SIMAI
e-Lecture Notes
ISSN 1970-4429

THERMODYNAMICS OF FLUIDS

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DOI: 10.1685/SELN09002 - Vol. 2 - 2009

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Abstract. The main steps used in deriving the mathematical models governing the fluid flows are shown. The fluids and their motions presented are more complicate than those studied by fluid dynamics. However, the relative position of fluid dynamics vs. thermodynamics of fluids is analyzed. The mathematical technicalities are avoided but the rigor was kept by the attentive explanation of the mathematical meaning involved in the concepts and results. The mathematical tools used are described separately from the physical hypotheses. Nevertheless, the mathematical content of basic physical notions is carefully revealed.

1. Thermodynamics of fluids as a physical domain

In this domain the concept of fluid goes far beyond the common sense of the word. Thus, among the fluids in the sense of thermodynamics we quote: ordinary fluids (water, air), oil, mud, blood, bitumen, maionesis, flour, wet argil, suspensions, sets of grains, sets of stars, cars in a traffic, people on the crowded street, honey.

There are **materials with partial fluid properties**, e.g. elastoviscoplastic materials.

In various circumstances, the same piece of matter can be one type of material or other. For instance, the air at velocity smaller than 1/3 of the sound velocity is incompressible. Beyond this limit it is compressible. Water at various temperatures has different material properties: solid, viscoplastic material, fluid, vapors, rarefied gas.

While (classical) fluid dynamics is still entitled to deal with the motion of ordinary fluids in usual conditions, the study of nonstandard fluids and of materials with partial fluid properties moving in nonstandard conditions is to be the concern of thermodynamics of fluids. In both these domains large classes of fluid flows are studied. Here are the most important classes of fluids with the corresponding models.

Large classes of fluids	**Corresponding models**
ideal (i.e. incompressible and inviscid)	incompressible Euler
incompressible and viscous	incompressible Navier-Stokes
compressible and inviscid	compressible Euler
compressible and viscous plus temperature influence	Navier-Stokes-Fourier

All the four models are studied by fluid dynamics but the last two have no theoretical basis in the fluid dynamics framework.

In fluid dynamics the derivation of the mathematical models governing the fluid flows is brief. In exchange, the methods to solve the involved equations and a lot of particular solutions are extensively presented. On the contrary, in thermodynamics of fluids the attention is focused on the derivation of models. This is natural because there are only a few situations when those equations can be solved exactly.

By a mathematical model we understand a set of equations the solutions of which are subject to some constraints (e.g. boundary conditions). Further we show the important facts to take into account when constructing such a model.

The **definition of materials** must take into account characteristics of dynamics, kinematics and conditions in which the phenomena in them occur (e.g. temperature field).

Flows (motions) can be found in three **regimes**: laminar, transitional (last stage of which is the deterministic chaos), turbulent.

Applications of fluid dynamics can be encountered in: aviation, naval industry, meteorology, hydrology, astrophysics, car industry, chemical industry, food industry, to name only the most important.

Fluid mechanics belongs to physics because its laws are physical.

Every physical science has two related domains:

- statistical physics, based on discrete models (e.g. kinetic theory of gases)
- thermodynamics, based on continuous models (e.g. thermodynamics of fluids).

Fluid dynamics is studied by physicists, mathematicians and engineers from various points of view. We related them by means of mathematical models for fluid, motion and for various quantities used in physical experiments.

Mathematical studies of fluids flows are: quantitative, namely theoretical, numerical (CFD = Computational Fluid Dynamics) and qualitative, e.g. stability, bifurcation, asymptotic behavior.

Numerical studies used are of various level of abstraction: numerical analysis (belongs to nonlinear functional analysis), numerical codes (used in important applications), experimental (simulation) computation (which can lead to great discoveries, e.g. fractals, but which, in general, have no mathematical relevance).

In thermodynamics the geometric model of the material is a continuum.

Mathematical continuum is a closed connected set with respect to a given topology. It is called a **body**.

Physical continua are mathematical continua endowed with specific quantities expressed by sufficiently smooth functions, up to sets of null Lebesgue measure: **mechanical** ($\boldsymbol{v}, p, \rho$), where p has not the physical significance); **thermal** (T, ρ, p), **thermomechanical** ($\boldsymbol{v}$, p, ρ, T), **chemical** ($\boldsymbol{v}, p, \rho, C$), where $\boldsymbol{v}, p, \rho, T, C$ are the velocity, pressure, density, temperature and concentration respectively.

Mathematical models are so numerous in fluid mechanics as well as in thermodynamics of fluids, such that the courses in differential equations and operator theories can be presented by using them. They are of various types and have various domains of applicability. We quote: **fundamental** (Navier-Stokes, Navier-Stokes-Fourier, Euler, Prandtl, Stokes, Oseen), **particular** (1D, 2D, some physical characteristics are absent, e.g. temperature), **of**

approximation, e.g. of asymptotic approximation. For instance: Euler and Prandtl models approximate Navier-Stokes model as $\mathrm{Re} \to \infty$ $(\upsilon \to 0)$ far from or near the walls respectively; Stokes and Oseen models approximate Navier-Stokes model as $\mathrm{Re} \to 0$ ($\upsilon \to \infty$) near or far from the wall respectively. Here $\mathrm{Re} = UV / \upsilon =$ Reynolds number, U and L are the characteristic velocity and length respectively and υ is the coefficient of kinematic viscosity. L, U large and/or υ small implies $\mathrm{Re} \to \infty$ (fast flows and/or fluids with small viscosity); L, U small and/or υ large implies $\mathrm{Re} \to 0$ (creeping (slow) flows and/or very viscous fluids).

Nonlinearity in models is of three kinds: geometrical (small displacements); physical (of constitutive relations), mathematical (in equations, e.g. nonlinear advective terms $u \cdot \nabla u$, $u \cdot \nabla T$; sometimes they are called the convective or transport terms).

Geometric and physical characterization of a material point are given by: $\boldsymbol{x}$ (position), ρ (inertia characteristic); $\boldsymbol{p}$ (spin in materials with microstructure, e.g. Cosserat materials, as dipols)); supplementary microscopic characteristics are used for *micropolar fluids*.

Fluid particle is a volume V_c (limit of continuum) such that for smaller volumes the averaged fluid flow characteristics are random, while for V larger than V_c they are smooth almost everywhere.

The models which have to take into account this huge variety of requirements have a corresponding degree of generality. Thus mathematical models result from: physical laws; constitutive relations and state equations; initial and boundary conditions; other constraints.

2. Balance equations

2.1 Three geometrical descriptions of position (and the body configurations)

Here we introduce the functions of time and space, referred to as fields, and present some of their descriptions, derivatives and types.

Field is a function $f(t, \boldsymbol{x})$. It may be scalar, vectorial, tensorial.

There are the three descriptions

Eulerian description of fields: $f(t, \boldsymbol{x})$, where $\boldsymbol{x}$ is a **geometric** point, t is the actual, i.e. present, time.

