldsymbol{Q}_{xx} - 2\boldsymbol{Q}^3 = 0_{(m+n)\times(m+n)}.$$

Translating this information into our treatment of the matrix Zakharov-Shabat system (2.1), we get

$$\boldsymbol{Q} = \begin{pmatrix} 0_{m\times m} & q \\ -r & 0_{n\times n} \end{pmatrix}.$$

Thus we have the Lax pair equations

$$L\Psi = \lambda\Psi, \qquad \Psi_t = A\Psi.$$

where

$$L = \begin{pmatrix} i\partial_x I_m & -iq \\ -ir & -i\partial_x I_n \end{pmatrix}, \qquad A = \begin{pmatrix} 2i\partial_x^2 I_m + iqr & -2iq\partial_x + iq_x \\ -2ir\partial_x + ir_x & -2i\partial_x^2 I_n - irq \end{pmatrix}.$$

Letting Ψ stand for a Jost solution [which obviously solves $L\Psi = \lambda\Psi$], the second equation $\Psi_t = A\Psi$ tells us how this Jost solution evolves in time.

We now consider the abstract scattering problem (using a fictitious time τ, while t is viewed as a parameter) for the matrix Zakharov-Shabat system. A "particle" subject to the matrix Zakharov-Shabat hamiltonian L is supposed to be asymptotically subject to the free matrix Zakharov-Shabat hamiltonian L_0 in the remote past ($\tau \to -\infty$) and in the far future ($\tau \to +\infty$). This means that, as $\tau \to \pm\infty$, L and L_0 have the same fictitious time evolution. For any vector ϕ, there are supposed to exist vectors $Z_\pm\phi$ and $W_\pm\phi$ such that

$$\begin{aligned} e^{-i\tau L}\phi &\simeq e^{-i\tau L_0} Z_\pm\phi, \qquad \tau \to \pm\infty, \\ e^{-i\tau L_0}\phi &\simeq e^{-i\tau L} W_\pm\phi, \qquad \tau \to \pm\infty. \end{aligned}$$

The linear operators $Z_\pm$ and $W_\pm$ are called *wave operators* or *Møller operators.* They are defined as the strong limits

$$Z_\pm\phi = \lim_{\tau\to\pm\infty} e^{i\tau L_0} e^{-i\tau L}\phi, \qquad W_\pm\phi = \lim_{\tau\to\pm\infty} e^{i\tau L} e^{-i\tau L_0}\phi. \tag{2.52}$$

The following result is known [71] [cf. [45] in the defocusing case]:

Theorem 2.4 *Suppose there are no spectral singularities. Then $W_\pm\phi$ is defined for any vector $\phi \in L^2(\mathbb{R})^{m+n}$, is one-to-one, and has a closed complement $\mathcal{M}(L)$ of the finite dimensional subspace spanned by all eigenvectors and generalized eigenvectors of L as its image. On the other hand, $Z_\pm\phi$ is only defined for $\phi \in \mathcal{M}(L)$ and*

$$\begin{aligned} Z_\pm W_\pm\phi &= \phi, \qquad \phi \in L^2(\mathbb{R})^{m+n}, \\ W_\pm Z_\pm\phi &= \phi, \qquad \phi \in \mathcal{M}(L). \end{aligned}$$

We also have the intertwining relations

$$L_0 Z_\pm = Z_\pm L, \qquad L W_\pm = W_\pm L_0.$$

The trick now is to map the remote past asymptotics (described by $Z_-\phi$) into the far future asymptotics (described by $Z_+\phi$) by applying the *scattering operator* S defined by

$$S = Z_+ W_-,$$

as clarified by the diagram

$$L^2(\mathbb{R})^{m+n} \xrightarrow{W_-} \mathcal{M}(L) \xrightarrow{Z_+} L^2(\mathbb{R})^{m+n}.$$

Then S is a boundedly invertible operator on $L^2(\mathbb{R})^{m+n}$. Obviously,

$$\begin{aligned} L_0S &= L_0(Z_+W_-) = (L_0Z_+)W_- = (Z_+L)W_- \\ &= Z_+(LW_-) = Z_+(W_-L_0) = (Z_+W_-)L_0 = SL_0, \end{aligned}$$

so that the scattering operator S and the free hamiltonian L_0 commute. In the defocusing case, the wave operators $Z_\pm$ and $W_\pm$ are isometries and the scattering operator S is unitary. In the focusing case, $Z_\pm J$ and $JW_\pm$ are isometries and $S^{-1} = JS^\dagger J$.

Time dependent scattering theory has been developed from the 1950's to describe the past and future asymptotics of quantum mechanical states where particles are moving under the effect of a hamiltonian H and behave as if they were moving under the effect of a free hamiltonian in the remote past and the far future. This is the typical situation of scattering under the influence of a localized interaction. We refer the reader to the textbooks [68, 99, 96].

Introducing the Fourier transform operator $\mathbb{F}$ by

$$(\mathbb{F}\phi)(\lambda) = \int_{-\infty}^{\infty} dx\, e^{i\lambda Jx}\phi(x), \qquad (\mathbb{F}^{-1}\hat{\phi})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda Jx}\hat{\phi}(\lambda),$$

it is clear that

$$(\mathbb{F}L_0\phi)(\lambda) = \lambda(\mathbb{F}\phi)(\lambda), \qquad \phi \in L^2(\mathbb{R})^{m+n}.$$

Exploiting that S and L_0 commute [and hence that $\mathbb{F}S\mathbb{F}^{-1}$ and the operator of multiplication by the independent variable commute], it is easily understood [88] that $\mathbb{F}S\mathbb{F}^{-1}$ is the operator of premultiplication by an $(m+n)\times(m+n)$ matrix function $S(\lambda)$. In [45], it has been shown that $S(\lambda)$ coincides with the scattering matrix introduced before. Using a diagonalization $\mathbb{G}$ of L which turns the absolutely continuous part of L into the multiplication by the independent variable on $L^2(\mathbb{R})^{m+n}$, we arrive at the commutative diagram

$$\begin{tikzcd} L^2(\mathbb{R})^{m+n} \arrow[r, "W_-"] \arrow[d, "\mathbb{F}"'] & \mathcal{M}(L) \arrow[r, "Z_+"] \arrow[d, "\mathbb{G}"'] & L^2(\mathbb{R})^{m+n} \arrow[d, "\mathbb{F}"] \\ L^2(\mathbb{R})^{m+n} \arrow[r, "\mathbb{G}W_-\mathbb{F}^{-1}"] & L^2(\mathbb{R})^{m+n} \arrow[r, "\mathbb{F}Z_+\mathbb{G}^{-1}"] & L^2(R)^{m+n} \end{tikzcd}$$

Let us now consider time dependence (using t, not τ). As discussed before [cf. (1.2)], the Lax pair equation (1.1) for $\{L, A\}$ can be written as

$$U(t,s)L(s) = L(t)U(t,s),$$

where $(\partial/\partial t)U(t,s) = A(t)U(t,s)$, $U(s,s) = I$, and the t-dependence of L and A has been written explicitly. Then

$$U(t,s)W_\pm(s) = W_\pm(t)U_0(t,s), \qquad U_0(t,s)Z_\pm(s) = Z_\pm(t)U(t,s),$$

where $(\partial/\partial t)U_0(t,s) = A_0U_0(t,s)$ and $U_0(s,s) = I$ [Lax pair equations for $\{L_0, A_0\}$]. Thus

$$\begin{aligned} U_0(t,s)S(s) &= U_0(t,s)Z_+(s)W_-(s) = Z_+(t)U(t,s)W_-(s) \\ &= Z_+(t)W_-(t)U_0(t,s) = S(t)U_0(t,s). \end{aligned}$$

Since

$$\mathbb{F}U_0(t,s)\mathbb{F}^{-1} = \mathbb{F}e^{(t-s)A_0}\mathbb{F}^{-1} = e^{-2i(t-s)\lambda^2 J},$$

we obtain the time evolution

$$e^{-2i(t-s)\lambda^2 J}S(\lambda;s) = S(\lambda;t)e^{-2i(t-s)\lambda^2 J}. \tag{2.53}$$

Using reflection and transmission matrices, we obtain

$$\begin{aligned} &\begin{pmatrix} e^{-2i(t-s)\lambda^2}I_m & 0_{m\times n} \\ 0_{n\times m} & e^{2i(t-s)\lambda^2} \end{pmatrix} \begin{pmatrix} T_r(\lambda;s) & L(\lambda;s) \\ R(\lambda;s) & T_l(\lambda;s) \end{pmatrix} \\ &= \begin{pmatrix} T_r(\lambda;t) & L(\lambda;t) \\ R(\lambda;t) & T_l(\lambda;t) \end{pmatrix} \begin{pmatrix} e^{-2i(t-s)\lambda^2}I_m & 0_{m\times n} \\ 0_{n\times m} & e^{2i(t-s)\lambda^2} \end{pmatrix}. \end{aligned}$$

Thus

$$T_l(\lambda;t) = T_l(\lambda;s), \qquad T_r(\lambda;t) = T_r(\lambda;s), \tag{2.54a}$$

$$e^{4it\lambda^2}R(\lambda;t) = e^{4is\lambda^2}R(\lambda;s), \qquad e^{-4it\lambda^2}L(\lambda;t) = e^{-4is\lambda^2}L(\lambda;s). \tag{2.54b}$$

In other words, the transmission coefficients do not depend on time, but the reflection coefficients do, though in an elementary way. Finally, since the time evolution (2.53) also holds for the dual scattering matrix

$$\breve{S}(\lambda;t) = S(\lambda;t)^{-1} = \begin{pmatrix} \breve{T}_l(\lambda;t) & \breve{R}(\lambda;t) \\ \breve{L}(\lambda;t) & \breve{T}_r(\lambda;t) \end{pmatrix},$$

we obtain in a similar way

$$\breve{T}_l(\lambda;t) = \breve{T}_l(\lambda;s), \qquad \breve{T}_r(\lambda;t) = \breve{T}_r(\lambda;s), \tag{2.55a}$$

$$e^{-4it\lambda^2}\breve{R}(\lambda;t) = e^{-4is\lambda^2}\breve{R}(\lambda;s), \qquad e^{4it\lambda^2}\breve{L}(\lambda;t) = e^{4is\lambda^2}\breve{L}(\lambda;s). \tag{2.55b}$$

Using (2.43) and the time evolution of the reflection coefficients, we obtain

$$\begin{aligned} \rho(x;t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{i\lambda x}R(\lambda;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{i\lambda x}e^{4it\lambda^2}R(\lambda;0), \\ \ell(x;t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda x}L(\lambda;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda x}e^{-4it\lambda^2}L(\lambda;0). \end{aligned}$$

Thus

$$\rho_t + 4i\rho_{xx} = 0_{n\times m},$$
$$\ell_t - 4i\ell_{xx} = 0_{m\times n}.$$

It can be proved that the corresponding Marchenko kernels satisfy the same time evolution PDE's:

$$[\Omega_l]_t + 4i[\Omega_l]_{xx} = 0_{n\times m}, \tag{2.56a}$$
$$[\Omega_r]_t - 4i[\Omega_r]_{xx} = 0_{m\times n}. \tag{2.56b}$$

In a similar way we obtain

$$[\breve{\Omega}_l]_t - 4i[\breve{\Omega}_l]_{xx} = 0_{m\times n}, \tag{2.56c}$$
$$[\breve{\Omega}_r]_t + 4i[\breve{\Omega}_r]_{xx} = 0_{n\times m}. \tag{2.56d}$$

The inverse scattering transform for the matrix NLS system can now be expressed in various ways. In the classical way, written in the focusing case, assuming that the transmission coefficients have only simple poles, and using reflection coefficients, we have

$$\begin{array}{ccc} q(x,0) & \xrightarrow{\text{direct scattering}} & \{R(\lambda), \{ia_j, \Gamma_j\}_{j=1}^N\} \\ {\scriptstyle\text{matrix NLS}}\big\downarrow & & \big\downarrow{\scriptstyle\text{time evolution}} \\ q(x,t) & \xleftarrow{\text{inverse scattering}} & \{R(\lambda)e^{4i\lambda^2 t}, \{ia_j, \Gamma_j e^{-4ia_j^2 t}\}_{j=1}^N\} \end{array}$$

In the focusing way, using Marchenko integral kernels, we have

$$\begin{array}{ccc} q(x,0) & \xrightarrow{\text{direct scattering}} & \Omega_l(x+y;0) \\ {\scriptstyle\text{matrix NLS}}\big\downarrow & & \big\downarrow{\scriptstyle [\Omega_l]_t+4i[\Omega_l]_{xx}=0} \\ q(x,t) & \xleftarrow{\text{inverse scattering}} & \Omega_l(x+y;t) \end{array}$$

Without symmetries, the latter diagram has to be modified as follows:

$$\begin{array}{ccc} \{q(x,0), r(x,0)\} & \xrightarrow{\text{direct scattering}} & \{\Omega_l(x+y;0), \breve{\Omega}_l(x+y;0)\} \\ {\scriptstyle\text{matrix NLS}}\big\downarrow & & \big\downarrow{\scriptstyle \begin{matrix}[\Omega_l]_t+4i[\Omega_l]_{xx}=0\\ [\breve{\Omega}_l]_t-4i[\breve{\Omega}_l]_{xx}=0\end{matrix}} \\ \{q(x,t), r(x,t)\} & \xleftarrow{\text{inverse scattering}} & \{\Omega_l(x+y;t), \breve{\Omega}_l(x+y;t)\} \end{array}$$

The quantities Ω_l and $\breve{\Omega}_l$ can be viewed as position and momentum of a hamiltonian system with hamiltonian density

$$\mathcal{H} = 4i[\Omega_l]_x[\breve{\Omega}_l]_x.$$

The Hamilton equations reduce to (2.56a) and (2.56b). The two variables Ω_l and $\breve{\Omega}_l$ are canonical, because

$$\{\Omega_l, \Omega_l\} = \{\breve{\Omega}_l, \breve{\Omega}_l\} = 0, \qquad \{\breve{\Omega}_l, \Omega_l\} = 1.$$

Finally, the horizontal arrows in the last diagram are canonical transformations which are each other's inverses. The focusing case can be obtained by imposing the constraint $r(x) = q(x)^\dagger$ (before transformation) or the constraint $\breve{\Omega}_l(x) = -\Omega_l(x)^\dagger$ after transformation. The defocusing case can be obtained by imposing the constraint $r(x) = -q(x)^\dagger$ (before transformation) or the constraint $\breve{\Omega}_l(x) = \Omega_l(x)^\dagger$ after transformation.

The IST can be formulated with minimal changes for the matrix mKdV equation and the sine-Gordon equation. In these cases the time evolution of the scattering dataisentirely determined by the "free" Lax pair $\{L_0, A_0\}$ given by

$$L_0 = iJ\partial_x,$$
$$A_0 = \begin{cases} -4\partial_x^3 = -4iJL_0^3, & \text{matrix mKdV,} \\ \frac{1}{4}\partial_x^{-1} = \frac{1}{4}iJL_0^{-1}, & \text{sine-Gordon,} \end{cases}$$

which is obtained from the actual Lax pair by allowing the potential $\boldsymbol{Q}$ to vanish. Following the above arguments for the matrix NLS equation, we see that

$$\mathbb{F}U_0(t-s)\mathbb{F}^{-1} = \mathbb{F}e^{(t-s)A_0}\mathbb{F}^{-1} = \begin{cases} e^{-4iJ\lambda^3(t-s)}, & \text{matrix mKdV,} \\ e^{\frac{1}{4\lambda}iJ(t-s)}, & \text{sine-Gordon,} \end{cases}$$

intended as a premultiplication by a matrix function depending on λ. Thus

$$R(\lambda;t) = e^{8i\lambda^3 t}R(\lambda;0), \quad L(\lambda;t) = e^{-8i\lambda^3 t}L(\lambda;0), \quad \text{matrix mKdV,} \tag{2.57a}$$
$$R(\lambda;t) = e^{-it/2\lambda}R(\lambda;0), \; L(\lambda;t) = e^{it/2\lambda}L(\lambda;0), \quad \text{sine-Gordon.} \tag{2.57b}$$

Thus, instead of (2.56), we get the PDE

$$[\Omega_l]_t + 8[\Omega_l]_{xxx} = 0_{n\times m}, \quad \text{matrix mKdV,} \tag{2.58a}$$
$$[\Omega_r]_t + 8[\Omega_r]_{xxx} = 0_{m\times n}, \quad \text{matrix mKdV,} \tag{2.58b}$$
$$[\Omega_l]_{xt} - \tfrac{1}{2}\Omega_l = 0, \quad \text{sine-Gordon,} \tag{2.58c}$$
$$[\Omega_r]_{xt} - \tfrac{1}{2}\Omega_r = 0, \quad \text{sine-Gordon.} \tag{2.58d}$$

2.6 Summarizing the IST

Let us summarizing the IST method in two different ways, either involving five steps. Here we subdivide the first and the last of the three steps in the classical description of the IST in two each. This will facilitate describing a numerical method for solving the matrix NLS equation based on the IST.

Using scattering data on right half-lines, the inverse scattering transform for solving the matrix NLS consists of the following five steps:

1. **From initial data to initial Marchenko solution**: Solve the Volterra integral equations (2.17).

2. **From initial Marchenko solution to initial Marchenko kernel**: Compute the Marchenko kernels $\Omega_l(x+y)$ and $\breve{\Omega}_l(x+y)$ from the solution of the Marchenko equations (2.50a)-(2.50b).

3. **Propagating the Marchenko kernel**: Propagate the Marchenko kernels by solving

$$[\Omega_l]_t + 4i[\Omega_l]_{xx} = 0_{n\times m},$$
$$[\breve{\Omega}_l]_t - 4i[\breve{\Omega}_l]_{xx} = 0_{m\times n}.$$

4. **From final Marchenko kernel to final Marchenko solution**: Solve the Marchenko equations (2.50a) and (2.50b) for $\overline{K}(x,y;t)$ and $K(x,y;t)$.