Lagrangean (or **material**) description follows the material point which at $t = 0$, in a reference frame S had the position $\boldsymbol{x}_0$, and it is $\tilde{f}_S(t, \boldsymbol{x_0})$. Let $\boldsymbol{X} = \boldsymbol{x}(t, \boldsymbol{x}_0)$ be the position at time t of the point which at $t = 0$ had the position $\boldsymbol{x}_0$. By definition the **velocity** field is

$\boldsymbol{u} = \frac{\partial \boldsymbol{X}}{\partial t}$ and it can be described in Eularian (E) or Lagrangian (L) coordinates. By definition the **acceleration** is $\boldsymbol{a} = \frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial^2 \boldsymbol{X}}{\partial t^2}$; this means that $\boldsymbol{u}$ is in Lagrangian coordinates. Being a field, $\boldsymbol{a}$ can be expressed in E or L coordinates. Usually S is omitted.

Total derivative of $f(t,\boldsymbol{x})$ is $\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial t} = \frac{\partial f}{\partial t} + \boldsymbol{u} \cdot grad\, f$.

The field f is expressed in E coordinates, otherwise we cannot apply to it the operator $grad = \frac{\partial}{\partial x_1}\boldsymbol{i} + \frac{\partial}{\partial x_2}\boldsymbol{j} + \frac{\partial}{\partial x_3}\boldsymbol{k} \equiv \nabla$; $\frac{\partial f}{\partial t}$ = local derivative of $f(t,\boldsymbol{x})$, $\frac{df}{dt}$ = material or convective derivative, or the derivative along trajectories.

The field $\boldsymbol{X} = \boldsymbol{x}(t,\boldsymbol{x}_0)$ is referred to as a motion of $\boldsymbol{x}_0$ (fixed, chosen) while the image in $\mathbf{R}^3$ of this function of t is the **trajectory of** $\boldsymbol{x}_0$. The velocity field can be also denoted by $\boldsymbol{v}$. The total derivative expresses the fact that f is convected by the material continuum in its motion with the velocity $\boldsymbol{u}$ along the trajectories. We also say that the continuum in its motion transports f with velocity $\boldsymbol{u}$. The E description is appropriate to experiments “in situ”. In E description, $\boldsymbol{a} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$.

Relative description: $f(\tau, \boldsymbol{X}(t,\boldsymbol{x_0}))$ is used to study materials with memory and nonlocal deformations.

We mention the most important types of fields.

Stationary (steady) field $f(\boldsymbol{x})$, i.e. it does not depend explicitely on time, or equivalently, is the same at every time.

***n*-dimensional** field: $\boldsymbol{f} = (f_1,...,f_n)$. **Homogenous** field: $f(t)$. **Uniform** field = constant field.

2.2 Transport theorems

They are identities expressing the time derivatives of integrals which are defined over moving domains. They generalize the following Leibnitz-Newton formula

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)dx = \frac{db}{dt} f(t,b(t)) - \frac{da}{dt} f(t,a(t)) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(t,x)dx,$$

which corresponds to a moving segment [$a(t)$, $b(t)$]. In the case of moving curves $C(t)$, surfaces $S(t)$ and domains $\Omega(t)$, for scalar, vector and tensor fields, they read

$$\frac{d}{dt}\int_{C(t)} f(t,x_1,x_2,x_3)dx_k = \int_{C(t)} \frac{\partial f}{\partial t}dx_k + f\frac{\partial u_k}{\partial x^l}dx^l,$$

$$\frac{d}{dt}\int_{C(t)} \boldsymbol{f}\cdot\boldsymbol{\tau}ds = \int_{C(t)}\left[\frac{\partial \boldsymbol{f}}{\partial t} + \boldsymbol{f}\nabla\boldsymbol{u}\right]\cdot\boldsymbol{\tau}ds$$

$$\frac{d}{dt}\int_{C(t)} \boldsymbol{T}\boldsymbol{\tau}ds = \int_{C(t)}\left[\frac{\partial \boldsymbol{T}}{\partial t} + \boldsymbol{T}\nabla\boldsymbol{u}\right]\boldsymbol{\tau}ds,$$

$$\frac{d}{dt}\int_{S(t)} f(t,\boldsymbol{x})n_k dS = \int_{S(t)}\left[\frac{\partial f}{\partial t}n_k + f\left(-\frac{\partial u_i}{\partial x_k}n_i + \frac{\partial u_i}{\partial x_i}n_k\right)\right]dS,$$

$$\frac{d}{dt}\int_{S(t)} \boldsymbol{f}\cdot\boldsymbol{n}dS = \int_{S(t)}\left[\frac{\partial \boldsymbol{f}}{\partial t} + rot(\boldsymbol{f}\times\boldsymbol{u}) + \boldsymbol{u}div\boldsymbol{f}\right]\cdot\boldsymbol{n}dS,$$

$$\frac{d}{dt}\int_{S(t)} \boldsymbol{T}\boldsymbol{n}dS = \int_{S(t)}\left\{\frac{\partial \boldsymbol{T}}{\partial t} + \left[\boldsymbol{T}((div\boldsymbol{u})\boldsymbol{I} - \nabla\boldsymbol{u}^T)\right]\right\}\boldsymbol{n}dS,$$

$$\frac{d}{dt}\int_{\Omega(t)} f(t,\boldsymbol{x})d\Omega = \int_{\Omega(t)}\left[\frac{\partial f}{\partial t} + div(f\,\boldsymbol{u})\right]d\Omega,$$

(for $f \equiv 1$ it follows the relation $\dfrac{d\,mes\,\Omega(t)}{dt} = \displaystyle\int_{\Omega(t)} div\,\boldsymbol{u}d\Omega$),

$$\frac{d}{dt}\int_{\Omega(t)} \boldsymbol{f}(t,\boldsymbol{x})d\Omega = \int_{\Omega(t)}\left[\frac{\partial \boldsymbol{f}}{\partial t} + div(\boldsymbol{f}\otimes\boldsymbol{u})\right]d\Omega,$$

$$\frac{d}{dt}\int_{\Omega(t)} \boldsymbol{T}(t,\boldsymbol{x})d\Omega = \int_{\Omega(t)}\left[\frac{\partial \boldsymbol{T}}{\partial t} + div(\boldsymbol{T}\otimes\boldsymbol{u})\right]d\Omega,$$

where $\boldsymbol{u}$ is the velocity field; $dx_k = \tau_k ds$; $\boldsymbol{\tau}$ is the unit vector tangent to $C(t)$; s is the arc along $C(t)$; and $\boldsymbol{n}$ is the unit vector normal to $S(t)$.

Proof of Leibnitz-Newton formula

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,x)dx = \frac{d}{dt}\int_{a_0}^{b_0}\tilde{f}(t,\boldsymbol{x_0})Jdx_0 = \int_{a_0}^{b_0}\frac{d}{dt}\left(\tilde{f}\,J\right)dx_0 = \int_{a_0}^{b_0}\left(J\frac{d\tilde{f}}{dt} + \tilde{f}\frac{dJ}{dt}\right)dx_0 = \int_a^b\frac{df}{dt}dx + \int_{a_0}^{b_0}\tilde{f}\frac{du}{dx_0}dx_0 =$$
$$\int_a^b\frac{df}{dt}dx + [f(t,x)\frac{dx}{dt}]_{a(t)}^{b(t)}, \text{ where } Jdiv\,\boldsymbol{u} = \frac{dJ}{dt},\ J_{ij} = \frac{\partial x_i}{\partial x_{0j}}, J = \det\boldsymbol{J}.$$

Alternative proof

$$\int_{a_0}^{b_0} \frac{d}{dt}\left[f(t,x(t,x_0))\frac{dx}{dx_0}\right]dx_0 = \int_{a_0}^{b_0}\left[\left(\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}\right)\frac{\partial x}{\partial x_0}+f\frac{\partial}{\partial t}\frac{\partial x}{\partial x_0}\right]dx_0 = \int_a^b \frac{df}{dt}dx + \int_{a_0}^{b_0}\left(\frac{\partial f}{\partial x_0}\frac{dx}{dt}+f\frac{\partial}{\partial x_0}\frac{\partial x}{\partial t}\right)dx_0 =$$

$$\int_a^b \frac{\partial f}{\partial t}dx + \int_{a_0}^{b_0}\frac{\partial}{\partial x_0}(f\,u)dx_0 = \int_a^b \frac{df}{dt}dx + \left[f(t,x(t,x_0))\frac{\partial x(t,x_0)}{\partial t}\right]\Bigg|_{a_0}^{b_0}.$$

The proof of the other transport theorems use as basic fact the transformation of the integral over a moving domain into an integral over a fixed domain, the last integral commuting with the time derivative. Then formulae from tensor analysis are used. At the end the obtained integrals over fixed domains are transformed into integrals over moving domains.

2.3 Flux-divergence formula

We recall this well-know theorem

$$\int_{\Omega} div\,\boldsymbol{f}\,d\Omega = \int_{\partial\Omega} \boldsymbol{f}\cdot\boldsymbol{n}\,d\sigma;\quad \int_{\Omega}\frac{\partial f}{\partial x_i}d\Omega = \int_{\partial\Omega} f\,n_i\,d\sigma.$$

It is valid for sufficiently “good” $\partial\Omega$.

2.4 Continuity formula

In the simplest case this theorem reads

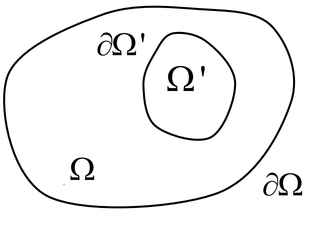
$$f:[a,b]\subseteq R\to R, f\in C[a,b], \int_{a'}^{b'} f(x)dx = 0, \forall[a',b']\subset[a,b]\Rightarrow f(x)\equiv 0\quad on\quad [a,b].$$

$$h:\Omega\to R, h\in C(\Omega), \int_{\Omega'} h(t,\mathrm{x})d\Omega' = 0, \forall\Omega'\subseteq\Omega\Rightarrow h\equiv 0\quad on\quad \Omega.$$

The physical laws connect quantities characterizing parts of a material and not a point in it. This means that in its first stage a mathematical model is not expressed in terms of fields but in terms of the so-called global quantities. The three theorems from Sections 2.2 – 2.4 permit just to pass from the global to the local form.

2.5 Local and global quantities

We introduce the local and global quantities and comment the relationships between them. **Local quantities** are fields, i.e. functions of t and $\boldsymbol{x}$, $\forall\boldsymbol{x}\in\Omega, \forall t\in[a,b]\subseteq R.$ **Global quantities** are functions of t and the arbitrary set $\Omega'\subset\Omega.$ Local and global quantities are alternative descriptions since Ω' can shrink around $\boldsymbol{x}$ as much



as possible. We define Ω = domain of motion, $\partial\Omega$ = frontier (or boundary) of Ω. Then we have $\Omega' \subseteq \Omega$, **control volume**, $\Omega' \in \wp(\Omega)$; $\Omega'(t) \subseteq \Omega$, **material volume,** locus of the same material points, $\Omega'(t) \in \wp(\Omega)$.
For the sake of brevity we write Ω' even for $\Omega'(t)$.

Relationship between global and local quantities reads:

$G(t,\Omega') = \int_{\Omega'} g(t,\boldsymbol{x})d\boldsymbol{x}, \forall \Omega' \subset \Omega$ (Ω' can be $\Omega'(t)$. In the case g=const, it follows $G = g\, mes\,\Omega' \to g = G / mes\,\Omega'$ =density. In general

$$g(t,\boldsymbol{x}) = \lim_{\substack{mes\,\Omega' \to 0 \\ \boldsymbol{x} \in \Omega'}} \frac{G(t,\Omega')}{mes\,\Omega'} = \frac{dG(t,\Omega')}{d(mes\,\Omega')}.$$

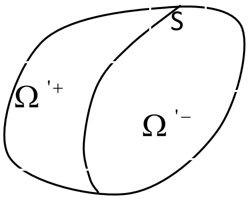
In the most general case when a global quantity G_1 has singularities along a surface *s* and a curve *C* moving independently of the material motion then the general form of G_1 is

$$G_1(t,\Omega',s,C) = \int_{\Omega'} g_d(t,\boldsymbol{x})d\Omega' + \int_{s(t)} g_s(t,\boldsymbol{x}^*)d\sigma' + \int_{C(t)} g_C(t,\boldsymbol{x}^{**})dl, \forall \Omega', s, C \in \Omega$$

If $C = \varnothing$, the transport formula reads

$$\frac{dG_1}{dt}(t,\Omega'(t),s(t)) = \int_{\Omega'(t)} \frac{\partial g_d}{\partial t}(t,\boldsymbol{x})d\Omega' + \int_{\partial\Omega'^{+} \cup \partial\Omega'^{-}} g_s(t,\boldsymbol{x}^*)\boldsymbol{u_1}(t,\boldsymbol{x}^*)\boldsymbol{n}'(t,\boldsymbol{x}^*)d\sigma' -$$

$$\int_{s(t)} [g_s(t,\boldsymbol{x}^*)]\boldsymbol{u_2}(t,\boldsymbol{x}^*)\boldsymbol{n}''(t,\boldsymbol{x}^*)ds,$$



where $[g] = g_+ - g_-$ is the jump across $s(t)$. $\boldsymbol{u}_1$, $\boldsymbol{u_2}$, n' and n'' are the velocity fields and the outwards normal vectors on *s* in the domains Ω'^{+} and Ω'^{-}, respectively [14].

Representation formula for G is: $G(t,\Omega') = \int_{\Omega'} \frac{dG}{d(mes\,\Omega')} d\Omega'$. It is similar to $u(x) = \int_0^x \frac{du}{dy}(y)dy$ for $u(0) = 0$, from the one-dimensional case.