5. **From final Marchenko solution to matrix NLS solution**: Compute $q(x;t)$ and $r(x;t)$ from

$$q(x;t) = -2K^{\mathrm{up}}(x,x;t), \qquad r(x;t) = 2\overline{K}^{\mathrm{dn}}(x,x;t).$$

Schematically,

$$\begin{tikzcd}
\{q(x,0),r(x,0)\} \arrow[r] \arrow[d, "\text{matrix NLS system}"] & \{\overline{K}(x,y;0),K(x,y;0)\} \arrow[r] & \{\Omega_l(x;0),\breve{\Omega}_l(x;0)\} \arrow[d, "{[\Omega_l]_t+4i[\Omega_l]_{xx}=0},\ {[\breve{\Omega}_l]_t-4i[\breve{\Omega}_l]_{xx}=0}"'] \\
\{q(x,t),r(x,t)\} & \{\overline{K}(x,y;t),K(x,y;t)\} \arrow[l] & \{\Omega_l(x;t),\breve{\Omega}_l(x;t)\} \arrow[l]
\end{tikzcd}$$

Using scattering data on left half-lines, the inverse scattering transform for solving the matrix NLS consists of the following five steps:

1. **From initial data to initial Marchenko solution**: Solve the Volterra integral equations (2.19).

2. **From initial Marchenko solution to initial Marchenko kernel**: Compute the Marchenko kernels $\Omega_r(x+y)$ and $\breve{\Omega}_r(x+y)$ from the solution of the Marchenko equations (2.50c)-(2.50d).

3. **Propagating the Marchenko kernel**: Propagate the Marchenko kernels by solving

$$[\Omega_r]_t - 4i[\Omega_r]_{xx} = 0_{m\times n},$$
$$[\breve{\Omega}_r]_t + 4i[\breve{\Omega}_r]_{xx} = 0_{n\times m}.$$

4. **From final Marchenko kernel to final Marchenko solution**: Solve the Marchenko equations (2.50c) and (2.50d) for $M(x,y;t)$ and $\overline{M}(x,y;t)$.

5. **From final Marchenko solution to matrix NLS solution**: Compute $q(x;t)$ and $r(x;t)$ from

$$q(x;t) = 2\overline{M}^{\mathrm{up}}(x,x;t), \qquad r(x;t) = -2M^{\mathrm{dn}}(x,x;t).$$

Schematically,

$$\begin{array}{ccccc}
\{q(x,0),r(x,0)\} & \longrightarrow & \{M(x,y;0),\overline{M}(x,y;0)\} & \longrightarrow & \{\Omega_r(x;0),\breve{\Omega}_r(x;0)\} \\
\Big\downarrow {\scriptstyle \text{matrix NLS system}} & & & & {\scriptstyle [\Omega_r]_t-4i[\Omega_r]_{xx}=0 \atop [\breve{\Omega}_r]_t+4i[\breve{\Omega}_r]_{xx}=0} \Big\downarrow \\
\{q(x,t),r(x,t)\} & \longleftarrow & \{M(x,y;t),\overline{M}(x,y;t)\} & \longleftarrow & \{\Omega_r(x;t),\breve{\Omega}_r(x;t)\}
\end{array}$$

The two five-step procedures to implement the IST for the matrix NLS equation are the basis of a numerical method to solve the matrix NLS equation, developed at the University of Cagliari. Discretization of the integral equations involved using the trapezoid integration rule leads to a Volterra system (step 1), a Toeplitz-plus-diagonal system (step 2), in principle a fast Fourier transform (step 3), a Hankel-plus-diagonal system (step 4), and writing down the diagonal elements of a square matrix (step 5). The fourth step has been well developed [95, 36]. Intermediate results on the numerical method appeared in [14, 13]. At present, the five steps are being integrated into one single program.

Chapter 3

Study of discrete models through linear difference systems

The two discrete integrable models studied most are the Toda lattice and the IDNLS system. The Toda lattice was first formulated in 1967 by Morikazu Toda [1917-2010] [91] to model an infinite sequence of nonlinearly coupled oscillators converging to a physical system described by the KdV equation as the distance between two consecutive oscillators vanishes. The IDNLS system was first formulated by Ablowitz and Ladik [3, 4] in 1975-1976 as a discretization of the NLS equation. In either case the nonlinear discrete equation is integrable. In Chapter 1 we have constructed a Lax pair for the Toda lattice equation and an AKNS pair for the matrix IDNLS system. Here we explain the IST for either model.

The Toda lattice equation was first proven to be integrable, by formulation of the inverse scattering transform, by Flaschka [57]. It has a Lax pair $\{L, A\}$ given by

$$\begin{aligned}(L\boldsymbol{x})_n &= a_{n+1}x_{n+1} + a_n x_{n-1} + b_n x_n,\\ (A\boldsymbol{x})_n &= a_{n+1}x_{n+1} - a_{n-1}x_{n-1},\end{aligned}$$

where $\boldsymbol{x} = \{x_n\}_{n=-\infty}^{\infty}$. The inverse scattering transform for the matrix IDNLS system can be found in the book by Ablowitz et al. [5, Ch. 5] and in the article by Tsuchida et al. [93]. As shown by Example 1.9, the matrix IDNLS system has an AKNS pair (X_n, T_n) given by

$$X_n = \boldsymbol{Z} + \boldsymbol{Q}_n, \quad T_n = -\tfrac{1}{2}iJ(\boldsymbol{Z} - \boldsymbol{Z}^{-1})^2 + iJ\boldsymbol{Q}_n\boldsymbol{Q}_{n-1} - iJ\boldsymbol{Z}\boldsymbol{Q}_n + iJ\boldsymbol{Z}^{-1}\boldsymbol{Q}_n,$$

$$\boldsymbol{Z} = zI_N \oplus z^{-1}I_M, \qquad \boldsymbol{Q}_n = \begin{pmatrix} 0_{N\times N} & Q_n \\ R_n & 0_{M\times M}\end{pmatrix}.$$

Recently [47, 48], a different inverse scattering transform for the matrix IDNLS system has been found which avoids many of the mathematical difficulties inherent in the Ablowitz-Ladik approach but for which, at present, no Lax pair or AKNS pair is known. The difference regards the choice of the operator L appearing in the linear eigenvalue problem. When replacing the differentiation in the matrix Zakharov-Shabat system

$$iJ\frac{\partial \boldsymbol{v}}{\partial x}(\lambda, x) - V(x)\boldsymbol{v}(\lambda, x) = \lambda \boldsymbol{v}(\lambda, x),$$

where $J = I_m \oplus (-I_n)$ and $JV(x) = -V(x)J$, by a finite difference scheme (with step h), we have the following two options:

1. **Forward differencing** (Ablowitz-Ladik [3, 4]):

$$iJ\frac{\boldsymbol{v}_{n+1} - \boldsymbol{v}_n}{h} - \boldsymbol{V}_n\boldsymbol{v}_n = \lambda \boldsymbol{v}_n.$$

This leads to the one-step forward difference equation system

$$\boldsymbol{v}_{n+1} = \begin{pmatrix} (1 - i\lambda h)I_N & hq_n \\ hr_n & (1 + i\lambda h)I_M \end{pmatrix} \boldsymbol{v}_n.$$

Ablowitz and Ladik then make the approximation [of order $O(h^2)$ as $h \to 0^+$] $1 - i\lambda h \simeq z$ and $1 + i\lambda h \simeq z^{-1}$ and write $Q_n = hq_n$ and $R_n = hr_n$. Their final equation is as follows:

$$\boldsymbol{v}_{n+1} = \begin{pmatrix} zI_N & Q_n \\ R_n & z^{-1}I_M \end{pmatrix} \boldsymbol{v}_n.$$

To be able to go backward in n, their system matrix must be invertible for each $n \in \mathbb{Z}$ and every $0 \neq z \in \mathbb{C}$. This requires assuming that

$$\det \begin{pmatrix} zI_N & Q_n \\ R_n & z^{-1}I_M \end{pmatrix} = \det(I_N - Q_nR_n) = \det(I_M - R_nQ_n) \neq 0,$$

where $n \in \mathbb{Z}$.[1]

2. **Central differencing** (Demontis-Van der Mee [48]):

$$iJ\frac{\boldsymbol{v}_{n+1} - \boldsymbol{v}_{n-1}}{2h} - \boldsymbol{V}_n\boldsymbol{v}_n = \lambda \boldsymbol{v}_n.$$

This is a two-step difference equation which can be uniquely extended in the forward as well as the backward direction, because it can also be written as

$$\boldsymbol{v}_{n+1} - \boldsymbol{v}_{n-1} = -2ihJ[\lambda I_{N+M} + \boldsymbol{V}_n]\boldsymbol{v}_n.$$

[1] Here we have used the following easily verified facts: (a) if $I - TS$ is invertible, then $I - ST$ is invertible with inverse $I + S(I - TS)^{-1}T$. (b) If $I - ST$ is invertible, then $I - TS$ is invertible with inverse $I + T(I - ST)^{-1}S$. (c) Either matrix is invertible iff $\left(\begin{smallmatrix} I & T \\ S & I \end{smallmatrix}\right)$ is, and in that case its inverse is $\begin{pmatrix} (I-TS)^{-1} & -T(I-ST)^{-1} \\ -S(I-TS)^{-1} & (I-ST)^{-1} \end{pmatrix} = \begin{pmatrix} (I-TS)^{-1} & -(I-TS)^{-1}T \\ -(I-ST)^{-1}S & (I-ST)^{-1} \end{pmatrix}$.

3.1 Jost Solutions as Fourier Series

To define the Jost solutions, we first consider the free hamiltonian L_0, because the solutions of $L_0\boldsymbol{v} = \lambda \boldsymbol{v}_0$ are the asymptotic expressions for the Jost solutions of $L\boldsymbol{v} = \lambda\boldsymbol{v}$ as $n \to \pm\infty$. We have the following:

1. **Flaschka-Toda**: For $\boldsymbol{x} = \{x_n\}_{n=-\infty}^{\infty}$ we define L_0 by

   $$(L_0\boldsymbol{x})_n = \tfrac{1}{2}x_{n+1} + \tfrac{1}{2}x_{n-1}.$$

   Then the (absolutely continuous) spectrum $\sigma(L_0) = [-1, 1]$. Further,

   $$[(L - L_0)\boldsymbol{x}]_n = \alpha_{n+1}x_{n+1} + \alpha_n x_{n-1} + b_n x_n,$$

   where $\alpha_n = a_n - \frac{1}{2}$ and for some $s = 0, 1, 2$ the following condition is assumed:

   $$\sum_{n=-\infty}^{\infty} (1 + |n|)^s (2|\alpha_n| + |b_n|) < +\infty, \qquad s = 0, 1, 2 \qquad \text{Hyp}_s$$

2. **Ablowitz-Ladik**: It is not clear how to define L_0. Instead the $n \to \pm\infty$ asymptotic solutions of the Jost solutions follow from the equation $\boldsymbol{v}_{n+1} = \boldsymbol{Z}\boldsymbol{v}_n$, where $\boldsymbol{Z} = zI_N \oplus z^{-1}I_M$. We assume that for some $s = 0, 1, 2$

   $$\begin{cases} \sum_{n=-\infty}^{\infty} (1 + |n|)^s \{\|Q_n\| + \|R_n\|\} < +\infty, \\ \det(I_N - Q_nR_n) \neq 0 \text{ for each } n \in \mathbb{Z}. \end{cases} \qquad s = 0, 1, 2. \qquad \text{Hyp}_s$$

   One could argue for $\{z \in \mathbb{C} : z \in \mathbb{T}\}$, $\mathbb{T}$ standing for the unit circle in the complex plane, as the "spectrum" of the free Ablowitz-Ladik system $\boldsymbol{v}_{n+1} = \boldsymbol{Z}\boldsymbol{v}_n$, by using $\boldsymbol{Z}$ as the "spectral parameter."

3. **Central differencing**: For $\boldsymbol{x} = \{x_n\}_{n=-\infty}^{\infty}$ we define L_0 by

   $$(L_0\boldsymbol{x})_n = iJ\frac{x_{n+1} - x_{n-1}}{2h}.$$

   Then the (absolutely continuous) spectrum $\sigma(L_0) = [-\frac{1}{h}, \frac{1}{h}]$. Also,

   $$[(L - L_0)\boldsymbol{x}]_n = -\boldsymbol{V}_n x_n = -\begin{pmatrix} 0_{N\times N} & Q_n \\ R_n & 0_{M\times M} \end{pmatrix} v_n,$$

   where for some $s = 0, 1, 2$ the following condition is assumed:

   $$\sum_{n=-\infty}^{\infty} (1 + |n|)^s \{\|Q_n\| + \|R_n\|\} < +\infty, \qquad s = 0, 1, 2. \qquad \text{Hyp}_s$$

In the Flaschka-Toda case we define the Jost solutions as follows:

$$\phi_n(z) \sim z^n, \quad \overline{\phi}_n(z) \sim z^{-n}, \qquad n \to -\infty, \tag{3.1a}$$

$$\psi_n(z) \sim z^{-n}, \quad \overline{\psi}_n(z) \sim z^{n}, \qquad n \to +\infty, \tag{3.1b}$$

where $\lambda = \frac{1}{2}(z + z^{-1})$. Thus $\overline{\phi}_n(z) = \phi_n(z^{-1})$ and $\overline{\psi}_n(z) = \psi_n(z^{-1})$. For this reason, Flaschka [57] has only defined two different Jost solutions.

In the second and the third cases we define the Jost solutions as follows:

$$\phi_n(z) \sim z^n \begin{pmatrix} I_N \\ 0_{MN} \end{pmatrix}, \quad \overline{\phi}_n(z) \sim z^{-n} \begin{pmatrix} 0_{NM} \\ I_M \end{pmatrix}, \qquad n \to -\infty, \tag{3.2a}$$

$$\psi_n(z) \sim z^{-n} \begin{pmatrix} 0_{NM} \\ I_M \end{pmatrix}, \quad \overline{\psi}_n(z) \sim z^{n} \begin{pmatrix} I_N \\ 0_{MN} \end{pmatrix}, \qquad n \to +\infty, \tag{3.2b}$$

where $\lambda = \frac{1}{2}(z + z^{-1})$ [second case] and $\lambda = [z^{-1} - z]/(2ih)$ [third case].

The next thing to do is to prove the existence of the Jost solutions for $z \in \mathbb{T}$ by deriving Volterra summation equations and proving their unique solvability. The potential difficulties are the cases $z = \pm 1$ [Ablowitz-Ladik and Flaschka-Toda] where $z^n = z^{-n} = (\pm 1)^n$ may (but does not always) lead to linearly dependent Jost solutions. A similar problem exists for $z = \pm i$ in the central differencing case. The existence of the Jost solutions for $\pm 1 \neq z \in \mathbb{T}$ requires the hypothesis Hyp_0; their existence at $z = \pm 1$ requires assuming Hyp_1.[2]

Next, we write the Jost solutions as the sums of absolutely convergent Fourier series as follows:

$$\psi_n(z) = \sum_{j=n}^{\infty} z^{-j} \boldsymbol{K}(n,j), \tag{3.3a}$$

$$\overline{\psi}_n(z) = \sum_{j=n}^{\infty} z^{j} \overline{\boldsymbol{K}}(n,j), \tag{3.3b}$$

$$\phi_n(z) = \sum_{j=-\infty}^{n} z^{j} \boldsymbol{L}(n,j), \tag{3.3c}$$

$$\overline{\phi}_n(z) = \sum_{j=-\infty}^{n} z^{-j} \overline{\boldsymbol{L}}(n,j), \tag{3.3d}$$

where

$$\sum_{j=n}^{\infty} \left(\|\boldsymbol{K}(n,j)\| + \|\overline{\boldsymbol{K}}(n,j)\| \right) + \sum_{j=-\infty}^{n} \left(\|\boldsymbol{L}(n,j)\| + \|\overline{\boldsymbol{L}}(n,j)\| \right) < +\infty.$$

[2]Similar assumptions have to be made regarding the Schrödinger on the line [53, 38, 29]. Here the Jost solutions $f_l(k,x)$ and $f_r(k,x)$ exist for $0 \neq k \in \mathbb{R}$ if the real potential belongs to $L^1(\mathbb{R}; (1+|x|)dx)$. They exist for $k = 0$ if the real potential belongs to $L^1(\mathbb{R}; (1+|x|)^2 dx)$.

In the Flaschka-Toda case, the functions $\boldsymbol{K}$, $\overline{\boldsymbol{K}}$, $\boldsymbol{L}$, and $\overline{\boldsymbol{L}}$ are scalar, while $\boldsymbol{K}(n,j) = \overline{\boldsymbol{K}}(n,j)$ and $\boldsymbol{L}(n,j) = \overline{\boldsymbol{L}}(n,j)$.

The Ablowitz-Ladik and central differencing systems display the following *parity symmetry*:

If $\boldsymbol{v}_n(z)$ is a solution, then also

$$\tilde{\boldsymbol{v}}_n(z) = (-1)^n J \boldsymbol{v}_n(-z) J$$

is a solution. Here we note that $z \mapsto -z$ implies $\lambda \mapsto -\lambda$ as far as the central differencing system is concerned. The uniqueness of the Jost solutions then lead to the parity symmetry relations

$$\begin{pmatrix}\phi_n(z) & \overline{\phi}_n(z)\end{pmatrix} = (-1)^n J \begin{pmatrix}\phi_n(-z) & \overline{\phi}_n(-z)\end{pmatrix} J, \tag{3.4a}$$

$$\begin{pmatrix}\overline{\psi}_n(z) & \psi_n(z)\end{pmatrix} = (-1)^n J \begin{pmatrix}\overline{\psi}_n(-z) & \psi_n(-z)\end{pmatrix} J. \tag{3.4b}$$

These two systems also display the following *inversion symmetry*:

If $\boldsymbol{v}_n(z)$ is a solution, then also

$$\tilde{\boldsymbol{v}}_n(z) = (-1)^n J \boldsymbol{v}_n(z) J$$

is a solution of the corresponding system with λ replaced by $-\lambda$ (or, in the case of the Ablowitz-Ladik system, with z replaced by $-z$). Thus the spectrum of either system is inversion symmetric.