The role of inertia throughout physics is materialized by the occurrence of ρ. Let $G = \int \rho g$. Then *g* is called the **specific** G and ρg the **density** of G. Here G is a scalar, a vector or a tensor.

Physically, we characterize the fluid motion by means of global quantities. In well-determined conditions, there exist some fields in terms of which the global quantities can be expressed. These conditions are given by

Radon-Nikodym (R-N) theorem. *If the global quantity* ***mass*** *is a countably additive, σ-finite and absolutely continuous function with respect to a measure μ, then there exists a* *unique field* ρ *such that* $m(t,\Omega') = \int_{\Omega'} \rho(t,\boldsymbol{x})d\mu, \forall\Omega' \in A,$ *where* A *is a* $\sigma-$*algebra on* $\boldsymbol{R}^3$ *and* μ *is the Lebesgue measure.*

If G is not continuous but it is absolutely continuous as required by the R-N theorem, then $G = G_1 + \sum_{\alpha} g_{\alpha}$, where g_{α} are functions defined at some singularities.

Other two global quantities are

Momentum: $\boldsymbol{M}(t,\Omega') = \int_{\Omega'} \rho(t,\boldsymbol{x})\boldsymbol{u}(t,\boldsymbol{x})d\Omega'$;

Stress: $\boldsymbol{f}(t,\partial\Omega') = \int_{\partial\Omega'} \boldsymbol{Tn}d\sigma'$ where $\boldsymbol{T}$ is the stress tensor.

This representation does not follow from R-N theorem but from the Cauchy hypotesis that $\boldsymbol{T}$ depends on a single geometric characteristic of the surface, namely the normal $\boldsymbol{n}$. The corresponding continuum is referred to as the **Cauchy medium**.

Other global quantities of interest in thermodynamics foundation are:

body force $\boldsymbol{f}(t,\Omega'(t)) = \int_{\Omega'(t)} \rho(t,\boldsymbol{x})\boldsymbol{F}(t,\boldsymbol{x})d\Omega'(t),$ where $\boldsymbol{F}$ is the effort = specific force;

internal energy $E\ (t,\Omega'(t)) = \int_{\Omega'(t)} \rho(t,\boldsymbol{x})\varepsilon(t,\boldsymbol{x})d\Omega'(t);$

kinetic energy $\int_{\Omega'(t)} \rho(t,\boldsymbol{x})\frac{\boldsymbol{u}^2}{2}(t,\boldsymbol{x})d\Omega'(t).$

2.6 Balance equations

These are the matematical expression of the most general balance law: *the rate of change of a global quantity* G *is equal to, i.e. is balanced by, the sum of the flux* ${}^{G}\Phi$ *of* G *through* $\partial\Omega'$, *the production* ${}^{G}P$ *of* G *in* Ω' *and the supply* ${}^{G}S$ *of* G *from outside* Ω', i.e.

$$\frac{dG}{dt} = {}^{G}\Phi + {}^{G}P + {}^{G}S,\ \forall\Omega' \subset \Omega.$$

If ${}^{G}P = 0,$ the balance law is called the *conservation law.*

The integral form of a balance law is called the *integral balance equation.* It is obtained by replacing the expression of the global quantities

$$G(t,\Omega'(t)) = \int\limits_{\Omega'(t)} g(t,\boldsymbol{x})d\Omega', {}^{G}\Phi(t,\partial\Omega'(t)) = -\int\limits_{\partial\Omega'(t)} {}^{G}\Phi_i n_i d\sigma',$$

$${}^{G}P(t,\Omega'(t)) = \int\limits_{\Omega'(t)} {}^{G}p(t,\boldsymbol{x})d\Omega', {}^{G}S(t,\Omega'(t)) = \int\limits_{\Omega'(t)} {}^{G}s(t,\boldsymbol{x})d\Omega'(t)$$

into the above balance laws, i.e.

$$\frac{d}{dt}\int\limits_{\Omega'(t)} g(t,\boldsymbol{x})d\Omega' = -\int\limits_{\partial\Omega'(t)} {}^{G}\Phi_i n_i d\sigma' + \int\limits_{\Omega'(t)} {}^{G}p(t,\boldsymbol{x})d\Omega' + \int\limits_{\Omega'(t)} {}^{G}s(t,\boldsymbol{x})d\Omega'(t), \forall\Omega'\subset\Omega.$$

It is expressed as an equality of integrals of the corresponding fields.

The differential form of a balance law is called the **local balance equation** and it follows from the above balance equation by using: the transport theorems (in order to introduce $\frac{d}{dt}$ under the integral sign), the flux-divergence theorem (in order to transform the surface integrals into volume integrals) and the continuity theorem (enabling us to equate to zero the integrand, due to arbitrarity of Ω'). Thus, the local balance equation are the following partial differential equations for every local scalar quantity g

$$\frac{\partial g}{\partial t} + div(g\,\boldsymbol{u}) = -div\,{}^{G}\Phi + {}^{G}P + {}^{G}S,\ t,\boldsymbol{x}\in(0,t_1)\times\Omega.$$

For vector global quantities we have

$$\frac{\partial \boldsymbol{g}}{\partial t} + div(\boldsymbol{g}\otimes\boldsymbol{u}) = -div\boldsymbol{G}_{\Phi} + \boldsymbol{G}_P + \boldsymbol{G}_S, \quad t,\boldsymbol{x}\in(0,t)\times\Omega$$

We present a table with a few global quantities and the related local quantities.

Global quantity	Density	Flux density	Production density	Supply density
mass	ρ	0	0	0
momentum	$\rho\boldsymbol{u}$	$-\boldsymbol{T}$	$\boldsymbol{0}$	$\rho\boldsymbol{F}$
momentum of momentum	$\rho\boldsymbol{x}\times\boldsymbol{u}$	$-\boldsymbol{x}\times\boldsymbol{T}$	$\boldsymbol{RT}$	$\rho\boldsymbol{x}\times\boldsymbol{F}$
spin	$\rho\,\boldsymbol{\sigma}$	$\boldsymbol{\Sigma}$ (stress couple)	$-\boldsymbol{RT}$	$\rho^{\sigma}\boldsymbol{S}$
angular momentum	$\rho(\boldsymbol{\sigma}+\boldsymbol{x}\times\boldsymbol{u})$	$\boldsymbol{\Sigma} - \boldsymbol{x}\times\boldsymbol{T}$	$\boldsymbol{0}$	$\rho({}^{\sigma}\boldsymbol{S}+\boldsymbol{x}\times\boldsymbol{F})$
kinetic energy	$\rho\boldsymbol{u}^2/2$	$-\boldsymbol{uT}$	$-\boldsymbol{T}\cdot\nabla\boldsymbol{u}$	$\rho\boldsymbol{u}\cdot\boldsymbol{F}$
internal energy	$\rho\varepsilon$	$\boldsymbol{q}$ (heat flux)	$\boldsymbol{T}\cdot\nabla\boldsymbol{u}$	$\boldsymbol{0}$
total energy	$\rho(\varepsilon+\boldsymbol{u}^2/2)$	$\boldsymbol{q}-\boldsymbol{uT}$	0	$\rho\boldsymbol{u}\cdot\boldsymbol{F}+\rho r$
entropy	$\rho\eta$	$\boldsymbol{\Phi}$	σ	${}^{\eta}\boldsymbol{S}$

In components $(\boldsymbol{x}\times\boldsymbol{T})_{ij}=\varepsilon_{ikl}x_kT_{lj}$ and $(RT)_i=\varepsilon_{ijk}T_{jk}$.