The Flaschka-Toda system does not display either symmetry.

3.2 Transition and scattering coefficients

In this section we define the transition and scattering coefficients, derive their continuity and analyticity properties, and discuss their symmetries. This will be done for the Flaschka-Toda and Ablowitz-Ladik systems, where $z = \pm 1$ play a special role. A few words will be devoted to the central differencing system, where $z = \pm i$ play a special role.

Under the hypothesis Hyp_0 and for $\pm 1 \neq z \in \mathbb{T}$, there exist so-called *transition matrices*

$$\boldsymbol{T}(z) = \begin{pmatrix}\boldsymbol{a}(z) & \overline{\boldsymbol{b}}(z)\\ \boldsymbol{b}(z) & \overline{\boldsymbol{a}}(z)\end{pmatrix}, \qquad \overline{\boldsymbol{T}}(z) = \begin{pmatrix}\overline{\boldsymbol{c}}(z) & \boldsymbol{d}(z)\\ \overline{\boldsymbol{d}}(z) & \boldsymbol{c}(z)\end{pmatrix}, \tag{3.5}$$

such that

$$\begin{pmatrix}\phi_n(z) & \overline{\phi}_n(z)\end{pmatrix} = \begin{pmatrix}\overline{\psi}_n(z) & \psi_n(z)\end{pmatrix} \boldsymbol{T}(z), \tag{3.6a}$$

$$\begin{pmatrix}\overline{\psi}_n(z) & \psi_n(z)\end{pmatrix} = \begin{pmatrix}\phi_n(z) & \overline{\phi}_n(z)\end{pmatrix} \overline{\boldsymbol{T}}(z). \tag{3.6b}$$

The matrices in (3.5) are each other's inverses. The reason for the exceptional position of the points $z = \pm 1$ is that the sequences $\{z^n\}_{n=-\infty}^{\infty}$ and $\{z^{-n}\}_{n=-\infty}^{\infty}$ appearing in the asymptotics of the Jost functions are linearly independent if and only if $z = \pm 1$. Thus for $\pm 1 \neq z \in \mathbb{T}$ the Jost functions $\phi_n(z)$ and $\overline{\phi}_n(z)$ are linearly independent, as are the Jost functions $\overline{\psi}_n(z)$ and $\psi_n(z)$. Also, the asymptotic form

$$\boldsymbol{Z}^n = z^n I_N \oplus z^{-n} I_M$$

of the square matrices in (3.6) composed of Jost functions implies that, for $\pm 1 \neq z \in \mathbb{T}$, the transition matrices are nonsingular.

Proposition 3.1 *Under the above hypotheses, for $\pm 1 \neq z \in \mathbb{T}$ we have the identities*

$$\det \boldsymbol{a}(z) = \det \boldsymbol{c}(z), \qquad \det \overline{\boldsymbol{a}}(z) = \det \overline{\boldsymbol{c}}(z). \tag{3.7}$$

Points $\pm 1 \neq z \in \mathbb{T}$ for which at least one of the determinants in (3.7) vanishes, will be called *spectral singularities.*

Proof. From $\boldsymbol{T}(z)\overline{\boldsymbol{T}}(z) = I_{N+M} = \overline{\boldsymbol{T}}(z)\boldsymbol{T}(z)$ we get

$$\begin{aligned}
\boldsymbol{a}(z)\overline{\boldsymbol{c}}(z) + \overline{\boldsymbol{b}}(z)\overline{\boldsymbol{d}}(z) &= I_N = \overline{\boldsymbol{c}}(z)\boldsymbol{a}(z) + \boldsymbol{d}(z)\boldsymbol{b}(z),\\
\overline{\boldsymbol{a}}(z)\boldsymbol{c}(z) + \boldsymbol{b}(z)\boldsymbol{d}(z) &= I_M = \boldsymbol{c}(z)\overline{\boldsymbol{a}}(z) + \overline{\boldsymbol{d}}(z)\overline{\boldsymbol{b}}(z),\\
\boldsymbol{a}(z)\boldsymbol{d}(z) + \overline{\boldsymbol{b}}(z)\boldsymbol{c}(z) &= 0_{N\times M} = \overline{\boldsymbol{c}}(z)\overline{\boldsymbol{b}}(z) + \boldsymbol{d}(z)\overline{\boldsymbol{a}}(z),\\
\boldsymbol{b}(z)\overline{\boldsymbol{c}}(z) + \overline{\boldsymbol{a}}(z)\overline{\boldsymbol{d}}(z) &= 0_{M\times N} = \overline{\boldsymbol{d}}(z)\boldsymbol{a}(z) + \boldsymbol{c}(z)\boldsymbol{b}(z).
\end{aligned}$$

If $\det \boldsymbol{a}(z) \neq 0$, then

$$\boldsymbol{d}(z) = -\boldsymbol{a}(z)^{-1}\overline{\boldsymbol{b}}(z)\boldsymbol{c}(z), \qquad \overline{\boldsymbol{d}}(z) = -\boldsymbol{c}(z)\boldsymbol{b}(z)\boldsymbol{a}(z)^{-1},$$

and hence

$$(\overline{\boldsymbol{a}}(z) - \boldsymbol{b}(z)\boldsymbol{a}(z)^{-1}\overline{\boldsymbol{b}}(z))\boldsymbol{c}(z) = I_M = \boldsymbol{c}(z)(\overline{\boldsymbol{a}}(z) - \boldsymbol{b}(z)\boldsymbol{a}(z)^{-1}\overline{\boldsymbol{b}}(z)),$$

which implies the invertibility of $\boldsymbol{c}(z)$. On the other hand, if $\det \boldsymbol{c}(z) \neq 0$, then

$$\boldsymbol{b}(z) = -\boldsymbol{c}(z)^{-1}\overline{\boldsymbol{d}}(z)\boldsymbol{a}(z), \qquad \overline{\boldsymbol{b}}(z) = -\boldsymbol{a}(z)\boldsymbol{d}(z)\boldsymbol{c}(z)^{-1},$$

and hence

$$(\overline{\boldsymbol{c}}(z) - \boldsymbol{d}(z)\boldsymbol{c}(z)^{-1}\overline{\boldsymbol{d}}(z))\boldsymbol{a}(z) = I_N = \boldsymbol{a}(z)(\overline{\boldsymbol{c}}(z) - \boldsymbol{d}(z)\boldsymbol{c}(z)^{-1}\overline{\boldsymbol{d}}(z)),$$

which proves the invertibility of $\boldsymbol{a}(z)$. Taking determinants on either side, we obtain

$$\begin{aligned}
\det \boldsymbol{T}(z) &= [\det \boldsymbol{a}(z)][\det(\overline{\boldsymbol{a}}(z) - \boldsymbol{b}(z)\boldsymbol{a}(z)^{-1}\overline{\boldsymbol{b}}(z))] = \frac{\det \boldsymbol{a}(z)}{\det \boldsymbol{c}(z)},\\
\det \overline{\boldsymbol{T}}(z) &= [\det(\overline{\boldsymbol{c}}(z) - \boldsymbol{d}(z)\boldsymbol{c}(z)^{-1}\overline{\boldsymbol{d}}(z))][\det \boldsymbol{c}(z)] = \frac{\det \boldsymbol{c}(z)}{\det \boldsymbol{a}(z)}.
\end{aligned}$$

Since the square matrices containing Jost functions behave as $\boldsymbol{Z}^n$ (which satisfies $\det(\boldsymbol{Z}^n) = 1$) as $n \to \pm\infty$, we have $\det \boldsymbol{T}(z) = \det \overline{\boldsymbol{T}}(z) = 1$. ∎

Proposition 3.1 constitutes a general linear algebra result: If $\boldsymbol{T} = \left(\begin{smallmatrix} \boldsymbol{a} & \overline{\boldsymbol{b}} \\ \boldsymbol{b} & \overline{\boldsymbol{a}} \end{smallmatrix}\right)$ and $\det \boldsymbol{a} \neq 0$, then $\det \boldsymbol{T} = [\det \boldsymbol{a}][\det(\overline{\boldsymbol{a}} - \boldsymbol{b}\boldsymbol{a}^{-1}\overline{\boldsymbol{b}})]$. The matrix $\overline{\boldsymbol{a}} - \boldsymbol{b}\boldsymbol{a}^{-1}\overline{\boldsymbol{b}}$ is called the *Schur complement* of $\boldsymbol{a}$ in $\boldsymbol{T}$. In fact,

$$\boldsymbol{T} = \begin{pmatrix} I & 0 \\ \boldsymbol{b}\boldsymbol{a}^{-1} & I \end{pmatrix} \begin{pmatrix} \boldsymbol{a} & 0 \\ 0 & \overline{\boldsymbol{a}} - \boldsymbol{b}\boldsymbol{a}^{-1}\overline{\boldsymbol{b}} \end{pmatrix} \begin{pmatrix} I & \boldsymbol{a}^{-1}\overline{\boldsymbol{b}} \\ 0 & I \end{pmatrix}.$$

The results of Section 3.2 so far are also true for the central differencing system, provided we replace $\pm 1 \neq z \in \mathbb{T}$ by $\pm i \neq z \in \mathbb{T}$.

As to the Ablowitz-Ladik and central differencing systems, we now observe that the parity symmetry relations (3.4) imply

$$\boldsymbol{T}(z) = J\boldsymbol{T}(-z)J, \qquad \overline{\boldsymbol{T}}(z) = J\overline{\boldsymbol{T}}(-z)J. \tag{3.8}$$

Thus $\boldsymbol{a}(z)$, $\boldsymbol{c}(z)$, $\overline{\boldsymbol{a}}(z)$, and $\overline{\boldsymbol{c}}(z)$ are even functions of z and $\boldsymbol{b}(z)$, $\boldsymbol{d}(z)$, $\overline{\boldsymbol{b}}(z)$, and $\overline{\boldsymbol{d}}(z)$ are odd functions of z. The Flaschka-Toda system does not have any parity symmetry.

Swapping the Jost solutions in (3.6) to create square matrices containing only Jost solutions that are analytic in either the unit disk or in the unit exterior disk, we obtain the Riemann-Hilbert problems

$$\begin{pmatrix} \overline{\psi}_n(z) & \overline{\phi}_n(z) \end{pmatrix} = \begin{pmatrix} \phi_n(z) & \psi_n(z) \end{pmatrix} J\boldsymbol{S}(z)J, \tag{3.9a}$$
$$\begin{pmatrix} \phi_n(z) & \psi_n(z) \end{pmatrix} = \begin{pmatrix} \overline{\psi}_n(z) & \overline{\phi}_n(z) \end{pmatrix} J\overline{\boldsymbol{S}}(z)J, \tag{3.9b}$$

where $J = I_N \oplus (-I_M)$ and

$$\boldsymbol{S}(z) = \begin{pmatrix} \boldsymbol{t}_r(z) & \boldsymbol{\ell}(z) \\ \boldsymbol{\rho}(z) & \boldsymbol{t}_t(z) \end{pmatrix}, \qquad \overline{\boldsymbol{S}}(z) = \begin{pmatrix} \overline{\boldsymbol{t}}_l(z) & \overline{\boldsymbol{\rho}}(z) \\ \overline{\boldsymbol{\ell}}(z) & \overline{\boldsymbol{t}}_r(z) \end{pmatrix},$$

are called *scattering matrices*. Their diagonal and off-diagonal blocks are called *transmission coefficients* and *reflection coefficients*, respectively. The two scattering matrices are each other's inverses. Their existence is only guaranteed if $\overline{\psi}_n(z)$ and $\overline{\phi}_n(z)$ are linearly independent and $\phi_n(z)$ and $\psi_n(z)$ are linearly independent. Assuming $\boldsymbol{a}(z)$ and $\boldsymbol{c}(z)$ to be invertible, we get

$$\boldsymbol{S}(z) = \begin{pmatrix} \boldsymbol{a}(z)^{-1} & \boldsymbol{d}(z)\boldsymbol{c}(z)^{-1} \\ \boldsymbol{b}(z)\boldsymbol{a}(z)^{-1} & \boldsymbol{c}(z)^{-1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}(z)^{-1} & -\boldsymbol{a}(z)^{-1}\overline{\boldsymbol{b}}(z) \\ -\boldsymbol{c}(z)^{-1}\overline{\boldsymbol{d}}(z) & \boldsymbol{c}(z)^{-1} \end{pmatrix}. \tag{3.10a}$$

Assuming $\overline{\boldsymbol{a}}(z)$ and $\overline{\boldsymbol{c}}(z)$ to be invertible, we get instead

$$\overline{\boldsymbol{S}}(z) = \begin{pmatrix} \overline{\boldsymbol{c}}(z)^{-1} & \overline{\boldsymbol{b}}(z)\overline{\boldsymbol{a}}(z)^{-1} \\ \overline{\boldsymbol{d}}(z)\overline{\boldsymbol{c}}(z)^{-1} & \overline{\boldsymbol{a}}(z)^{-1} \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{c}}(z)^{-1} & -\overline{\boldsymbol{c}}(z)^{-1}\boldsymbol{d}(z) \\ -\overline{\boldsymbol{a}}(z)^{-1}\boldsymbol{b}(z) & \overline{\boldsymbol{a}}(z)^{-1} \end{pmatrix}. \tag{3.10b}$$

As a result, for $\pm 1 \neq z \in \mathbb{T}$, the scattering matrices $\boldsymbol{S}(z)$ and $\overline{\boldsymbol{S}}(z)$ are defined and are each other's inverses if and only if z is not a spectral singularity. The results of this paragraph are also true for the central differencing system, provided we replace $\pm 1 \neq z \in \mathbb{T}$ by $\pm i \neq z \in \mathbb{T}$.

We now observe that the Ablowitz-Ladik and the central differencing system display the parity symmetry

$$\boldsymbol{S}(z) = J\boldsymbol{S}(-z)J, \qquad \overline{\boldsymbol{S}}(z) = J\overline{\boldsymbol{S}}(-z)J. \tag{3.11}$$

Thus for these two systems the transmission coefficients are even functions of z and the reflection coefficients are odd functions of z. As to the Flaschka-Toda system, the scattering matrices do not have parity symmetry, but either matrix has two identical diagonal elements. A similar situation occurs for the Schrödinger equation on the line [53, 38].

In the literature it has not been worked out how the Jost solutions, transition coefficients, and reflection and transmission coefficients for the Flaschka-Toda (FT), Ablowitz-Ladik (AL), and central differencing (CD) systems behave as $z \to \pm 1$ (FT, AL) or $z \to \pm i$ (CD). Well-known properties of the Schödinger equation on the line [38, 29] suggest the following:

1. Under the hypothesis Hyp_0, the so-called Faddeev functions $z^{-n}\phi_n(z)$ and $z^n\psi_n(z)$ are analytic in z in the exterior unit disk (∞ included) and have continuous limits as z approaches the circle, except possibly at $z = \pm 1$ (FT, AL) or $z = \pm i$ (CD). The Faddeev functions $z^n\overline{\phi}_n(z)$ and $z^{-n}\overline{\psi}_n(z)$ are analytic in the disk and have continuous limits as z approaches the circle, except possibly at $z = \pm 1$ (FT, AL) or $z = \pm i$ (CD). To also have continuous limits at $z = \pm 1$ (FT, AL) or $z = \pm i$ (CD), the hypothesis Hyp_1 is needed.

2. Under the hypothesis Hyp_1, we have to distinguish between the *generic case* where $\phi_n(\pm 1)$ and $\psi_n(\pm 1)$ are linearly independent (and therefore also $\overline{\psi}_n(\pm 1)$ and $\overline{\phi}_n(\pm 1)$ are linearly independent), and various *exceptional cases*, where at $z = \pm 1$ (FT, AL) or $z = \pm i$ (CD), the two square matrices composed of the Jost functions do not have full rank. Because of parity symmetry, the same case must occur at either point $z = \pm 1$ (AL) or $z = \pm i$ (CD).

3. Under the hypothesis Hyp_1 and in the generic case, the scattering matrices are continuous at $z = \pm 1$ (FT, AL) or $z = \pm i$ (CD).

4. Under the hypothesis Hyp_2 and in an exceptional case, the scattering matrices are continuous at $z = \pm 1$ (FT, AL) or $z = \pm i$ (CD).

3.3 Wronskian relations

In this section we derive the Wronskian relations for the three discrete models and explain their impact on the cojugation symmetry relations of the transition matrices and scattering matrices in the focusing and defocusing cases. The three models require different proofs.

1. Flaschka-Toda system. Given two solutions $\boldsymbol{\psi} = \{\psi_n\}_{n=-\infty}^{\infty}$ and $\tilde{\boldsymbol{\psi}} = \{\tilde{\psi}_n\}_{n=-\infty}^{\infty}$ of the difference equation

$$a_{n+1}\psi_{n+1} + a_n\psi_{n-1} + b_n\psi_n = \lambda\psi_n,$$

where a_n are positive constants and b_n are real constants such that $a_n \to \frac{1}{2}$ and $b_n \to 0$ as $n \to \pm\infty$, we define their *Wronskian* as follows:

$$W(\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}}) = a_n(\psi_n\tilde{\psi}_{n+1} - \psi_{n+1}\tilde{\psi}_n). \tag{3.12}$$

Then

$$\begin{aligned} W(\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}}) &= a_n(\psi_n\tilde{\psi}_{n+1} - \psi_{n+1}\tilde{\psi}_n) \\ &= \psi_n[\lambda\tilde{\psi}_n - a_{n-1}\tilde{\psi}_{n-1} - b_n\tilde{\psi}_n] - [\lambda\psi_n - a_{n-1}\psi_{n-1} - b_n\psi_n]\tilde{\psi}_n \\ &= a_{n-1}(\psi_{n-1}\tilde{\psi}_n - \psi_n\tilde{\psi}_{n-1}). \end{aligned}$$

Hence, the Wronskian does not depend on $n \in \mathbb{Z}$. Moreover, since a_n does not vanish, the Wronskian $W(\boldsymbol{\psi}, \tilde{\boldsymbol{\psi}})$ vanishes if and only if the two solutions $\boldsymbol{\psi} = \{\psi_n\}_{n=-\infty}^{\infty}$ and $\tilde{\boldsymbol{\psi}} = \{\tilde{\psi}_n\}_{n=-\infty}^{\infty}$ are linearly dependent. In particular, the Wronskians of the Jost solutions are as follows:

$$W(\phi, \overline{\phi}) = \tfrac{1}{2}(z^{-1} - z), \tag{3.13a}$$
$$W(\overline{\psi}, \psi) = \tfrac{1}{2}(z^{-1} - z), \tag{3.13b}$$
$$W(\phi, \psi) = \tfrac{1}{2}(z^{-1} - z)\boldsymbol{a}(z), \tag{3.13c}$$
$$W(\overline{\psi}, \overline{\phi}) = \tfrac{1}{2}(z^{-1} - z)\overline{\boldsymbol{a}}(z), \tag{3.13d}$$

which turns $z = \pm 1$ into special points where the Jost matrices are singular. Since it is immediate from the definitions that $\overline{\psi}_n(z) = \psi_n(z^{-1})$ and $\overline{\phi}_n(z) = \phi_n(z^{-1})$, we have $\overline{\boldsymbol{a}}(z) = \boldsymbol{a}(z)^*$. Similar conjugation symmetry properties of the remaining transition coefficients imply that the transition matrices are J-unitary and the scattering matrices are unitary.