If electrical phenomena are present we must consider, in addition, the electric charge. We do not take into account electromagnetic phenomena, radiation (r) and chemical reactions. Neither, we consider mixtures involving the concentration field. Also our scheme is not appropriate to sedimentation, when production of mass is present. For a more general situation the reader is kindly send to [14].

The corresponding local balance equations (some of them are conservation equations) read, for $(t,\boldsymbol{x})\in(0,t_1)\times\Omega$,

$$(1)\quad \frac{\partial\rho}{\partial t}+div(\rho\,\boldsymbol{u})=0,$$

$$(2)\quad \frac{\partial(\rho\boldsymbol{u})}{\partial t}+div(\rho\boldsymbol{u}\otimes\boldsymbol{u})-div\boldsymbol{T}-\boldsymbol{\rho F}=0,$$

$$(3)\quad \frac{\partial(\,\rho\boldsymbol{x}\times\boldsymbol{u})}{\partial t}+div[(\,\rho\boldsymbol{x}\times\boldsymbol{u})\otimes\boldsymbol{u}\,]-div(\,\boldsymbol{x}\times\boldsymbol{T})-\boldsymbol{RT}-\rho\boldsymbol{x}\times\boldsymbol{F}=0,$$

$$(4)\quad \frac{\partial(\rho\boldsymbol{\sigma})}{\partial t}+div(\rho\boldsymbol{\sigma}\otimes\boldsymbol{u})+div\boldsymbol{\Sigma}+\boldsymbol{RT}-\boldsymbol{\rho}^{\sigma}\boldsymbol{S}=0,$$

$$(5)\quad \frac{\partial[\,\rho(\boldsymbol{\sigma}+\boldsymbol{x}\times\boldsymbol{u})]}{\partial t}+div[\,\rho(\boldsymbol{\sigma}+\boldsymbol{x}\times\boldsymbol{u}\,)\otimes\boldsymbol{u}\,]-div(\,\boldsymbol{\Sigma}-\boldsymbol{x}\times\boldsymbol{T})-\rho({}^{\sigma}\boldsymbol{S}+\boldsymbol{x}\times\boldsymbol{F})=0,$$

$$(6)\quad \frac{\partial[\rho\,|\,\boldsymbol{u}\,|^2\;/\,2\,)]}{\partial t}+div(\rho\frac{\boldsymbol{u}^2}{2}\boldsymbol{u})-div(\boldsymbol{uT})+\boldsymbol{T}\cdot\nabla\boldsymbol{u}-\rho\boldsymbol{u}\cdot\boldsymbol{F}=0,$$

$$(7)\quad \frac{\partial(\rho\varepsilon)}{\partial t}+div(\rho\varepsilon\boldsymbol{u})-div\boldsymbol{q}-\boldsymbol{T}\cdot\nabla\boldsymbol{u}=0,$$

$$(8)\quad \frac{\partial[\rho(\varepsilon+|\,\boldsymbol{u}\,|^2\;/\,2\,)]}{\partial t}+div[\rho(\varepsilon+\frac{\boldsymbol{u}^2}{2})\boldsymbol{u}]+div[\boldsymbol{q}-\boldsymbol{uT}]-\rho\boldsymbol{u}\cdot\boldsymbol{F}=0,$$

$$(9)\quad \frac{\partial(\rho\eta)}{\partial t}+div(\rho\eta\boldsymbol{u})-div\boldsymbol{\Phi}-\sigma-{}^{\sigma}S=0.$$

They are called: the continuity, first Cauchy, second Cauchy, spin, angular momentum of momentum, kinetic energy, internal energy, total internal energy and entropy equation, respectively.

In equations (1)-(9) we took into account only mechanical and thermal effects, therefore they are appropriate to a thermomechanical phenomenon.

Equation (5) is obtained summing (3) and (4). Equation (8) is derived from (6) and (7). Equation (9) is not influencing equations (1)-(8). It follows that the independent balance equations, where equation (1) is taken into account, are

$$(1)\qquad \frac{\partial\rho}{\partial t}+div(\rho\,\boldsymbol{u})=0,$$

$$(10)\qquad \frac{\partial\boldsymbol{u}}{\partial t}+div(\boldsymbol{u}\otimes\boldsymbol{u})-\frac{1}{\rho}div\boldsymbol{T}-\boldsymbol{F}=0,$$

$$(11)\qquad \frac{\partial(\boldsymbol{\sigma}+\boldsymbol{x}\times\boldsymbol{u})}{\partial t}+div[(\boldsymbol{\sigma}+\boldsymbol{x}\times\boldsymbol{u})\otimes\boldsymbol{u}]+\frac{1}{\rho}div(\boldsymbol{\Sigma}-\boldsymbol{x}\times\boldsymbol{T})-({}^{\sigma}\boldsymbol{S}+\boldsymbol{x}\times\boldsymbol{F})=0,$$

$$(12)\qquad \frac{\partial[(\varepsilon+|\boldsymbol{u}|^{2}/2)]}{\partial t}+div[(\varepsilon+\frac{\boldsymbol{u}^{2}}{2})\boldsymbol{u}]+\frac{1}{\rho}div[\boldsymbol{q}-\boldsymbol{u}\boldsymbol{T}]-\boldsymbol{u}\cdot\boldsymbol{F}=0,$$

$$(13)\qquad \frac{\partial(\eta)}{\partial t}+div(\eta\boldsymbol{u})+\frac{1}{\rho}div\boldsymbol{\Phi}-\frac{1}{\rho}\sigma-\frac{1}{\rho}{}^{\sigma}S=0.$$

If the spin is absent, i.e. we do not take into account the micropolarity, i.e. $\boldsymbol{\sigma}=\boldsymbol{0},\boldsymbol{\Sigma}=\boldsymbol{0},{}^{\sigma}\boldsymbol{S}=\boldsymbol{0}$. From (4) it follows that $T_{ij}=T_{ji}$ (the stress tensor is symmetric). Then (10) multiplied by $\boldsymbol{u}$ gives (11). Equation (10) is called the **equation of motion.** If $\boldsymbol{u}$ is a homogenous field and fluxes are absent, then (10) becomes the second Newton law.

Sometimes all conservation equations (1), (10), (12) and the balance equation (13) are called *equations of motion.* They are valid for all Cauchy continua, which have not microstructure, in which do not take place electromagnetic, chemical phenomena and the shocks are absent.

In the electromagnetic case we define: the magnetic flux $\boldsymbol{B}=\mu_0\boldsymbol{H}$, magnetic field (or potential of current) $\boldsymbol{H}$, electric field $\boldsymbol{E}$, total current in the surface $\boldsymbol{J}$, polarization charge $\boldsymbol{P}$, potential charge (or dielectric displacement) $\boldsymbol{D}=\varepsilon_0\boldsymbol{E}$, density of electric charge q.

The global electric charge is $\int_{\Omega'(t)} q(t,\boldsymbol{x})d\Omega'(t)=\int_{\Omega'(t)} q^{F}(t,\boldsymbol{x})d\Omega'(t)-\int_{\partial\Omega'(t)}\boldsymbol{P}\boldsymbol{n}d\boldsymbol{\sigma}'(t)$ hence it is the general form of a global quantity, with $g_C=0$. In this case the mechanical momentum law and mechanical energy law are not conservation equations.

They read

$$\frac{d}{dt}\int_{\Omega'(t)}\rho\boldsymbol{u}\,d\Omega'(t)=\int_{\partial\Omega'(t)}\boldsymbol{T}\boldsymbol{n}d\sigma'(t)+\underline{\int_{\Omega'(t)}[q\boldsymbol{E}+(\boldsymbol{j}\times\boldsymbol{B})]d\Omega'(t)}+\int_{\Omega'(t)}\rho\boldsymbol{F}\,d\Omega'(t),$$

$$\frac{d}{dt}\int_{\Omega'(t)}[\rho(\varepsilon+\frac{\boldsymbol{u}^{2}}{2})-\boldsymbol{T}\cdot\nabla\boldsymbol{u}]d\Omega'(t)=-\int_{\partial\Omega'(t)}\boldsymbol{q}\cdot\boldsymbol{n}d\sigma'(t)+\underline{\int_{\Omega'(t)}(\boldsymbol{j}\cdot\boldsymbol{E})d\Omega'(t)}.\ \text{(We took } \boldsymbol{F}=\boldsymbol{0}\text{)}$$

The underlined terms are productions. In exchange, if we define the momentum of matter (more precisely, substance) and field (i.e. matter which is not in the form of a substance) as $\int_{\Omega'(t)}(\rho\boldsymbol{u}+\boldsymbol{D}\times\boldsymbol{B})d\Omega'(t)$ and the energy of matter and field as $\int_{\Omega'(t)}[\rho(\varepsilon+\frac{\boldsymbol{u}^{2}}{2})+\frac{\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{B}\cdot\boldsymbol{H}}{\boldsymbol{2}}]d\Omega'(t)$ then, for these global quantities the balance laws are conservation laws, namely [14]

$$\int_{\Omega'(t)}\{\frac{\partial}{\partial t}(\rho\boldsymbol{u}+\boldsymbol{D}\times\boldsymbol{B})]+div[\rho\boldsymbol{u}\otimes\boldsymbol{u}-\boldsymbol{T}+\frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{B}\cdot\boldsymbol{H})\boldsymbol{I}]-\boldsymbol{E}div\boldsymbol{D}-\boldsymbol{B}div\boldsymbol{H}\}d\Omega'(t)=0,$$

$$\int_{\Omega'(t)} \left\{ \frac{\partial}{\partial t}[\rho\left(\boldsymbol{\varepsilon}+\frac{\boldsymbol{u}^2}{2}\right)+\frac{\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{B}\cdot\boldsymbol{H}}{\boldsymbol{2}}]+div\left[\rho\left(\boldsymbol{\varepsilon}+\frac{\boldsymbol{u}^2}{2}\right)\boldsymbol{u}+\boldsymbol{q}-\boldsymbol{Tu}+\boldsymbol{E}\times\boldsymbol{H}\right]\right\} d\Omega'(t)=0.$$

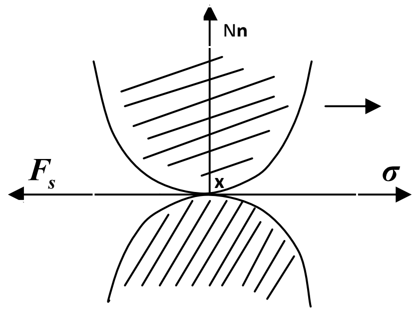
In the five equations (1), (10), (12) there occur the unknown functions $\rho,\boldsymbol{u},\boldsymbol{T},\varepsilon,\boldsymbol{q}$, i.e. 13 unknown scalar functions. To have a closed system of equations we must choose five **basic unknown functions**; the remained ones must be expressed in terms of the basic quantities. On the other hand, physically, we must supply the equations of motion with information about the physical continuum, other than that contained in the definition of the stress. This information is contained in the constitutive equations and the state equations, involving the main characteristics of *deformable media*, namely the deformations.

All integrals in the above were defined with respect to the Lebesgue measure. This is why the obtained models are valid for Lebesgue measurable domains of motion. As a consequence, fractal domains or domains with fractal boundaries, are not appropriately modeled in our framework.

3. Laws of material

3.1 Viscosity

The characteristics of motions are constrained to satisfy in addition the relations of definition of the material, i.e. **the constitutive equations**, which express the response of the medium to the excitation. The constitutive equations distinguish between various continuous media.

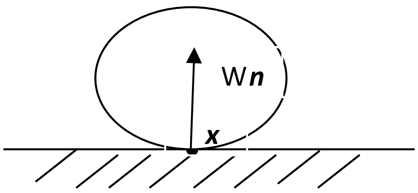


The viscosity, the most important characteristic for fluids, evidentiates the position of fluid mechanics versus the mechanics of rigids, thermomechanics and statistical mechanics. The force of dry friction of sliding or rolling occurs at the surface of contact of two bodies which have a relative motion and is produced by the reciprocal action of the small irregularities (roughness) of the surface and opposes to the motion. It is

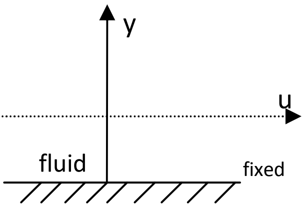
(14) $\boldsymbol{F}_s(\boldsymbol{x})=-\mu_s N(\boldsymbol{x})\boldsymbol{\sigma}(\boldsymbol{x}),\ \boldsymbol{F}_r(\boldsymbol{x})=\mu_r\frac{1}{r}W(\boldsymbol{x})\boldsymbol{\sigma}(\boldsymbol{x}).$

The proportionality coefficients μ_s and μ_r depend on: $\nabla\boldsymbol{v}$ ($\boldsymbol{v}$ is the relative velocity of two bodies), roughness of the contact surface, the normal forces and their motion along normal $\boldsymbol{n}$, the deformation of the contact surface, molecular attraction of the points in contact, the wear produced by the plastic deformation of the contact surface.