2. Central differencing system. Let $\boldsymbol{v}_n(z)$ and $\tilde{\boldsymbol{v}}_n(z)$ be two solutions of the central differencing system

$$\boldsymbol{v}_{n+1}(z) - \boldsymbol{v}_{n-1}(z) = 2ihJ\begin{pmatrix} \lambda I_N & Q_n \\ R_n & \lambda I_M \end{pmatrix}\boldsymbol{v}_n(z)$$

in the focusing case, where $R_n = -Q_n^\dagger$. Then for $z \neq \pm 1$ on the unit circle we have

$$\begin{aligned} J\tilde{\boldsymbol{v}}_{n+1}(z)^\dagger J - J\tilde{\boldsymbol{v}}_{n-1}(z)^\dagger J &= -2ih[J\tilde{\boldsymbol{v}}_n(z)^\dagger J]J\begin{pmatrix} \lambda I_N & R_n^\dagger \\ Q_n^\dagger & \lambda I_M \end{pmatrix} \\ &= -2ih[J\tilde{\boldsymbol{v}}_n(z)^\dagger J]\begin{pmatrix} \lambda I_N & -R_n^\dagger \\ -Q_n^\dagger & \lambda I_M \end{pmatrix}J \\ &= -2ih[J\tilde{\boldsymbol{v}}_n(z)^\dagger J]\begin{pmatrix} \lambda I_N & Q_n \\ R_n & \lambda I_M \end{pmatrix}J. \end{aligned}$$

Therefore,

$$[J\tilde{\boldsymbol{v}}_{n+1}(z)^\dagger J - J\tilde{\boldsymbol{v}}_{n-1}(z)^\dagger J]J\boldsymbol{v}_n(z) = -J\tilde{\boldsymbol{v}}_n(z)^\dagger[\boldsymbol{v}_{n+1}(z) - \boldsymbol{v}_{n-1}(z)].$$

Reordering this identity, we see that the Wronskian expression

$$\tilde{\boldsymbol{v}}_{n+1}(z)^\dagger \boldsymbol{v}_n(z) + \tilde{\boldsymbol{v}}_n(z)^\dagger(z)\boldsymbol{v}_{n+1}(z) = \tilde{\boldsymbol{v}}_n(z)^\dagger \boldsymbol{v}_{n-1}(z) + \tilde{\boldsymbol{v}}_{n-1}(z)^\dagger(z)\boldsymbol{v}_n(z) \tag{3.14}$$

does not depend on $n \in \mathbb{Z}$. Applying (3.14) for all four combinations of $\boldsymbol{v}_n(z)$ and $\tilde{\boldsymbol{v}}_n(z)$ taken from $\big(\overline{\psi}_n(z), \psi_n(z)\big)$ and $\big(\phi_n(z), \overline{\phi}_n(z)\big)$ while equating the asymptotic expressions as $n \to \pm\infty$, we get

$$\begin{aligned}
\boldsymbol{T}(z)^\dagger(\boldsymbol{Z} + \boldsymbol{Z}^{-1})\boldsymbol{T}(z) &= \boldsymbol{Z} + \boldsymbol{Z}^{-1},\\
\overline{\boldsymbol{T}}(z)^\dagger(\boldsymbol{Z} + \boldsymbol{Z}^{-1})\overline{\boldsymbol{T}}(z) &= \boldsymbol{Z} + \boldsymbol{Z}^{-1},\\
\boldsymbol{T}(z)^\dagger(\boldsymbol{Z} + \boldsymbol{Z}^{-1}) &= (\boldsymbol{Z} + \boldsymbol{Z}^{-1})\overline{\boldsymbol{T}}(z),\\
\overline{\boldsymbol{T}}(z)^\dagger(\boldsymbol{Z} + \boldsymbol{Z}^{-1}) &= (\boldsymbol{Z} + \boldsymbol{Z}^{-1})\boldsymbol{T}(z).
\end{aligned}$$

Using that $\boldsymbol{Z} + \boldsymbol{Z}^{-1}$ is a multiple of the identity matrix which can only be zero if $z = \pm i$, we obtain for $z \neq \pm i$ belonging to the unit circle

$$\overline{\boldsymbol{T}}(z) = \boldsymbol{T}(z)^\dagger, \qquad \boldsymbol{T}(z)^\dagger \boldsymbol{T}(z) = I_{N+M} = \overline{\boldsymbol{T}}(z)^\dagger \overline{\boldsymbol{T}}(z). \tag{3.15}$$

Thus in the focusing case the transition matrices are unitary and therefore the scattering matrices are J-unitary in the sense that

$$\boldsymbol{S}(z)^{-1} = J\boldsymbol{S}(z)^\dagger J, \qquad \overline{\boldsymbol{S}}(z)^{-1} = J\overline{\boldsymbol{S}}(z)^\dagger J. \tag{3.16}$$

In the same way we prove that in the defocusing case the transition matrices are J-unitary and the scattering matrices are unitary.

3. Ablowitz-Ladik system. In the focusing case where $R_n = -Q_n^\dagger$, we have for each $z \neq \pm 1$ belonging to the unit circle and each solution $\boldsymbol{v}_n(z)$ of the Ablowitz-Ladik system

$$\begin{aligned}
\boldsymbol{v}_{n+1}(z)^\dagger \boldsymbol{v}_{n+1}(z) &= \boldsymbol{v}_n(z)^\dagger \begin{pmatrix} z^{-1}I_N & -Q_n \\ -R_n & zI_M \end{pmatrix} \begin{pmatrix} zI_N & Q_n \\ R_n & z^{-1}I_M \end{pmatrix} \boldsymbol{v}_n(z)\\
&= \boldsymbol{v}_n(z)^\dagger \begin{pmatrix} I_N - Q_nR_n & 0_{N\times M} \\ 0_{M\times N} & I_M - R_nQ_n \end{pmatrix} \boldsymbol{v}_n(z).
\end{aligned}$$

In [5, Ch. 5] and [93] it is assumed that

$$\forall n \in \mathbb{Z}\ \exists 0 \neq \alpha_n \in \mathbb{R} : I_N - Q_nR_n = \alpha_n I_N \text{ and } I_M - R_nQ_n = \alpha_n I_M. \tag{3.17}$$

Observe that (3.17) implies that $N = M$ whenever there exists some n for which $1 - \alpha_n \neq 0$. Indeed, $Q_nR_n = (1 - \alpha_n)I_N$ and $R_nQ_n = (1 - \alpha_n)I_M$ would make $R_n/(1 - \alpha_n)$ into the two-sided inverse of Q_n and this is only possible if Q_n is a square matrix. Under the hypothesis (3.17), we can continue the above calculations and perform the analogous calculation in the defocusing case where $R_n = Q_n^\dagger$, to arrive at the following identities:

$$\begin{cases} \boldsymbol{v}_{n+1}(z)^\dagger \boldsymbol{v}_{n+1}(z) = \alpha_n \boldsymbol{v}_n(z)^\dagger \boldsymbol{v}_n(z), & \text{focusing case,} \\ \boldsymbol{v}_{n+1}(z)^\dagger J \boldsymbol{v}_{n+1}(z) = \alpha_n \boldsymbol{v}_n(z)^\dagger J \boldsymbol{v}_n(z), & \text{defocusing case,} \end{cases} \tag{3.18}$$

where the infinite products converge absolutely if Hyp_0 is true. Iterating these identities forward for $\boldsymbol{v}_n(z) = \begin{pmatrix}\overline{\psi}_n(z) & \psi_n(z)\end{pmatrix}$, we get

$$\begin{cases} \begin{pmatrix}\overline{\psi}_n(z) & \psi_n(z)\end{pmatrix}^\dagger \begin{pmatrix}\overline{\psi}_n(z) & \psi_n(z)\end{pmatrix} = \left(\prod_{j=n}^{\infty} \alpha_j\right)^{-1} I_{N+M}, & \text{focusing,} \\ \begin{pmatrix}\overline{\psi}_n(z) & \psi_n(z)\end{pmatrix}^\dagger J \begin{pmatrix}\overline{\psi}_n(z) & \psi_n(z)\end{pmatrix} = \left(\prod_{j=n}^{\infty} \alpha_j\right)^{-1} I_{N+M}, & \text{defocusing.} \end{cases}$$

Iterating these identities backward for $\boldsymbol{v}_n(z) = \begin{pmatrix}\phi_n(z) & \overline{\phi}_n(z)\end{pmatrix}$, we get

$$\begin{cases} \begin{pmatrix}\phi_n(z) & \overline{\phi}_n(z)\end{pmatrix}^\dagger \begin{pmatrix}\phi_n(z) & \overline{\phi}_n(z)\end{pmatrix} = \left(\prod_{j=-\infty}^{n-1} \alpha_j\right) I_{N+M}, & \text{focusing,} \\ \begin{pmatrix}\phi_n(z) & \overline{\phi}_n(z)\end{pmatrix}^\dagger J \begin{pmatrix}\phi_n(z) & \overline{\phi}_n(z)\end{pmatrix} = \left(\prod_{j=-\infty}^{n-1} \alpha_j\right) I_{N+M}, & \text{defocusing.} \end{cases}$$

Taking $n \to -\infty$ in the former cases and $n \to +\infty$ in the latter cases, we arrive at the following conjugation symmetry properties: In the focusing case,

$$\boldsymbol{T}(z)^\dagger \boldsymbol{T}(z) = \left(\prod_{j=-\infty}^{\infty} \alpha_j\right) I_{N+M}, \quad \overline{\boldsymbol{T}}(z)^\dagger \overline{\boldsymbol{T}}(z) = \left(\prod_{j=-\infty}^{\infty} \alpha_j\right)^{-1} I_{N+M},$$

and in the defocusing case

$$\boldsymbol{T}(z)^\dagger J \boldsymbol{T}(z) = \left(\prod_{j=-\infty}^{\infty} \alpha_j\right) I_{N+M}, \quad \overline{\boldsymbol{T}}(z)^\dagger J \overline{\boldsymbol{T}}(z) = \left(\prod_{j=-\infty}^{\infty} \alpha_j\right)^{-1} I_{N+M}.$$

These (weighted) unitarity and J-unitarity properties for the transition matrices can be converted into (weighted) J-unitarity and unitarity properties for the scattering matrices. Here the sign of the (real and nonzero) infinite product $\prod_{j=-\infty}^{\infty} \alpha_j$ determines the exact nature of these (weighted) conjugation symmetries as well. Without the hypothesis (3.17) it is not clear how conjugation symmetry properties are to be derived.

3.4 Marchenko equations

In this section we derive the Marchenko equations for the three linear difference systems.

3.4.1 Flaschka-Toda system

The derivation of the Marchenko equations from the Riemann-Hilbert problem (3.9a) is very simple, because the poles z of the transmission coefficients are always simple, real, and different from $z=0$, and often finite in number. Assuming Hyp_1 and writing (3.9a) in the form

$$\begin{pmatrix}\overline{\psi}_n(z) & \overline{\phi}_n(z)\end{pmatrix} = \begin{pmatrix}\phi_n(z) & \psi_n(z)\end{pmatrix}\begin{pmatrix}\boldsymbol{t}(z) & -\boldsymbol{\ell}(z)\\ -\boldsymbol{\rho}(z) & \boldsymbol{t}(z)\end{pmatrix},$$

where $\boldsymbol{t}(z) \overset{\text{def}}{=} \boldsymbol{t}_l(z) = \boldsymbol{t}_r(z)$ and the transmission coefficient $\boldsymbol{t}(z)$ is given by

$$\boldsymbol{t}(z) = \boldsymbol{t}_0(z) + \sum_{s=1}^{N} \frac{\tau_s}{z - z_s}$$

for finitely many distinct $z_1, \ldots, z_N \in \mathbb{R} \setminus [-1,1]$ and $\boldsymbol{t}_0(z)$ that is analytic in the exterior unit disk, we arrive at the following Marchenko equations:

$$\begin{aligned}
\boldsymbol{K}(n,m) &= -\sum_{s]1}^{N} \frac{\tau_s \boldsymbol{C}_s}{z_s^{n+m+1}} \boldsymbol{K}(n,n) - \sum_{s=1}^{N}\sum_{j=n+1}^{\infty} \frac{\tau_s \boldsymbol{C}_s}{z_s^{j+m+1}} \boldsymbol{K}(n,j)\\
&\quad - \hat{\boldsymbol{\rho}}(n+m)\boldsymbol{K}(n,n) - \sum_{j=n+1}^{\infty} \hat{\boldsymbol{\rho}}(j+m)\boldsymbol{K}(n,j),\\
\boldsymbol{L}(n,m) &= -\sum_{s=1}^{N} \frac{\tau_s z_s^{n-m-1}}{\boldsymbol{C}_s} - \sum_{s=1}^{N}\sum_{j=-\infty}^{n-1} \frac{\tau_s z_s^{j+m-1}}{\boldsymbol{C}_s} \boldsymbol{L}(n,j)\\
&= -\hat{\boldsymbol{\ell}}(n+m)\boldsymbol{L}(n,n) - \sum_{r=1}^{N} \hat{\boldsymbol{\ell}}(j+m)\boldsymbol{L}(n,j).
\end{aligned}$$

Here $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_N$ are the proportionality constants defined by

$$\phi_n(z_s) = \boldsymbol{C}_s \psi_n(z_s), \qquad s = 1, \ldots, N, \quad n \in \mathbb{Z}.$$

It can now be shown that $\boldsymbol{K}(n,n)$ and $\boldsymbol{L}(n,n)$ are real nonzero scalars. Putting

$$\kappa(n,m) = \frac{\boldsymbol{K}(n,m)}{\boldsymbol{K}(n,n)}, \qquad \ell(n,m) = \frac{\boldsymbol{L}(n,m)}{\boldsymbol{L}(n,n)},$$

the two Marchenko equations can be written in their final form

$$\kappa(n,m) + \boldsymbol{F}_r(n+m) + \sum_{j=n+1}^{\infty} \boldsymbol{F}_r(m+j)\kappa(n,j) = 0, \quad m \geq n+1, \tag{3.19a}$$

$$\ell(n,m) + \boldsymbol{F}_l(n+m) + \sum_{j=-\infty}^{n-1} \boldsymbol{F}_l(m+j)\ell(n,j) = 0, \quad m \leq n-1, \tag{3.19b}$$

where the Marchenko kernels are given by

$$\boldsymbol{F}_r(n+m) = \hat{\boldsymbol{\rho}}(n+m) + \sum_{s=1}^{N} \frac{\tau_s \boldsymbol{C}_s}{z_s^{n+m+1}}, \tag{3.20a}$$

$$\boldsymbol{F}_l(n+m) = \hat{\boldsymbol{\ell}}(n+m) + \sum_{s=1}^{N} \frac{\tau_s z_s^{n+m-1}}{\boldsymbol{C}_s}. \tag{3.20b}$$

Let us discuss how to compute a_n and b_n from the solutions of either Marchenko equation (3.19). First we compute one of $\boldsymbol{K}(n,n)$ or $\boldsymbol{L}(n,n)$ from the identities

$$\frac{1}{\boldsymbol{K}(n,n)^2} = 1 + \boldsymbol{F}_r(2n) + \sum_{j=n+1}^{\infty} \boldsymbol{F}_r(n+j)\kappa(n,j), \tag{3.21a}$$

$$\frac{1}{\boldsymbol{L}(n,n)^2} = 1 + \boldsymbol{F}_l(2n) + \sum_{j=-\infty}^{n-1} \boldsymbol{F}_l(n+j)\ell(n,j), \tag{3.21b}$$

where $\boldsymbol{K}(n,n)$ and $\boldsymbol{L}(n,n)$ are to be positive. Next we compute the coefficients a_n and b_n from

$$a_n = \frac{\boldsymbol{K}(n+1,n+1)}{2\boldsymbol{K}(n,n)}, \qquad a_n = \frac{\boldsymbol{L}(n,n)}{2\boldsymbol{L}(n+1,n+1)}, \tag{3.22}$$

and

$$b_n = \tfrac{1}{2}\left\{\kappa(n,n+1) - \kappa(n-1,n)\right\}, \tag{3.23a}$$

$$b_n = \tfrac{1}{2}\left\{\ell(n,n-1) - \ell(n+1,n)\right\}. \tag{3.23b}$$

3.4.2 Ablowitz-Ladik and central differencing systems

The derivation of the Marchenko equations for these two difference systems is greatly affected by their parity symmetry. The parity symmetry relations imply that, on writing $\begin{pmatrix}\phi_n(z) & \overline{\phi}_n(z)\end{pmatrix}$ and $\begin{pmatrix}\overline{\psi}_n(z) & \psi_n(z)\end{pmatrix}$ as 2×2 block matrices in the usual way, the diagonal blocks are even functions of z and the off-diagonal blocks are odd functions of z if n is even, whereas the diagonal blocks are odd functions of z and the off-diagonal blocks are even functions of z if n is odd. If we now turn to the Fourier series (3.3), we obtain the following results:

a. $\overline{\boldsymbol{K}}^{\mathrm{up}}(n,m)$, $\boldsymbol{K}^{\mathrm{dn}}(n,m)$, $\boldsymbol{L}^{\mathrm{up}}(n,m)$, and $\overline{\boldsymbol{L}}^{\mathrm{dn}}(n,m)$ are zero matrices if $m-n$ is odd;

b. $\overline{\boldsymbol{K}}^{\mathrm{dn}}(n,m)$, $\boldsymbol{K}^{\mathrm{up}}(n,m)$, $\boldsymbol{L}^{\mathrm{dn}}(n,m)$, and $\overline{\boldsymbol{L}}^{\mathrm{up}}(n,m)$ are zero matrices if $m-n$ is even.