In mechanics of rigid bodies (14) represents material (constitutive) laws, which define the rigid material.



The tangential force emerging at the surface of fluid layers in their motion and which opposes to this motion is the *internal friction force*, or the **viscosity force**, or the *wet force.* It is a result of the relative motion of microparticles (deformation of fluid bodies), due to external forces. *It exists only for fluid flow*, not for fluids at rest. The **viscosity** is the physical property of deformable bodies, to oppose resistance to their deformation, giving rise to tangential stress. It is characteristic also to media other than fluids.



The material fluids have as a specific feature their response to excitation when they are at rest: **a material is a fluid if at rest it does not oppose any resistance to the change of its form**. It follows that at rest the (tangential) shear stresses are null; this represents the IV Stokes postulate (Section 3.2).

The corresponding coefficients are called the coefficients of viscosity. The first measurement of such a coefficient was carried out by Newton (see the figure): the upper boundary of the fluid is moving with a constant velocity u. Two horizontal infinite plates, constant pressure, velocity field $(u(y),0,0)$. Whence

Newton friction law

(15) $\tau_{12} = \mu \frac{du}{dy}$,

where μ is the coefficient of dynamic viscosity (Newton wrote $\tau = \mu \frac{\Delta u}{\Delta y}$, where the symbol Δ stands for the increment), τ_{12} is the shear stress. In general (16) $\mu = \mu(p,T)$.

This relation is found experimentally and is given by tables or graphs.

For ordinary liquids, μ is almost independent of p, but fast decreasing with T.

For ordinary gases, μ increases with T and increases very slow (but more than for liquids) with p.

Usually we take $\mu =$ const. at

$20^0 C: \mu_{water} = 1010 \cdot 10^{-5} g/m \cdot sec,\ \mu_{air} = 17 \cdot 2 \cdot 10^{-5} g/m \cdot sec;\ \mu_{\text{oils}} = 9{,}86 g/m \cdot \text{sec}.$

Another possibility to get μ is from some discret model. For instance

Kinetico-molecular theories yield (for binary collisions and Maxwell statistics)

$$\mu = m_m \bar{v}_m \bar{l}\, n/6 = 2^{-3/2} m \bar{v}_m / (4\pi r^2),$$

where m_m = mass of a microparticle, n = number of microparticles per unit volume, $\bar{l}$ = free mean path; $\bar{v}_m$ = mean velocity of microparticles, r = their radius (the microparticles are assumed as elastic spheres). This expression of μ changes for other types of collisions, high temperature, turbulent regime. The generalization of (15) from continuum mechanics reads

$$\boldsymbol{F}_{\text{viscous}} \equiv \frac{\partial}{\partial x_j}[(T_{ij} - p\delta_{ij})\boldsymbol{e}_i] = \frac{\partial}{\partial x_j}(2\mu D_{ij}\boldsymbol{e}_i). \text{ If } \mu = \text{const. } \boldsymbol{F}_{\text{viscous}} = \mu\Delta u_i \boldsymbol{e}_i] = \mu\Delta\boldsymbol{u}.$$

Here D_{ij} are the components of the tensor $\boldsymbol{D}$, the simmetric part of the gradient of the velocity $\boldsymbol{u}$.

Viscosity ensures the momentum transport transversally to the motion. Indeed, $d(mu) \approx m\frac{\Delta u}{\Delta y}\Delta y$ is the increment of momentum between two horizontal adjacent layers. In particular, the motion of the walls induces the transversal motion of the fluid. The inviscid fluids cannot transmit the motion across the fluid. It follows the boundary conditions depend on the type of fluid.
Coefficient of kinematic viscosity is $\nu = \mu / \rho$.

The Newton law is the first constitutive equation for (viscous) fluids. It is linear. In fact, in their earlier formulation, the main constitutive equations were linear (e.g. Fick, Fourier), because they were inspired by the experiments, where there was the proportionality between two physical quantities.

3.2 Newtonian and nonNewtonian fluids

In this section, a few particular fluids are defined.
Stokesian fluid is described by

(17) $\boldsymbol{T} = \alpha_1\boldsymbol{I} + \alpha_2\boldsymbol{D} + \alpha_3\boldsymbol{D}^2$,

where $\alpha_1, \alpha_2, \alpha_3$ are functions of the main invariants $\boldsymbol{I}_k (k = 1,2,3)$ of the tensor $\boldsymbol{D}$, $\boldsymbol{I}_1$ is the trace of $\boldsymbol{D}$ (i.e. $\operatorname{div}\boldsymbol{u}$), $\boldsymbol{I}_2$ is the sum of main minors of second order of $\boldsymbol{D}$ (i.e. $\boldsymbol{I}_2 = D_{11}D_{22} + D_{11}D_{33} + D_{22}D_{33} - D_{12}^2 - D_{13}^2 - D_{23}^2$) and $\boldsymbol{I}_3$ is det$\boldsymbol{D}$. It is the most general continuum satisfying:

i. $\boldsymbol{T}$ is continuous function of $\boldsymbol{D}$ and does not depend on other kinematic variables;
ii. $\boldsymbol{T}$ does not depend explicitly on $\boldsymbol{x}$ (i.e. homogeneity);
iii. in $\mathbf{R}^3$ there are no privileged directions (isotropy);
iv. if $\boldsymbol{D} = \boldsymbol{0}$, then $\boldsymbol{T} = -p\boldsymbol{I}$ (ideal fluid).

Newtonian fluid is characterized by the fact that the *viscous tensor* $\boldsymbol{\sigma} \equiv \boldsymbol{T} - \alpha_1\boldsymbol{I}$ is linear in $\boldsymbol{D}$ (i.e. $\alpha_3 = 0, \alpha_2 =$ constant, in the sense that α_2 does not depend on kinematic quantities, but can depend on p and $\boldsymbol{T}$). It follows as a first order approximation as $|\boldsymbol{D}| \to 0$ from the Stokesian fluid. The viscosity μ at Newtonian fluids depends only on temperature.

Viscous incompressible fluid is defined by

(18) $\boldsymbol{\sigma} \equiv 2\mu\boldsymbol{D},\ \boldsymbol{T} = -p\boldsymbol{I} + 2\mu\boldsymbol{D},$ for $\mu > 0, p > 0,\ \operatorname{div}\boldsymbol{u} = 0.$

The relation (18) follows from (17) for $\alpha_1 = -p$, $\alpha_2 = 2\mu$ and $\operatorname{div}\boldsymbol{u} = 0$. For $\boldsymbol{u} = (u(y), 0, 0)$, the fluid becomes Newtonian. This is why (18) is said **the generalized Newton law**.

Viscous compressible fluid is defined by

(19) $\boldsymbol{T} = (-p + \frac{2}{3}\mu \operatorname{div}\boldsymbol{u})\boldsymbol{I} + 2\mu\boldsymbol{D}$ (friction law of Stokes). It follows from the Cauchy-Poisson law where $\gamma = \frac{2}{3}\mu$.