The parity symmetry relations (3.8) and (3.11) imply that the diagonal transition coefficients $\boldsymbol{a}(z)$, $\overline{\boldsymbol{a}}(z)$, $\boldsymbol{c}(z)$, and $\overline{\boldsymbol{c}}(z)$ and the transmission coefficients are even functions of z and the off-diagonal transition coefficients $\boldsymbol{b}(z)$, $\overline{\boldsymbol{b}}(z)$, $\boldsymbol{d}(z)$, and $\overline{\boldsymbol{d}}(z)$ and the reflection coefficients are odd functions of z. Writing the absolutely convergent Fourier series

$$\boldsymbol{\rho}(z) = \sum_{s=-\infty}^{\infty} z^s \hat{\boldsymbol{\rho}}(s), \qquad \overline{\boldsymbol{\rho}}(z) = \sum_{s=-\infty}^{\infty} z^{-s} \hat{\overline{\boldsymbol{\rho}}}(s), \tag{3.24a}$$

$$\overline{\boldsymbol{\ell}}(z) = \sum_{s=-\infty}^{\infty} z^s \hat{\overline{\boldsymbol{\ell}}}(s), \qquad \boldsymbol{\ell}(z) = \sum_{s=-\infty}^{\infty} z^{-s} \hat{\boldsymbol{\ell}}(s), \tag{3.24b}$$

we see that $\hat{\boldsymbol{\rho}}(s)$, $\hat{\overline{\boldsymbol{\rho}}}(s)$, $\hat{\overline{\boldsymbol{\ell}}}(s)$, and $\hat{\boldsymbol{\ell}}(s)$ are zero matrices if s is even. The absolute convergence of these Fourier series has only been established under rather restrictive assumptions.

We now arrive at the Marchenko equations[3]

$$\overline{\boldsymbol{K}}(n,m) + \sum_{j=n}^{\infty} \boldsymbol{K}(n,j)\boldsymbol{F}_r(j+m) = \begin{pmatrix} I_N \\ 0_{M\times N} \end{pmatrix} \delta_{n,m}, \tag{3.25a}$$

$$\boldsymbol{K}(n,m) + \sum_{j=n}^{\infty} \overline{\boldsymbol{K}}(n,j)\overline{\boldsymbol{F}}_r(j+m) = \begin{pmatrix} 0_{N\times M} \\ I_M \end{pmatrix} \delta_{n,m}, \tag{3.25b}$$

for $m \geq n$, and

$$\boldsymbol{L}(n,m) + \sum_{j=-\infty}^{n} \overline{\boldsymbol{L}}(n,j)\overline{\boldsymbol{F}}_l(j+m) = \begin{pmatrix} I_N \\ 0_{M\times N} \end{pmatrix} \delta_{n,m}, \tag{3.25c}$$

$$\overline{\boldsymbol{L}}(n,m) + \sum_{j=-\infty}^{n} \boldsymbol{L}(n,j)\boldsymbol{F}_l(j+m) = \begin{pmatrix} 0_{N\times M} \\ I_M \end{pmatrix} \delta_{n,m}, \tag{3.25d}$$

for $m \leq n$. Here $\delta_{n,m}$ is the Kronecker delta. In the case where the transmission coefficients have only simple poles, the Marchenko kernels are given by

$$\boldsymbol{F}_r(s) = \hat{\boldsymbol{\rho}}(s) + \sum_k z_k^{-(s+1)} \boldsymbol{C}_{rk}, \qquad \overline{\boldsymbol{F}}_r(s) = \hat{\overline{\boldsymbol{\rho}}}(s) + \sum_k \overline{z}_k^{s-1} \overline{\boldsymbol{C}}_{rk},$$

$$\overline{\boldsymbol{F}}_l(s) = \hat{\overline{\boldsymbol{\ell}}}(s) + \sum_k \overline{z}_k^{-(s+1)} \overline{\boldsymbol{C}}_{lk}, \qquad \boldsymbol{F}_l(s) = \hat{\boldsymbol{\ell}}(s) + \sum_k z_k^{s-1} \boldsymbol{C}_{lk},$$

where $\boldsymbol{C}_{lk}$ etc. are the norming constants. Even though we do not discuss the parity symmetry properties of the norming constants, we wish to point out that $\boldsymbol{F}_r(s)$, $\overline{\boldsymbol{F}}_r(s)$, $\overline{\boldsymbol{F}}_l(s)$, and $\boldsymbol{F}_l(s)$ are zero matrices if s is even.

[3]Only (3.25a) and (3.25b) appear in [5, Ch. 5] and [93] for the Ablowitz-Ladik system. For the Ablowitz-Ladik system, the right-hand sides of (3.25c) and (3.25d) are to be modified.

At this point, the derivations of the Marchenko equations are different. In the case of the *central differencing system*, the asymptotic properties of the Faddeev functions $z^{-n}\phi_n(z)$ and $z^n\psi_n(z)$ as $z \to \infty$ and the Faddeev functions $z^{-n}\overline{\psi}_n(z)$ and $z^n\overline{\phi}_n(z)$ as $z \to 0$ imply that

$$\overline{\boldsymbol{K}}^{\mathrm{up}}(n,n) = I_N, \qquad \overline{\boldsymbol{K}}^{\mathrm{dn}}(n,n) = 0_{M\times N}, \tag{3.26a}$$
$$\boldsymbol{K}^{\mathrm{up}}(n,n) = 0_{N\times M}, \qquad \boldsymbol{K}^{\mathrm{dn}}(n,n) = I_M, \tag{3.26b}$$
$$\boldsymbol{L}^{\mathrm{up}}(n,n) = I_N, \qquad \boldsymbol{L}^{\mathrm{dn}}(n,n) = 0_{M\times N}, \tag{3.26c}$$
$$\overline{\boldsymbol{L}}^{\mathrm{up}}(n,n) = 0_{N\times M}, \qquad \overline{\boldsymbol{L}}^{\mathrm{dn}}(n,n) = I_M. \tag{3.26d}$$

We may therefore write the Marchenko equations in the customary form

$$\overline{\boldsymbol{K}}(n,m) + \begin{pmatrix} 0_{N\times M} \\ I_M \end{pmatrix} \boldsymbol{F}_r(n+m) + \sum_{j=n+1}^{\infty} \boldsymbol{K}(n,j)\boldsymbol{F}_r(j+m) = \boldsymbol{0}, \tag{3.27a}$$
$$\boldsymbol{K}(n,m) + \begin{pmatrix} I_N \\ 0_{M\times N} \end{pmatrix} \overline{\boldsymbol{F}}_r(n+m) + \sum_{j=n+1}^{\infty} \overline{\boldsymbol{K}}(n,j)\overline{\boldsymbol{F}}_r(j+m) = \boldsymbol{0}, \tag{3.27b}$$

for $m \geq n+1$, and

$$\boldsymbol{L}(n,m) + \begin{pmatrix} 0_{N\times M} \\ I_M \end{pmatrix} \overline{\boldsymbol{F}}_l(n+m) + \sum_{j=-\infty}^{n-1} \overline{\boldsymbol{L}}(n,j)\overline{\boldsymbol{F}}_l(j+m) = \boldsymbol{0}, \tag{3.27c}$$
$$\overline{\boldsymbol{L}}(n,m) + \begin{pmatrix} I_N \\ 0_{M\times N} \end{pmatrix} \boldsymbol{F}_l(n+m) + \sum_{j=-\infty}^{n-1} \boldsymbol{L}(n,j)\boldsymbol{F}_l(j+m) = \boldsymbol{0}, \tag{3.27d}$$

for $m \leq n-1$. The zero matrices on the right-hand sides have $N+M$ rows and either N or M columns. The potentials follow from the solutions as follows:

$$Q_n = \frac{i}{2h}\boldsymbol{K}^{\mathrm{up}}(n-1,n), \qquad R_n = -\frac{i}{2h}\overline{\boldsymbol{K}}^{\mathrm{dn}}(n-1,n), \tag{3.28a}$$
$$Q_n = \frac{i}{2h}\overline{\boldsymbol{L}}^{\mathrm{up}}(n+1,n), \qquad R_n = -\frac{i}{2h}\boldsymbol{L}^{\mathrm{dn}}(n+1,n). \tag{3.28b}$$

Using that the Marchenko kernels $\boldsymbol{F}(s)$ are zero matrices for s even and certain up and down blocks of $\boldsymbol{K}(n,m)$ etc. are zero matrices depending on whether $m-n$ is even or odd, we arrive at the final Marchenko integral equations. We refer to [48, App. B] for details.

Let us now consider the Ablowitz-Ladik system. We should now replace (3.26) by

$$\overline{\boldsymbol{K}}^{\mathrm{up}}(n,n) = [\ldots(I_N - Q_{n+1}R_{n+1})(I_N - Q_nR_n)]^{-1}, \tag{3.29a}$$
$$\overline{\boldsymbol{K}}^{\mathrm{dn}}(n,n) = 0_{M\times N}, \tag{3.29b}$$
$$\boldsymbol{K}^{\mathrm{up}}(n,n) = 0_{N\times M}, \tag{3.29c}$$
$$\boldsymbol{K}^{\mathrm{dn}}(n,n) = [\ldots(I_M - R_{n+1}Q_{n+1})(I_M - R_nQ_n)]^{-1}, \tag{3.29d}$$

$$\boldsymbol{L}^{\mathrm{up}}(n,n) = I_N, \tag{3.29e}$$
$$\boldsymbol{L}^{\mathrm{dn}}(n,n) = 0_{M\times N}, \tag{3.29f}$$
$$\overline{\boldsymbol{L}}^{\mathrm{up}}(n,n) = 0_{N\times M}, \tag{3.29g}$$
$$\overline{\boldsymbol{L}}^{\mathrm{dn}}(n,n) = I_M. \tag{3.29h}$$

Instead of (3.28), we get

$$Q_n = -\boldsymbol{K}^{\mathrm{up}}(n,n+1)\boldsymbol{K}^{\mathrm{dn}}(n,n)^{-1} = \overline{\boldsymbol{L}}^{\mathrm{up}}(n+1,n), \tag{3.30a}$$
$$R_n = -\overline{\boldsymbol{K}}^{\mathrm{dn}}(n,n+1)\overline{\boldsymbol{K}}^{\mathrm{up}}(n,n)^{-1} = \boldsymbol{L}^{\mathrm{up}}(n+1,n), \tag{3.30b}$$

because $\boldsymbol{L}^{\mathrm{up}}(n,n) = I_M$ and $\overline{\boldsymbol{L}}^{\mathrm{dn}}(n,n) = I_N$.

Equations (3.25a) and (3.25b) remain the same, but (3.25c) and (3.25d) are to be replaced by

$$\boldsymbol{L}(n,m) + \sum_{j=-\infty}^{n} \overline{\boldsymbol{L}}(n,j)\overline{\boldsymbol{F}}_l(j+m) = \begin{pmatrix} I_N \\ 0_{M\times N} \end{pmatrix} (I_N - Q_nR_n)^{-1}\delta_{n,m}, \tag{3.31a}$$
$$\overline{\boldsymbol{L}}(n,m) + \sum_{j=-\infty}^{n} \boldsymbol{L}(n,j)\boldsymbol{F}_l(j+m) = \begin{pmatrix} 0_{N\times M} \\ I_M \end{pmatrix} (I_M - R_nQ_n)^{-1}\delta_{n,m}, \tag{3.31b}$$

where $m \leq n$. Thus we apparently have to choose between two bad alternatives:

a. Being able to formulate and solve the Marchenko equations (3.25a) and (3.25b), but not being able to get Q_n and R_n from its solution [cf. (3.30), middle members].

b. Not being able to formulate the Marchenko equations (3.31a) and (3.31b) [because the inhomogeneous terms require us to know the potentials in advance], but being able to compute Q_n and R_n once the solution is known.

The apparent disparity is due to the lack of forward-backward symmetry of the Ablowitz-Ladik system.

To remedy the situation, in [5, Ch. 5] and [93] a new function $\boldsymbol{\kappa}(n,m)$ is introduced by multiplying $\boldsymbol{K}(n,m)$ and $\overline{\boldsymbol{K}}(n,m)$ on the left by the direct sum of the right-hand sides of (3.29a) and (3.29d). By assuming (3.17), these right-hand sides become scalar multiples of the identity matrix, leading to Marchenko equations whose solutions lead to Q_n and R_n.

Instead, (3.31a) and (3.31b) can always be written in the familiar form (3.27c)-(3.27d) and then the feared inhomogeneous terms drop out. Then Q_n and R_n can be computed exactly as in the finite differencing case. Even more so: When properly choosing $h = \frac{1}{2}$, exactly the same Marchenko equations and exactly the same formulas to pass from its solutions to the potentials are valid in the central differencing and Ablowitz-Ladik cases. As a bonus, in the central differencing case we also have the additional Marchenko equation pair (3.25a)-(3.25b).

3.5 Propagation of scattering data

In this section we discuss the propagation of the scattering data and the Marchenko kernels of the Flaschka-Toda, Ablowitz-Ladik, and finite differencing systems.

1. Flaschka-Toda system. The Toda lattice equation has a Lax pair $\{L, A\}$ given by (1.8). If $a_n \equiv \frac{1}{2}$ and $b_n \equiv 0$, this Lax pair reduces to $\{L_0, A_0\}$, where for $\boldsymbol{x} = \{x_n\}_{n=-\infty}^{\infty}$

$$(L_0\boldsymbol{x})_n = \tfrac{1}{2}(x_{n+1} + x_{n-1}), \qquad (A_0\boldsymbol{x})_n = \tfrac{1}{2}(x_{n+1} - x_{n-1}).$$

The Marchenko kernels have the following time evolution

$$[\boldsymbol{F}_r(n;t)]_t = -\boldsymbol{F}_r(n+1;t) + \boldsymbol{F}_r(n-1;t), \tag{3.32a}$$

$$[\boldsymbol{F}_l(n;t)]_t = \boldsymbol{F}_l(n+1;t) - \boldsymbol{F}_l(n-1;t). \tag{3.32b}$$

These are the Hamilton equations corresponding to the hamiltonian

$$\begin{aligned}\boldsymbol{H} &= \sum_{n=-\infty}^{\infty} \{\boldsymbol{F}_r(n+1) - \boldsymbol{F}_r(n-1)\}\, \boldsymbol{F}_l(n) \\ &= -\sum_{n=-\infty}^{\infty} \boldsymbol{F}_r(n)\, \{\boldsymbol{F}_l(n+1) - \boldsymbol{F}_l(n-1)\},\end{aligned}$$

where $\boldsymbol{F}_r(n)$ is "position" and $\boldsymbol{F}_l(n)$ is "momentum." Thus the inverse scattering transform is a canonical transformation from the original variables $\{q_n, p_n\}_{n=-\infty}^{\infty}$ to the variables $\{\boldsymbol{F}_r(n), \boldsymbol{F}_l(n)\}_{n=-\infty}^{\infty}$ which linearizes the Hamilton equations of motion.

Schematically, using κ and $\boldsymbol{F}_r$ we have the inverse scattering transform

$$\begin{tikzcd}
\{q_n(0), p_n(0)\} \arrow[r] \arrow[d, "\text{Toda lattice}"] & \{a_n(0), b_n(0)\} \arrow[r] \arrow[d, "\text{Toda lattice}"'] & \kappa(n,m;0) \arrow[r] & \boldsymbol{F}_r(n;0) \arrow[d, "{[\boldsymbol{F}_r]_t = -\boldsymbol{F}_r(n+1)+\boldsymbol{F}_r(n-1)}"'] \\
\{q_n(t), p_n(t)\} & \{a_n(t), b_n(t)\} \arrow[l] & \kappa(n,m;t) \arrow[l] & \boldsymbol{F}_r(n;t) \arrow[l]
\end{tikzcd}$$

Using ℓ and $\boldsymbol{F}_l$ we have the inverse scattering transform

$$\begin{tikzcd}
\{q_n(0), p_n(0)\} \arrow[r] \arrow[d, "\text{Toda lattice}"] & \{a_n(0), b_n(0)\} \arrow[r] \arrow[d, "\text{Toda lattice}"'] & \ell(n,m;0) \arrow[r] & \boldsymbol{F}_l(n;0) \arrow[d, "{[\boldsymbol{F}_l]_t = \boldsymbol{F}_l(n+1)-\boldsymbol{F}_l(n-1)}"'] \\
\{q_n(t), p_n(t)\} & \{a_n(t), b_n(t)\} \arrow[l] & \ell(n,m;t) \arrow[l] & \boldsymbol{F}_l(n;t) \arrow[l]
\end{tikzcd}$$

2. Ablowitz-Ladik and central differencing systems. The time evolution of the reflection coefficients is as follows (as far as the Ablowitz-Ladik system is concerned, see [5, Ch. 5] and [93]):

$$\boldsymbol{\rho}(z;t) = e^{i(z-z^{-1})^2 t}\boldsymbol{\rho}(z;0), \qquad \overline{\boldsymbol{\rho}}(z;t) = e^{-i(z-z^{-1})^2 t}\overline{\boldsymbol{\rho}}(z;0),$$

$$\overline{\boldsymbol{\ell}}(z;t) = e^{i(z-z^{-1})^2 t}\overline{\boldsymbol{\ell}}(z;0), \qquad \boldsymbol{\ell}(z;t) = e^{i(z-z^{-1})^2 t}\boldsymbol{\ell}(z;0).$$

Thus using (3.24) we find the difference equations

$$\begin{aligned}
[\hat{\boldsymbol{\rho}}]_t(n;t) &= +i\left(\hat{\boldsymbol{\rho}}(n+2;t) - 2\hat{\boldsymbol{\rho}}(n;t) + \hat{\boldsymbol{\rho}}(n-2;t)\right), \\
[\hat{\overline{\boldsymbol{\rho}}}]_t(n;t) &= -i\left(\hat{\overline{\boldsymbol{\rho}}}(n+2;t) - 2\hat{\overline{\boldsymbol{\rho}}}(n;t) + \hat{\overline{\boldsymbol{\rho}}}(n-2;t)\right), \\
[\hat{\overline{\boldsymbol{\ell}}}]_t(n;t) &= +i\left(\hat{\overline{\boldsymbol{\ell}}}(n+2;t) - 2\hat{\overline{\boldsymbol{\ell}}}(n;t) + \hat{\overline{\boldsymbol{\ell}}}(n-2;t)\right), \\
[\hat{\boldsymbol{\ell}}]_t(n;t) &= -i\left(\hat{\boldsymbol{\ell}}(n+2;t) - 2\hat{\boldsymbol{\ell}}(n;t) + \hat{\boldsymbol{\ell}}(n-2;t)\right).
\end{aligned}$$

Using the time evolution of the norming constants, we arrive at the time evolution of the Marchenko kernels

$$[\boldsymbol{F}_r]_t(n;t) = +i\left(\boldsymbol{F}_r(n+2;t) - 2\boldsymbol{F}_r(n;t) + \boldsymbol{F}_r(n-2;t)\right), \tag{3.33a}$$

$$[\overline{\boldsymbol{F}}_r]_t(n;t) = -i\left(\overline{\boldsymbol{F}}_r(n+2;t) - 2\overline{\boldsymbol{F}}_r(n;t) + \overline{\boldsymbol{F}}_r(n-2;t)\right), \tag{3.33b}$$

$$[\overline{\boldsymbol{F}}_l]_t(n;t) = +i\left(\overline{\boldsymbol{F}}_l(n+2;t) - 2\overline{\boldsymbol{F}}_l(n;t) + \overline{\boldsymbol{F}}_l(n-2;t)\right), \tag{3.33c}$$

$$[\boldsymbol{F}_l]_t(n;t) = -i\left(\boldsymbol{F}_l(n+2;t) - 2\boldsymbol{F}_l(n;t) + \boldsymbol{F}_l(n-2;t)\right). \tag{3.33d}$$

In other words,

> *Irrespective of which linear difference system is used, we have found the same inverse scattering transform to solve the matrix IDNLS system.*

Equations (3.33a) and (3.33b) are the Hamilton equations corresponding to the hamiltonian

$$\boldsymbol{H} = i\sum_{n=-\infty}^{\infty}(\boldsymbol{F}_r(n+2) - \boldsymbol{F}_r(n))(\overline{\boldsymbol{F}}_r(n+2) - \overline{\boldsymbol{F}}_r(n)),$$

provides $\boldsymbol{F}_r(n)$ is "position" and $\overline{\boldsymbol{F}}_r(n)$ is "momentum." Equations (3.33c) and (3.33d) follow from the hamiltonian

$$\boldsymbol{H} = -i\sum_{n=-\infty}^{\infty}(\boldsymbol{F}_l(n+2) - \boldsymbol{F}_l(n))(\overline{\boldsymbol{F}}_l(n+2) - \overline{\boldsymbol{F}}_l(n))$$

instead. We have not been able to find the hamiltonian leading to the original (matrix) IDNLS system.

Schematically, using $\boldsymbol{K}$ and $\boldsymbol{F}_r$ we have the inverse scattering transform

$$\begin{tikzcd}
\{Q_n(0),R_n(0)\} \arrow[r] \arrow[d, "\substack{\text{matrix IDNLS}\\ \text{system}}"] & \{\overline{\boldsymbol{K}}(n,m;0),\boldsymbol{K}(n,m;0)\} \arrow[r] & \{\boldsymbol{F}_r(n;0),\overline{\boldsymbol{F}}_r(n;0)\} \arrow[d, "\substack{[\boldsymbol{F}_r]_t=i(\boldsymbol{F}_r(n+2)-2\boldsymbol{F}_r(n)+\boldsymbol{F}_r(n-2))\\ [\overline{\boldsymbol{F}}_l]_t=-i(\overline{\boldsymbol{F}}_r(n+2)-2\overline{\boldsymbol{F}}_r(n)+\overline{\boldsymbol{F}}_r(n-2))}"'] \\
\{Q_n(t),R_n(t)\} & \{\overline{\boldsymbol{K}}(n,m;t),\boldsymbol{K}(n,m;t)\} \arrow[l] & \{\boldsymbol{F}_r(n;t),\overline{\boldsymbol{F}}_r(n;t)\} \arrow[l]
\end{tikzcd}$$

Using $\boldsymbol{L}$ and $\boldsymbol{F}_l$ we have the inverse scattering transform

$$\begin{tikzcd}
\{Q_n(0),R_n(0)\} \arrow[r] \arrow[d, "\substack{\text{matrix IDNLS}\\ \text{system}}"] & \{\boldsymbol{L}(n,m;0),\overline{\boldsymbol{L}}(n,m;0)\} \arrow[r] & \{\boldsymbol{F}_r(n;0),\overline{\boldsymbol{F}}_r(n;0)\} \arrow[d, "\substack{[\boldsymbol{F}_l]_t=-i(\boldsymbol{F}_l(n+2)-2\boldsymbol{F}_l(n)+\boldsymbol{F}_l(n-2))\\ [\overline{\boldsymbol{F}}_l]_t=i(\overline{\boldsymbol{F}}_l(n+2)-2\overline{\boldsymbol{F}}_l(n)+\overline{\boldsymbol{F}}_l(n-2))}"'] \\
\{Q_n(t),R_n(t)\} & \{\boldsymbol{L}(n,m;t),\overline{\boldsymbol{L}}(n,m;t)\} \arrow[l] & \{\boldsymbol{F}_l(n;t),\overline{\boldsymbol{F}}_l(n;t)\} \arrow[l]
\end{tikzcd}$$

Chapter 4

Closed form solutions

In this chapter we derive closed form solutions of various integrable equations by restricting ourselves to those cases in which the Marchenko equation is separable in the sense that for each $t \in \mathbb{R}$ its kernel satisfies

$$\begin{cases} \Omega(x+y;t) = \Omega_1(x;t)\Omega_2(y;t), & \text{continuous case,} \\ \Omega(n+m;t) = \Omega_1(n;t)\Omega_2(m;t), & \text{discrete case.} \end{cases}$$

These separable Marchenko kernels are in turn written in terms of a matrix triplet (A, B, C) in the form

$$\begin{cases} \Omega(x+y;t) = Ce^{-(x+y)A}e^{it\phi(A)}B = Ce^{-xA}e^{-yA}e^{it\phi(A)}B, & \text{continuous case,} \\ \Omega(n+m;t) = CA^{n+m}e^{it\phi(A)}B = CA^nA^me^{it\phi(A)}B, & \text{discrete case.} \end{cases}$$

Since in this case the Marchenko equation is very easy to solve, the result is a powerful method for deriving closed form solutions.

Matrix and operator triplets have been applied to derive solutions of nonlinear evolution equations by using a continuous multiplicative functional to pass from a solution in a large algebra to the actual solution. In this way solutions have been found of the KdV [7], NLS [17], Toda lattice [84], and SG [85] equations. This procedure has also been cast in the framework of bidifferential calculus in [51, 50], leading to explicit solutions of the NLS [51] and Ernst equations [50]. Matrix triplets have also been applied to obtain matrix NLS and mKdV solution formulas that were verified by direct substitution [63]. In all of these papers closed form solutions are obtained without any recourse to the IST. On the other hand, representations of Marchenko kernels in terms of matrix triplets have supplied an alternative route to closed form solutions. This method has been applied to getting closed form solutions of the KdV [11], NLS with vanishing boundary conditions [41, 8, 43], mKdV [40], SG [9], IDNLS [46, 47], and defocusing NLS with nonvanishing boundary conditions [42].

Solutions of integrable nonlinear evolution equations written in terms of matrix triplets (A, B, C) contain virtually all of the known explicit solutions to these equations, such as N-soliton and breather solutions. Their amenability to computer algebra makes these expressions into excellent tools to obtain solutions in unpacked analytical or graphical form. Also, these expressions can be used to check the accuracy and speed of numerical methods to compute the solution of an integrable nonlinear evolution equation.

4.1 Some important matrix equations

In this section we derive some well-known results on solutions of certain matrix equations [18, 62, 52]. For observability and controllability of matrix pairs we refer to [16, 52] and textbooks on mathematical control theory [98, 33].

1. Sylvester equations. Consider the so-called *Sylvester equation*

$$AX - XB = C, \tag{4.1}$$

where the known matrices A, B, and C have the sizes $p \times p$, $q \times q$, and $p \times q$ and the unknown matrix X is $p \times q$. Then for each $\lambda \in \mathbb{C}$ we have

$$X(\lambda - B) - (\lambda - A)X = C.$$

For those λ which are not eigenvalues of A or B, we get

$$(\lambda - A)^{-1}X - X(\lambda - B)^{-1} = (\lambda - A)^{-1}C(\lambda - B)^{-1}.$$

For any closed rectifiable contour Γ in the complex plane which does not contain any eigenvalues of A and B, we have

$$\begin{aligned}&\left[\frac{1}{2\pi i}\int_\Gamma d\lambda\,(\lambda - A)^{-1}\right]X\\ &- X\left[\frac{1}{2\pi i}\int_\Gamma d\lambda\,(\lambda - B)^{-1}\right] = \frac{1}{2\pi i}\int_\Gamma d\lambda\,(\lambda - A)^{-1}C(\lambda - B)^{-1}.\end{aligned}$$

If A and B do not have eigenvalues in common, we can choose the closed rectifiable contour Γ in such a way that Γ has winding number $+1$ with respect to each eigenvalue of A and winding number zero with respect to each eigenvalue of B. In other words,

$$\frac{1}{2\pi i}\int_\Gamma \frac{d\lambda}{\lambda - \lambda_0} = \begin{cases}1, & \lambda_0 \text{ is eigenvalue of } A,\\ 0, & \lambda_0 \text{ is eigenvalue of } B.\end{cases}$$

Using this fact, we get

$$\frac{1}{2\pi i}\int_\Gamma d\lambda\,(\lambda - A)^{-1} = I_p. \qquad \frac{1}{2\pi i}\int_\Gamma d\lambda\,(\lambda - B)^{-1} = 0_{q\times q}.$$

As a result,

$$X = \frac{1}{2\pi i} \int_\Gamma d\lambda \, (\lambda - A)^{-1} C (\lambda - B)^{-1}. \tag{4.2}$$

On the other hand, if Γ' is a closed rectifiable contour in the complex plane which has winding number zero with respect to each eigenvalue of A and winding number $+1$ with respect to each eigenvalue of B, then

$$\frac{1}{2\pi i} \int_{\Gamma'} d\lambda \, (\lambda - A)^{-1} = 0_{p\times p}. \qquad \frac{1}{2\pi i} \int_{\Gamma'} d\lambda \, (\lambda - B)^{-1} = I_q.$$

As a result,

$$X = -\frac{1}{2\pi i} \int_{\Gamma'} d\lambda \, (\lambda - A)^{-1} C (\lambda - B)^{-1}. \tag{4.3}$$

The following result is due to Sylvester [90], though the solution formulas (4.2) and (4.3) are due to Rosenblum [82]. Such results, though sometimes without their original historical context, can be found in modern notations in the paper by Rutherford [83] and in textbooks [18, 62, 52]. If C is a rank-one matrix, then explicit formulas for $\det X$ in terms of the eigenvalue structure of A and B have been given by Schiebold [86], based in part on a problem posed by Pólya and Szegő [80, Ex. IQ.7.3].

Theorem 4.1 *The Sylvester equation* (4.1) *has a unique solution X for each right-hand side C if and only if A and B do not have eigenvalues in common. The unique solution is given by either* (4.2) *or* (4.3).

Proof. It remains to prove that $AX = XB$ has at least one nontrivial solution X if A and B have eigenvalues in common. Indeed, let λ_0 be a common eigenvalue of A and B. Further, let $\boldsymbol{x}$ be a nontrivial column vector (of length p) such that $A\boldsymbol{x} = \lambda_0 \boldsymbol{x}$; also, let $\boldsymbol{\phi}$ be a nontrivial row vector (of length q) such that $\boldsymbol{\phi} B = \lambda_0 \boldsymbol{\phi}$. Put

$$X = \boldsymbol{x}\, \boldsymbol{\phi},$$

which is a nontrivial $p \times q$ matrix of rank one. Then

$$AX = A(\boldsymbol{x}\,\boldsymbol{\phi}) = (A\boldsymbol{x})\boldsymbol{\phi} = (\lambda_0 \boldsymbol{x})\boldsymbol{\phi} = \boldsymbol{x}(\lambda_0 \boldsymbol{\phi}) = \boldsymbol{x}(\boldsymbol{\phi} B) = (\boldsymbol{x}\,\boldsymbol{\phi})B = XB,$$

as claimed. ∎

The map Φ defined by

$$\Phi(X) = AX - XB$$

is a linear transformation defined on the pq-dimensional vector space of $p \times q$ matrices. Then λ_0 is an eigenvalue of Φ if and only if there exists a nontrivial $p \times q$ matrix such that

$$\Phi(X) - \lambda_0 X = (A - \lambda_0)X - XB$$

is the zero matrix. The latter implies that $A-\lambda_0$ and B have at least one common eigenvalue, μ_0 say. Thus λ_0 is the difference $(\lambda_0+\mu_0)-\mu_0$ of an eigenvalue of A and an eigenvalue of B. Thus the spectrum $\sigma(\Phi)$ of the linear transformation Φ is given by

$$\sigma(\Phi)=\{\lambda-\mu:\lambda\in\sigma(A)\text{ and }\mu\in\sigma(B)\}.$$

Now suppose that A has only eigenvalues of positive real part and B has only eigenvalues of negative real part. Then A and B cannot have common eigenvalues and hence the Sylvester equation (4.1) has a unique solution X for each $p\times q$ matrix C. It is now easily seen that the matrix groups e^{-xA} and e^{xB} are exponentially decreasing as $x\to+\infty$. Further,

$$\frac{d}{dx}\left[e^{-xA}Xe^{xB}\right]=-e^{-xA}(AX-XB)e^{xB}=-e^{-xA}Ce^{xB}.$$

Integrating this identity with respect to $x\in(0,+\infty)$ we get

$$X=\left[-e^{-xA}Xe^{xB}\right]_{x=0}^{\infty}=\int_0^\infty dx\,e^{-xA}Ce^{xB}.$$

In other words, we have found the following result of Heinz [65]:

$$X=\int_0^\infty dx\,e^{-xA}Ce^{xB}. \tag{4.4}$$

2. Stein equations. Consider the so-called Stein equation

$$X-AXB=C. \tag{4.5}$$

where the known matrices A, B, and C are $p\times p$, $q\times q$, and $p\times q$ and the unknown matrix C is $q\times q$. Then (4.5) can be rewritten in one of the following two forms:

$$X(I_q-\lambda B)+(\lambda-A)XB=C, \tag{4.6a}$$
$$(I_p-\lambda A)X+AX(\lambda-B)=C. \tag{4.6b}$$

Now suppose that there do not exist nonzero eigenvalues of A and B which have $+1$ as their product. In that case there exists a closed rectifiable contour Γ in the complex plane which has winding number $+1$ with respect to the eigenvalues of A and with respect to zero and has winding number zero with respect to the reciprocals of the nonzero eigenvalues of B. In that case the matrix equation (4.6a) can be written in the form

$$XB(I_q-\lambda B)^{-1}+(\lambda-A)^{-1}X=(\lambda-A)^{-1}C(I_q-\lambda B)^{-1},$$

where $\lambda \in \Gamma$. As a result,

$$X = \frac{1}{2\pi i} \int_{\Gamma} d\lambda \, (\lambda - A)^{-1} C (I_q - \lambda B)^{-1}. \tag{4.7}$$

On the other hand, there exists a closed rectifiable contour Γ' in the complex plane which has winding number $+1$ with respect to the eigenvalues of B and with respect to zero and has winding number zero with respect to the reciprocals of the nonzero eigenvalues of A. In that case the matrix equation (4.6b) can be written in the form

$$(I_p - \lambda A)^{-1} A X + X(\lambda - B)^{-1} = (I_p - \lambda A)^{-1} C (\lambda - B)^{-1},$$

where $\lambda \in \Gamma'$. As a result,

$$X = \frac{1}{2\pi i} \int_{\Gamma'} d\lambda \, (I_p - \lambda A)^{-1} C (\lambda - B)^{-1}. \tag{4.8}$$

Theorem 4.2 *The matrix equation* (4.5) *has a unique solution X for each right-hand side C if and only if there do not exist an eigenvalue of A and an eigenvalue of B having $+1$ as their product. The unique solution is given by either* (4.7) *or* (4.8).

Proof. It remains to prove the existence of a nontrivial $p \times q$ matrix X such that $X = AXB$ if A has a nonzero eigenvalue λ_0 such that $1/\lambda_0$ is an eigenvalue of B. In that case there exists a nontrivial column vector (of length p) $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda_0 \boldsymbol{x}$; also, there exists a nontrivial row vector (of length q) $\boldsymbol{\phi}$ such that $\lambda_0 \phi B = \boldsymbol{\phi}$. Put

$$X = \boldsymbol{x} \, \boldsymbol{\phi},$$

which is a nontrivial $p \times q$ matrix. Then

$$AXB = (A\boldsymbol{x})(\boldsymbol{\phi} B) = (\lambda_0 \boldsymbol{x})(\lambda_0^{-1} \boldsymbol{\phi}) = \boldsymbol{x} \, \boldsymbol{\phi} = X,$$

as claimed. ∎

Consider the linear transformation

$$\Psi(X) = X - AXB$$

defined on the pq-dimensional vector space of $p \times q$ matrices. Then the spectrum of Ψ is given by

$$\sigma(\Psi) = \{1 - \lambda\mu : 0 \neq \lambda \in \sigma(A) \text{ and } 0 \neq \mu \in \sigma(B)\}.$$

Let us now suppose that at least one of the matrices A and B is nilpotent, A say. Then $A^r = 0_{p\times p}$ for some nonnegative integer r. Then it is easily verified (by iteration) that

$$X = \sum_{j=0}^{r-1} A^j C B^j$$

is the unique solution to (4.5). The same unique solution formula holds if $B^r = 0_{q\times q}$.

Now suppose that A and B both have a spectral radius of less than $+1$. Then there do not exist eigenvalues of A and B having 1 as their product. Thus the matrix equation (4.5) has a unique solution X for each right-hand side C. In fact, X is given by the absolutely convergent series

$$X = \sum_{n=0}^{\infty} A^n C B^n, \tag{4.9}$$

as is easily verified. In fact,

$$AXB = \sum_{n=0}^{\infty} A^{n+1} C B^{n+1} = \sum_{n=1}^{\infty} A^n C B^n = X - C.$$

3. Lyapunov equations. Consider the Lyapunov equations

$$A^\dagger Q + QA = C^\dagger C, \tag{4.10a}$$

$$AN + NA^\dagger = BB^\dagger. \tag{4.10b}$$

Then either Lyapunov equation has a unique solution if and only if A and $-A^\dagger$ do not have common eigenvalues, i.e., if and only A does not have imaginary eigenvalues or eigenvalue pairs $\{\lambda_0, -\lambda_0^*\}$. When A has only eigenvalues with positive real part, we have

$$Q = \int_0^\infty dx\, e^{-xA^\dagger} C^\dagger C e^{-xA}, \tag{4.11a}$$

$$N = \int_0^\infty dx\, e^{-xA} BB^\dagger e^{-xA^\dagger}. \tag{4.11b}$$

As a result, for each $\boldsymbol{x} \in \mathbb{C}^p$ we get

$$(Q\boldsymbol{x}, \boldsymbol{x}) = \int_0^\infty dx\, \|Ce^{-xA}\boldsymbol{x}\|^2, \tag{4.12a}$$

$$(N\boldsymbol{x}, \boldsymbol{x}) = \int_0^\infty dx\, \|B^\dagger e^{-xA^\dagger}\boldsymbol{x}\|^2. \tag{4.12b}$$

Hence Q and N are nonnegative hermitian matrices. Also,

Proposition 4.3 *The matrix Q is invertible if and only if the pair of matrices (C, A) is observable in the sense that*

$$\bigcap_{n=0}^{\infty} Ker(CA^n) = \{0\}.$$

The matrix N is invertible if and only if the pair of matrices (A, B) is controllable in the sense that

$$span \bigcup_{n=0}^{\infty} Im(A^n B) = \{0\}.$$

Proof. Clearly, for $\boldsymbol{x} \in \mathbb{C}^p$ we have $Q\boldsymbol{x} = 0$ if and only if $(Q\boldsymbol{x}, \boldsymbol{x}) = 0$, which is true if and only if $CA^n\boldsymbol{x} = 0$ $(n = 0, 1, 2, \ldots)$, i.e., if and only if the pair (C, A) is observable. On the other hand, N is invertible if and only if the pair of matrices $(B^\dagger, A^\dagger)$ is observable. However, $\mathrm{Ker}\,(B^\dagger {A^\dagger}^n)$ and $\mathrm{Im}\,(A^n B)$ are each other's orthogonal complement. Hence, N is invertible if and only if the pair (A, B) is controllable. ∎

The matrix triplet (A, B, C) is called *minimal* if and only if the pair (C, A) is observable and the pair (A, B) is controllable. Thus the triplet (A, B, C) is minimal if and only if the Lyapunov solutions Q and N are both invertible.

The following result has appeared in a different context in linear control theory (cf. [98, 33, 52, 16] and references therein).

Theorem 4.4 *Suppose (A_1, B_1, C_1) and (C_2, A_2, B_2) are two minimal matrix triplets such that*

$$C_1 e^{-xA_1} B_1 = C_2 e^{-xA_2} B_2, \qquad x \in \mathbb{R}. \tag{4.13}$$

Expanding either side of this equation in powers of x, we can write (4.13) in the form

$$C_1 {A_1}^n B_1 = C_2 {A_2}^n B_2, \qquad n = 0, 1, 2, \ldots. \tag{4.14}$$

Then there exists a unique invertible matrix S, called similarity, such that

$$C_1 = C_2 S, \qquad SA_1 = A_2 S, \qquad SB_1 = B_2. \tag{4.15}$$

In fact, we can define S in the following way:

$$S\left(\sum_{n=0}^{\infty} A_1^n B_1 \boldsymbol{x}_n\right) = \sum_{n=0}^{\infty} A_2^n B_2 \boldsymbol{x}_n,$$

where all but finitely many $\boldsymbol{x}_n$ are nontrivial vectors. Then, as will be proven shortly, S is well-defined in the sense that it maps the zero vector to the zero vector. Equation (4.15) then is immediate.

Proof. Indeed, suppose that

$$\boldsymbol{z} = \sum_{n=0}^{\infty} A_1^n B_1 \boldsymbol{x}_n = \sum_{n=0}^{\infty} A_1^n B_1 \tilde{\boldsymbol{x}}_n.$$

Then for each $m = 0, 1, 2, \ldots$ we have

$$C_1 {A_1}^m \boldsymbol{z} = \sum_{n=0}^{\infty} C_1 A_1^{n+m} B_1 \boldsymbol{x}_n = \sum_{n=0}^{\infty} C_1 A_1^{n+m} B_1 \tilde{\boldsymbol{x}}_n.$$

Using (4.14), we get

$$C_2 {A_2}^m S \boldsymbol{z} = \sum_{n=0}^{\infty} C_2 A_2^{n+m} B_2 \boldsymbol{x}_n = \sum_{n=0}^{\infty} C_2 A_2^{n+m} B_2 \tilde{\boldsymbol{x}}_n,$$

which can also be written as [cf. (4.15)]

$$C_2 {A_2}^m S \boldsymbol{z} = C_1 {A_1}^m \boldsymbol{z} = C_2 {A_2}^m \sum_{n=0}^{\infty} A_2^n B_2 \boldsymbol{x}_n = C_2 {A_2}^m \sum_{n=0}^{\infty} A_2^n B_2 \tilde{\boldsymbol{x}}_n.$$

Because the pair (C_2, A_2) is observable, we have

$$S \boldsymbol{z} = \sum_{n=0}^{\infty} A_2^n B_2 \boldsymbol{x}_n = \sum_{n=0}^{\infty} A_2^n B_2 \tilde{\boldsymbol{x}}_n,$$

which proves that S is well-defined. Using the controllability of the pair (C_2, A_2), we can write any $\boldsymbol{y} \in \mathbb{C}^p$ in the form

$$\boldsymbol{y} = \sum_{n=0}^{\infty} {A_2}^n B_2 \boldsymbol{y}_n = S \left(\sum_{n=0}^{\infty} {A_1}^n B_1 \boldsymbol{y}_n \right),$$

where all but finitely many $\boldsymbol{y}_n$ vanish. As a result, S has full rank. Finally, the identities (4.18) for any similarity S imply that S must have the above form, thus proving the uniqueness of S. ■

4. Stein equations. Consider the hermitian *Stein equations*

$$Q - A^\dagger Q A = C^\dagger C, \tag{4.16a}$$

$$N - A N A^\dagger = B B^\dagger. \tag{4.16b}$$

Then either hermitian Stein equation has a unique solution if and only if A does not have eigenvalues of modulus 1 or eigenvalue pairs $\{\lambda_0, 1/\lambda_0^*\}$. When A has a spectral radius of less than 1, we get

$$Q = \sum_{n=0}^{\infty} {A^\dagger}^n C^\dagger C A^n, \tag{4.17a}$$

$$N = \sum_{n=0}^{\infty} A^n B B^\dagger {A^\dagger}^n. \tag{4.17b}$$

As a result, for each $\boldsymbol{x} \in \mathbb{C}^p$ we have

$$(Q\boldsymbol{x}, \boldsymbol{x}) = \sum_{n=0}^{\infty} \|CA^n\boldsymbol{x}\|^2, \tag{4.18a}$$

$$(N\boldsymbol{x}, \boldsymbol{x}) = \sum_{n=0}^{\infty} \|B^\dagger {A^\dagger}^n\boldsymbol{x}\|^2. \tag{4.18b}$$

Hence Q and N are nonnegative hermitian matrices.

The following result has appeared in a different context in linear control theory (cf. [98, 33, 52, 16] and references therein). The proof is very similar to that of Proposition 4.3.

Proposition 4.5 *The matrix Q is invertible if and only if the pair of matrices (C, A) is observable. The matrix N is invertible if and only if the pair of matrices (A, B) is controllable.*

In the same way as for Theorem 4.4 we can prove the following: Suppose (A_1, B_1, C_1) and (C_2, A_2, B_2) are two minimal matrix triplets such that (4.14) is true. Then there exists a unique invertible matrix S, called *similarity*, such that (4.15) is true.

Let us now derive an important invertibility property involving Lyapunov solutions.

Proposition 4.6 *Suppose A has only eigenvalues with positive real part, and let Q and N be given by* (4.11)*. Then the matrix*

$$\Gamma = I_p + QN$$

is invertible.

In the same way we prove the invertibility of Γ if Q and N are given by (4.17) of some matrix A with a spectral radius of less than 1.

Proof. There exist unique nonnegative hermitian matrices $Q^{1/2}$ and $N^{1/2}$ such that $[Q^{1/2}]^2 = Q$ and $[N^{1/2}]^2 = N$. Putting $W = Q^{1/2}N^{1/2}$, we see that $I_p + W^\dagger W$ is invertible. In fact,

$$([I_p + W^\dagger W]\boldsymbol{x}, \boldsymbol{x}) = \|\boldsymbol{x}\|^2 + \|W\boldsymbol{x}\|^2 \geq \|\boldsymbol{x}\|^2,$$

so that $\operatorname{Ker}[I_p + W^\dagger W] = \{0\}$. Solving $\Gamma\boldsymbol{x} = \boldsymbol{y}$, we get

$$\boldsymbol{x} = \boldsymbol{y} - QN^{1/2}N^{1/2}\boldsymbol{x},$$

where

$$(I_p + W^\dagger W)N^{1/2}\boldsymbol{x} = N^{1/2}\boldsymbol{y}.$$

As a result,

$$\boldsymbol{x} = \boldsymbol{y} - QN^{1/2}(I_p + W^\dagger W)^{-1}N^{1/2}\boldsymbol{y}.$$

Hence, Γ is invertible and

$$\Gamma^{-1} = I_p - QN^{1/2}(I_p + W^\dagger W)^{-1}N^{1/2},$$

as claimed. ■

Replacing the triplet (A, B, C) by $(A, Ce^{-xA}, e^{-xA}B)$ and observing that Q and N are to be replaced by $e^{-xA^\dagger}Qe^{-xA^\dagger}$ and $e^{-xA}Ne^{-xA}$, respectively, we obtain

Corollary 4.7 *Suppose A has only eigenvalues with positive real part, and let Q and N be given by* (4.11)*. Then for every $x \in \mathbb{R}$ the matrix*

$$\Gamma(x) = I_p + e^{-xA^\dagger}Qe^{-2xA}Ne^{-xA^\dagger}$$

is invertible.

4.2 Closed form solutions: Matrix NLS

In this section we derive the closed form matrix NLS solutions for which the corresponding reflection coefficients vanish. This will be done without assuming symmetries on the potentials. First we solve the inverse scattering problem. We then go on inserting the time factors. At the end, we impose symmetries on the potentials and derive the well-known solution formulas in the focusing case.

Let us solve the Marchenko equations (2.50a) and (2.50b) by writing

$$\Omega_l(x+y) = Ce^{-(x+y)A}B, \qquad \breve{\Omega}_l(x+y) = \breve{C}e^{-(x+y)\breve{A}}\breve{B},$$

where (A, B, C) and $(\breve{A}, \breve{B}, \breve{C})$ are two matrix triplets and the $p \times p$ matrix A and the $\breve{p} \times \breve{p}$ matrix $\breve{A}$ have only eigenvalues with positive real part. Then the matrix groups e^{-xA} and $e^{-x\breve{A}}$ are exponentially decreasing as $x \to +\infty$. We then write (2.50a) and (2.50b) in the form

$$\overline{K}(x, y) = -\left\{\begin{pmatrix} 0_{m\times n} \\ I_n \end{pmatrix} Ce^{-xA} + L(x)\right\} e^{-yA}B, \tag{4.19a}$$

$$K(x, y) = -\left\{\begin{pmatrix} I_m \\ 0_{n\times m} \end{pmatrix} \breve{C}e^{-x\breve{A}} + \overline{L}(x)\right\} e^{-y\breve{A}}\breve{B}, \tag{4.19b}$$

where

$$L(x) = \int_x^\infty dz\, K(x, z)Ce^{-zA}, \qquad \overline{L}(x) = \int_x^\infty dz\, \overline{K}(x, z)\breve{C}e^{-z\breve{A}}.$$

Postmultiplying (4.19a) by Ce^{-yA} and (4.19b) by $\breve{C}e^{-y\breve{A}}$ and integrating with respect to $y \in (x, +\infty)$, we obtain the linear systems of equations

$$
\begin{aligned}
\overline{L}(x) &= -\begin{pmatrix} 0_{m\times n} \\ I_n \end{pmatrix} Ce^{-xA}P(x) - L(x)P(x), \\
L(x) &= -\begin{pmatrix} I_m \\ 0_{n\times m} \end{pmatrix} \breve{C}e^{-x\breve{A}}\breve{P}(x) - \overline{L}(x)\breve{P}(x),
\end{aligned}
$$

where

$$
\begin{aligned}
P(x) &= \int_x^\infty dy\, e^{-yA}B\breve{C}e^{-y\breve{A}} = e^{-xA}P(0)e^{-x\breve{A}}, \\
\breve{P}(x) &= \int_x^\infty dy\, e^{-x\breve{A}}\breve{B}Ce^{-yA} = e^{-x\breve{A}}\breve{P}(0)e^{-xA}.
\end{aligned}
$$

These two equations can be written in the following form:

$$
\begin{pmatrix} L(x) & \overline{L}(x) \end{pmatrix} \begin{pmatrix} I_{\breve{p}} & P(x) \\ \breve{P}(x) & I_p \end{pmatrix} = -\begin{pmatrix} \breve{C}e^{-x\breve{A}}\breve{P}(x) & 0_{m\times p} \\ 0_{n\times\breve{p}} & Ce^{-xA}P(x) \end{pmatrix}.
$$

If $I_p - P(x)\breve{P}(x)$ [or, alternatively, $I_{\breve{p}} - \breve{P}(x)P(x)$] is invertible,[1] then

$$
\begin{aligned}
\begin{pmatrix} L(x) & \overline{L}(x) \end{pmatrix} &= -\begin{pmatrix} \breve{C}e^{-x\breve{A}}\breve{P}(x) & 0_{m\times p} \\ 0_{n\times\breve{p}} & Ce^{-xA}P(x) \end{pmatrix} \times \\
&\times \begin{pmatrix} (I_p - P(x)\breve{P}(x))^{-1} & -P(x)(I_{\breve{p}} - \breve{P}(x)P(x))^{-1} \\ -\breve{P}(x)(I_p - P(x)\breve{P}(x))^{-1} & (I_{\breve{p}} - \breve{P}(x)P(x))^{-1} \end{pmatrix}.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
K^{\mathrm{up}}(x,y) &= -\breve{C}e^{-x\breve{A}}(I_{\breve{p}} - \breve{P}(x)P(x))^{-1}e^{-y\breve{A}}\breve{B}, \\
K^{\mathrm{dn}}(x,y) &= Ce^{-xA}P(x)(I_{\breve{p}} - \breve{P}(x)P(x))^{-1}e^{-y\breve{A}}\breve{B}, \\
\overline{K}^{\mathrm{up}}(x,y) &= \breve{C}e^{-x\breve{A}}\breve{P}(x)(I_p - P(x)\breve{P}(x))^{-1}e^{-yA}B, \\
\overline{K}^{\mathrm{dn}}(x,y) &= -Ce^{-xA}(I_p - P(x)\breve{P}(x))^{-1}e^{-yA}B.
\end{aligned}
$$

Consequently [cf. (2.18a)-(2.18b)],

$$
\begin{aligned}
q(x) &= 2\breve{C}e^{-x\breve{A}}(I_{\breve{p}} - \breve{P}(x)P(x))^{-1}e^{-x\breve{A}}\breve{B}, \\
r(x) &= -2Ce^{-xA}(I_p - P(x)\breve{P}(x))^{-1}e^{-xA}B.
\end{aligned}
$$

Next, we derive matrix NLS solutions. To do so, we rely on the propagation equations (2.56a) and (2.56b) for the Marchenko kernels

$$
\Omega_l(x+y;t) = Ce^{-(x+y)A}e^{-4itA^2}B, \qquad \breve{\Omega}_l(x+y;t) = \breve{C}e^{-(x+y)\breve{A}}e^{4it\breve{A}^2}\breve{B}.
$$

[1] See the first footnote in Chapter 3.

Then the preceding evaluation of the potentials can be repeated, provided we modify the matrix triplets (A,B,C) and $(\breve{A},\breve{B},\breve{C})$ as follows:

$$(A,B,C)\mapsto(A,B,Ce^{-4itA^2}),$$
$$(\breve{A},\breve{B},\breve{C})\mapsto(\breve{A},e^{4it\breve{A}^2}\breve{B},\breve{C}).$$

In that case we replace $P(x)$ and $\breve{P}(x)$ as follows:

$$P(x)\mapsto P(x),\qquad \breve{P}(x)\mapsto\breve{P}(x;t)\stackrel{\text{def}}{=}e^{4it\breve{A}^2}\breve{P}(x)e^{-4itA^2}.$$

Thus we obtain the explicit matrix NLS solutions

$$q(x,t)=2\breve{C}e^{-x\breve{A}}(I_{\breve{p}}-\breve{P}(x;t)P(x))^{-1}e^{-x\breve{A}}e^{4it\breve{A}^2}\breve{B},$$
$$r(x,t)=-2Ce^{-4itA^2}e^{-xA}(I_p-P(x)\breve{P}(x;t))^{-1}e^{-xA}B.$$

In the *focusing* case, we have the symmetry relation

$$\breve{\Omega}_l(x+y)=-\Omega_l(x+y)^\dagger.$$

We thus relate the two matrix triplets as follows:

$$\breve{A}=A^\dagger,\qquad \breve{B}=-C^\dagger,\qquad \breve{C}=B^\dagger.$$

Then

$$P(x)=\int_x^\infty dy\,e^{-yA}BB^\dagger e^{-yA^\dagger}=e^{-xA}Ne^{-xA^\dagger},$$
$$\breve{P}(x;t)=-\int_x^\infty dy\,e^{-yA^\dagger}e^{4itA^{\dagger 2}}C^\dagger Ce^{-4itA^2}e^{-yA}$$
$$=-e^{-xA^\dagger}e^{4itA^{\dagger 2}}Qe^{-4itA^2}e^{-xA},$$

where

$$Q=\int_0^\infty dy\,e^{-yA^\dagger}C^\dagger Ce^{-yA},\qquad N=\int_0^\infty dy\,e^{-yA}BB^\dagger e^{-yA^\dagger}.$$

Writing

$$\Gamma(x;t)=I_p-P(x)\breve{P}(x;t)=I_p+e^{-xA}Ne^{-2xA^\dagger}e^{4itA^{\dagger 2}}Qe^{-4itA^2}e^{-xA},\tag{4.20}$$

we obtain

$$q(x,t)=-2B^\dagger e^{-xA^\dagger}\Gamma(x;t)^{\dagger^{-1}}e^{-xA^\dagger}e^{4itA^{\dagger 2}}C^\dagger,\tag{4.21}$$

where the inverse matrix exists for each $(x,t)\in\mathbb{R}^2$ [cf. Proposition 4.6].

For $A=(a)$ with $p=\operatorname{Re}a>0$ and $q=\operatorname{Im}a$, $B=(1)$, and $C=(c)$ with $0\neq c\in\mathbb{C}$, we obtain in the focusing case the one-soliton solution

$$q(x,t)=-\frac{2c^*e^{4it(a^*)^2}}{1+\dfrac{|c|^2e^{16pqt}}{4p^2}e^{-4px}}=\frac{c^*e^{2p(x-x_0)}e^{4it(p^2-q^2)}}{\cosh(2p[x-x_0-4qt])},$$

where $x_0 = \frac{1}{2p}\ln(|c|/2p)$. The so-called N-soliton solutions derived before [101] arise if A is a diagonal matrix having distinct diagonal elements with positive real parts.

In the case of the matrix mKdV equation, the same solution formulas hold with the following two modifications: (a) the triplets (A, B, C) consist of real matrices, and (b) the time factor e^{-4itA^2} is to be replaced by e^{8tA^3} [cf. (2.58a)].

4.3 Closed form solutions: Sine-Gordon

In this section we derive the closed form solutions of the sine-Gordon equation for which the corresponding reflection coefficients vanish. We first solve the inverse scattering problem by the Marchenko method and then insert the time factors.

The sine-Gordon equation (1.14) is a focusing real-valued problem in the sense that the solution $u(x,t)$ is real. Further, in the accompanying linear eigenvalue problem the focusing Zakharov-Shabat solution $q(x,t)$ is related to the sine-Gordon solution $u(x,t)$ as follows:

$$q(x,t) = \frac{1}{2}\frac{\partial u}{\partial x}(x,t).$$

In this case we solve the Marchenko equations with Marchenko kernel [cf. (2.58c)]

$$\Omega_l(x) = -\breve{\Omega}_l(x)^\dagger = Ce^{-xA}e^{-\frac{1}{2}tA^{-1}}B,$$

where (A, B, C) is a triplet of real matrices A, B, and C with a matrix A having only eigenvalues with positive real parts. Then in terms of the triplet (A, B, C) the solution of the sine-Gordon equation (1.14) satisfies

$$\frac{1}{2}u_x(x,t) = -2B^\dagger e^{-xA^\dagger}\Gamma(x,t)^{\dagger^{-1}}e^{-xA^\dagger}e^{-\frac{1}{2}tA^{-1}}C^\dagger,$$

where the matrix

$$\Gamma(x;t) = I_p + e^{-xA}Ne^{-2xA^\dagger}e^{-\frac{1}{2}tA^{-1}}Qe^{-\frac{1}{2}tA^{-1}}e^{-xA}$$

is invertible for each $(x,t) \in \mathbb{R}^2$ and the time factor $e^{4itA^{\dagger 2}}$ appearing in (4.21) has been replaced by $e^{-\frac{1}{2}tA^{-1}}$. Since the solution of the sine-Gordon equation $u(x,t)$ satisfies $u(+\infty,t) - u(-\infty,t) = 2\pi N$ for some integer N [cf. (1.14)], we can normalize our sine-Gordon solutions by the condition that $u(+\infty,t) = 0$. In that case we obtain

$$u(x,t) = 4\int_x^\infty dy\, B^\dagger e^{-yA^\dagger}\Gamma(y,t)^{\dagger^{-1}}e^{-yA^\dagger}e^{-\frac{1}{2}tA^{-1}}C^\dagger.$$

Using that the scalar Marchenko kernel $\Omega_l(x,t)$ is real, we have

$$\Omega_l(x,t)=\Omega_l(x,t)^\dagger=Ce^{-yA}e^{-\frac{1}{2}tA^{-1}}B.$$

Using this fact in the up components of the coupled Marchenko equations (2.50), i.e., in the equations

$$\begin{aligned}
&\overline{K}^{\mathrm{up}}(x,y;t)+\int_x^\infty dz\,K^{\mathrm{up}}(x,z;t)\Omega_l(z+y;t)=0,\\
&K^{\mathrm{up}}(x,y;t)-\Omega_l(x+y;t)-\int_x^\infty dz\,\overline{K}^{\mathrm{up}}(x,z;t)\Omega_l(z+y;t)=0,
\end{aligned}$$

we obtain the integral equation

$$\begin{aligned}
K^{\mathrm{up}}(x,y;t)&=\Omega_l(x+y;t)\\
&\quad-\int_x^\infty dv\int_x^\infty dw\,K^{\mathrm{up}}(x,v;t)\Omega_l(v+w;t)\Omega_l(w+y;t)=0.
\end{aligned}$$

Substituting the Marchenko kernel and solving this integral equation, we obtain

$$K^{\mathrm{up}}(x,y;t)=\left[Ce^{-xA}-L(x)e^{-xA}e^{-\frac{1}{2}tA^{-1}}Pe^{-xA}\right]e^{-\frac{1}{2}tA^{-1}}e^{-yA}B,\quad (4.22)$$

where

$$L(x)=\int_x^\infty dv\,K^{\mathrm{up}}(x,v;t)Ce^{-vA}$$

and

$$P=\int_0^\infty ds\,e^{-sA}BCe^{-sA}$$

is the unique solution to the Sylvester equation $AP+PA=BC$. Here we observe that this equation is uniquely solvable, because A and $-A$ do not have eigenvalues in common [cf. Theorem 4.1]. Postmultiplying (4.22) by Ce^{-yA} and integrating with respect to $y\in(x,+\infty)$, we obtain

$$L(x)=Ce^{-2xA}e^{-\frac{1}{2}tA^{-1}}Pe^{-xA}\left[I_p+e^{-xA}e^{-\frac{1}{2}tA^{-1}}Pe^{-2xA}e^{-\frac{1}{2}tA^{-1}}Pe^{-xA}\right]^{-1},$$

where the existence of the matrix inverse for each $(x,t)\in\mathbb{R}^2$ follows from the unique solvability of the Marchenko equations (2.50). As a result,

$$\begin{aligned}
K^{\mathrm{up}}(x,y;t)&=Ce^{-xA}\left[I_p+\left(e^{-xA}e^{-\frac{1}{2}tA^{-1}}Pe^{-xA}\right)^2\right]^{-1}e^{-\frac{1}{2}tA^{-1}}e^{-yA}B\\
&=Ce^{-xA}e^{-\frac{1}{4}tA^{-1}}\left[I_p+M(x,t)^2\right]^{-1}e^{-\frac{1}{4}tA^{-1}}e^{-yA}B,
\end{aligned}$$

where

$$M(x,t) = e^{-\frac{1}{4}tA^{-1}} e^{-xA} P e^{-xA} e^{-\frac{1}{4}tA^{-1}}.$$

Consequently,

$$u(x,t) = -4 \int_x^\infty dy \, C e^{-yA} e^{-\frac{1}{4}tA^{-1}} \left[I_p + M(y,t)^2\right]^{-1} e^{-\frac{1}{4}tA^{-1}} e^{-yA} B. \tag{4.23}$$

We now apply the scalarity of $u(x,t)$ and identities involving traces and determinants to simplify the solution formula (4.23). The invertibility of $I_p + M(x,t)^2$ implies the invertibility of $I_p \pm iM(x,t)$. It is then easy to prove that

$$u(x,t) = 2i \log\left(\frac{\det(I_p + iM(x,t))}{\det(I_p - iM(x,t))}\right) \tag{4.24a}$$

$$= 4 \arctan\left(i \frac{\det(I_p + iM(x,t)) - \det(I_p - iM(x,t))}{\det(I_p + iM(x,t)) + \det(I_p - iM(x,t))}\right), \tag{4.24b}$$

where the arctangent is defined as the inverse function of the tangent in such a way that $u(x,t)$ is continuous in $(x,t) \in \mathbb{R}^2$. Moreover, the logarithm is defined in such a way that $\log(1) = 0$. Indeed, the equivalence of (4.24a) and (4.24b) we employ the scalar identities

$$\arctan(z) = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right), \qquad \log(z) = 2i \arctan\left(\frac{i(1-z)}{1+z}\right),$$

where the branch cut of the arctangent is $(-\infty, -i] \cup [i, +i\infty)$. Using that matrix traces satisfy $\mathrm{Tr}(TS) = \mathrm{Tr}(ST)$, we obtain from (4.23)

$$\begin{aligned}
u(x,t) &= -4 \int_x^\infty dy \, \mathrm{Tr}\left\{C e^{-yA} e^{-\frac{1}{4}tA^{-1}} \left[I_p + M(y,t)^2\right]^{-1} e^{-\frac{1}{4}tA^{-1}} e^{-yA} B\right\} \\
&= -4 \int_x^\infty dy \, \mathrm{Tr}\left\{e^{-\frac{1}{4}tA^{-1}} e^{-yA} B C e^{-yA} e^{-\frac{1}{4}tA^{-1}} \left[I_p + M(y,t)^2\right]^{-1}\right\} \\
&= 4\mathrm{Tr}\left\{\int_x^\infty dy \left(\frac{\partial}{\partial y} M(y,t)\right) \left[I_p + M(y,t)^2\right]^{-1}\right\} \\
&= -4\mathrm{Tr}\arctan[M(x,t)].
\end{aligned}$$

Using that for any invertible matrix Z the identity $\mathrm{Tr}[\log(Z)] = \log\det(Z)$ holds, we obtain (4.24).

Choosing the real triplet (A,B,C) in a special way, the special sine-Gordon solutions appearing in [73] can be reproduced. The kink and antikink solutions arise for $A = (a)$ with $a > 0$, $B = (1)$, and $C = (c)$ with $c > 0$ and $c < 0$, respectively. In this case

$$P = \left(\frac{c}{2a}\right), \qquad M(x,t) = \left(\frac{c}{2a} e^{-2ax - t/(2a)}\right),$$

so that

$$u(x,t) = -4\arctan\left(\frac{c}{2a}e^{-2ax-t/(2a)}\right).$$

When $A = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ with $a > 0$ and $0 \neq b \in \mathbb{R}$, $B = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, and $C = \left(\begin{smallmatrix} c_2 & c_1 \end{smallmatrix}\right)$ with $c_1 \in \mathbb{R}$ and $0 \neq c_2 \in \mathbb{R}$, we obtain the breather solution

$$u(x,t) = -4\arctan\left[\frac{8a^2e^{a\zeta_+}[(ac_1 - bc_2)\cos(b\zeta_-) - (bc_1 + ac_2)\sin(b\zeta_-)]}{b^2(c_1^2 + c_2^2) + 16a^2(a^2 + b^2)e^{2a\zeta_+}}\right],$$

where $\zeta_\pm = 2x \pm [1/(2(a^2+b^2))]$. These solutions have been derived before by representing the solution as $u(x,t) = 4\arctan(U/V)$ for suitable functions $U(ax + a^{-1}t)$ and $V(ax + a^{-1}t)$ [102, 81, 85].

4.4 Closed form solutions: Toda lattice

In this section we derive the closed form solutions of the Toda lattice equation for which the corresponding reflection coefficients vanish. We first solve the inverse scattering problem by the Marchenko method and then stick in the time factors.

The discrete eigenvalues of the Flaschka-Toda system are those of the linear operator L: They are real, belong to $\mathbb{R} \setminus [-1,1]$, and are algebraically and geometrically simple. In the reflectionless case [i.e., if $\hat{\boldsymbol{\rho}}(n+m) \equiv 0$], we can write

$$\boldsymbol{F}_r(n+m;t) = CA^{n+m}e^{(A-A^{-1})t}B,$$

where C is $1 \times p$ and real, $B = C^\dagger$, A is a real symmetric $p \times p$ matrix with only simple eigenvalues and with a spectral radius of less than one. Then the unique solution to the Marchenko equation (3.19a) is given by

$$\kappa(n,m;t) = -CA^n\left[I_p + e^{t(A^{-1}-A)}A^{n+1}PA^{n+1}\right]^{-1}e^{t(A^{-1}-A)}A^mB,$$

where

$$P = \sum_{j=0}^{\infty} A^jBCA^j = \sum_{j=0}^{\infty} (CA^j)^\dagger CA^j$$

is the unique solution of the Stein equation

$$P - APA = BC. \tag{4.25}$$

We may, for instance, choose $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_p)$ for distinct $\alpha_1, \ldots, \alpha_p \in (-1,1)$ and $C = \begin{pmatrix} c_1 & \ldots & c_p \end{pmatrix} = B^\dagger$ for nonzero real $c_1, \ldots, c_p$, so that the solution P to (4.25) satisfies

$$P_{ij} = \frac{c_i^* c_j}{1 - \alpha_i\alpha_j}$$

for $i, j = 1, \ldots, p$.

The solution to the Toda lattice equation (1.19) is given by

$$q_n(t) = \det[\Gamma_{n+1}(t)\Gamma_{n+2}(t)^{-1}] = \frac{\det \Gamma_{n+1}(t)}{\det \Gamma_{n+2}(t)},$$

where

$$\Gamma_n(t) = I_p + e^{t(A^{-1}-A)} A^n P A^n.$$

Indeed, using (3.21a) and (4.25) and putting $E = e^{(A-A^{-1})t}$ we get

$$\begin{aligned}
\frac{1}{K(n,n)^2} &= 1 + \boldsymbol{F}_r(2n) + \sum_{j=n+1}^{\infty} \boldsymbol{F}_r(n+j)\kappa(n,j) \\
&= 1 + CA^{2n}EB - \sum_{j=n+1}^{\infty} CA^n A^j EBCA^n \Gamma_{n+1}^{-1} EA^j B \\
&= 1 + CA^{2n}EB - \sum_{j=n+1}^{\infty} CA^n \Gamma_{n+1}^{-1} EA^j BCA^n A^j EB \\
&= 1 + CA^{2n}EB - CA^n \Gamma_{n+1}^{-1} EA^{n+1} PA^{2n+1} EB \\
&= 1 + CA^{2n}EB - CA^n \Gamma_{n+1}^{-1} \left[\Gamma_{n+1} - I_p\right] A^n EB \\
&= 1 + CA^n \Gamma_{n+1}^{-1} A^n EB = \det(1 + CA^n \Gamma_{n+1}^{-1} A^n EB) \\
&= \det(I_p + A^n EBCA^n \Gamma_{n+1}^{-1}) \\
&= \det(I_p + A^n E[P - APA] A^n \Gamma_{n+1}^{-1}) \\
&= \det(I_p + [EA^n PA^n - EA^{n+1} PA^{n+1}]\Gamma_{n+1}^{-1}) \\
&= \det(I_p + [\Gamma_n - \Gamma_{n+1}]\Gamma_{n+1}^{-1}) = \det(\Gamma_n \Gamma_{n+1}^{-1}).
\end{aligned}$$

Therefore,

$$e^{q_{n-1}-q_n} = 4a_n^2 = \frac{K(n+1,n+1)^2}{K(n,n)^2}$$

and $L_n = \log\det\Gamma_n$ lead to the following:

$$\begin{aligned}
q_n - q_{n+k} &= \sum_{j=0}^{k-1} (q_{n+j} - q_{n+j+1}) = \sum_{j=0}^{k-1} (L_{n+j+1} - 2L_{n+j+2} + L_{n+j+3}) \\
&= L_{n+1} - L_{n+2} + L_{n+k+2} - L_{n+k+1},
\end{aligned}$$

while $L_m = \log\det\Gamma_m$ vanishes as $m \to +\infty$. Since $q_m \to 0$ as $m \to +\infty$, we obtain

$$q_n(t) = \log\det(\Gamma_{n+1}(t)\Gamma_{n+2}(t)^{-1}).$$

Using diagonal matrices A with simple real eigenvalues belonging to $(-1,1)$, we reporoduce the N-soliton solutions known in the literature [57].

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