Viscous fluids are physically linear materials. Fluids are characterized by high mobility of their microparticles and by the fact that they take totally or partially the form of the container. The first property implies that in their definition there occur the velocities of deformation and not the deformation itself (as for deformable solids).

NonNewtonian fluids (e.g. milk, synthetic materials, oils, colloidal suspensions, dyes, sediments, mud, inks, margarine, maionezis, solutions of polymers with big molecular mass) generalize Newtonian fluids in several respects: they are physically nonlinear, i.e. are defined by nonlinear relationship between the shear stress and $\boldsymbol{D}$; in general are defined by several constants of material which, in addition depend on $\nabla\boldsymbol{u}$, the constructive peculiarities of the body immersed in a fluid, on the material memory, on the form of the domain of motion. Important classes of nonNewtonian fluids are: Bingham fluids, pseudoplastic fluids, dilatant fluids, tixotropics, reopectic, viscoelastic (of Maxwell type, of Oldroyd type), materials with memory (e.g. Walters type fluid, where

$$\sigma_{ij}(t, \boldsymbol{x}) = -2\int_{-\infty}^{t} \Psi(t-s) \frac{\partial x_m^*}{\partial x_i} \frac{\partial x_r^*}{\partial x_j} A_{2mr}(s, x) ds,$$

where

$$\Psi(t-s) = \int_0^\infty \frac{N(x)}{x} \exp\{-(t-s)/x\} dx,$$

$\boldsymbol{A}_2$ is defined in Section 3.3, $\boldsymbol{x}^*(t, s, \boldsymbol{x})$ is the position at $t-s$ of the particle which at $t = t$ is at $\boldsymbol{x}$), second degree fluids. In fluid with memory it is used the relative description.

3.3 Constitutive relations (or equations)

In Sections 3.1, 3.2 a few particular classes of fluids were defined by specific laws of material. Here we present more general fluids.

First we consider as **basic quantities** defining the motion of a body: ρ (or p), $\boldsymbol{u}, \boldsymbol{T}, \boldsymbol{E}, \boldsymbol{H}$. The expressions of the fluxes $\boldsymbol{T}, \boldsymbol{q}, \boldsymbol{j}$ and of ε, q, η, called the **constitutive functions** $C(t, x_i)$ in terms of the basic quantities and/or their derivatives are called the **constitutive equations**. First they differ from a material to other, according to the type of deformations they can support. Further we introduce

Geometric transformations in $\mathbf{R}^3$. Consider two systems of coordinates $S = Ox_1x_2x_3$ and $S' = O'x'_1\,x'_2\,x'_3$ and let x_i and x'_i be the coordinates of a point P with respect to S and S' respectively. Let o'_i be the coordinates of O' with respect to S. Then $x'_i = a_{ij}x_j + o'_i\,, i \in \overline{1,3},$ $\det(a_{ij}) \neq 0,$ defines a transformation $T : \boldsymbol{R}^3 \to \boldsymbol{R}^3$. Assume that a_{ij} and o'_i do not depend on the time t. Then T is a **geometric** transformation, it is affine, and it carries the planes into planes. If $o'_i = 0, i = \overline{1,3}$ then T is **linear**. The affine and linear transformations invariate the distance iff $a_{ij}a_{ik} = \delta_{jk}$ (Kronecker delta). In this case, if T is linear, it is called the **centroisometry** or **homogenous orthogonal transformation**; if T is affine, then it is called the **isometry** or **nonhomogenous orthogonal transformation**. The linear T with $\det(a_{ij}) = 1$ are called the **proper rotations**. These transformations form a group.

Euclidean transformations are geometrical transformations depending on time, i.e. $a_{ij}(t), o'_i(t); x_i$ are the coordinates in a system of coordinates at rest with respect to a inertial system, x'_i are the coordinates in a system of coordinates with respect to a noninertial system; o'_i determine the motion of origin and the orthogonal matrix $(a_{ij}(t))$, the relative motion of the axes. The Euclidean transformations have a mechanical (inertia) aspect.

The inverse Euclidean transformation is $x_j = a'_{ij}(t)x'_i - o'_i(t).$ The scalar $s,$ vector $\boldsymbol{v}$ and tensor $\boldsymbol{T}$ is **objective** if $s' = s, v'_i = a_{ij}(t)v_j, T'_{ij} = a'_{ik}(t)a'_{jl}(t)T_{kl}.$ The velocity and accelerations are not objective, $\boldsymbol{D}$ is objective, the Rivlin-Ericksen tensors are objective [14], whence their use in defining nonNewtonian fluids. The constitutive equations must contain only objective quantities since a material is the same at all places on the Earth. We say that they obey the objectivity principle.

Experiments show that $C(t, x_i)$ depends on $\rho, \boldsymbol{X}, \boldsymbol{T}$ for all particles $\boldsymbol{x_0} \in \Omega_0$ and passed times $t - s,$ with $0 \le s < \infty,$ i.e.

$$(20)\quad C(t,x_i) = \underset{\substack{s=0\\ \Upsilon\in\Omega_0}}{\overset{\infty}{\{}}\ [\rho(t-s,\Upsilon_B), \boldsymbol{X}(t-s,\Upsilon_B), T(t-s,\Upsilon_B)],$$

where "{" is a functional and Υ is another point of components $\Upsilon_B (X_A$ are the components of $\boldsymbol{x_0}$). The memory occurs due to s and the nonlocality of deformations, due to the space derivatives of the basic fields $\rho, \boldsymbol{X}, \boldsymbol{T}$.

The fluid defined only by $\boldsymbol{D}$ has an infinitesimal memory, while if the constitutive law contains in addition some other Rivlin-Ericksen tensors, it has a better memory. It contains local effects if its constitutive law contains only first order space derivatives of $\boldsymbol{X}$ and nonlocal, otherwise.

Let us now define mathematically the deformations, in order to put into evidence these tensors, beginning with

Local and global deformations. In order to write the constitutive equations we define the tensors characterizing the deformation. They are space derivative of the displacement and velocity field. For local deformations we have

$$\boldsymbol{x}(t+dt,\boldsymbol{x}_0) \sim \boldsymbol{x}(t,\boldsymbol{x}_0) + \frac{d\boldsymbol{x}}{dt}dt = \boldsymbol{x}(t,\boldsymbol{x}_0) + \boldsymbol{u}(t,\boldsymbol{x})dt,$$

$$\boldsymbol{x}(t,\boldsymbol{x}_0+d\boldsymbol{x}_0) \sim \boldsymbol{x}(t,\boldsymbol{x}_0) + \frac{d\boldsymbol{x}}{d\boldsymbol{x}_0}d\boldsymbol{x}_0 = \boldsymbol{x}(t,\boldsymbol{x}_0) + \boldsymbol{J}(t,\boldsymbol{x}_0)d\boldsymbol{x}_0,$$

$$\boldsymbol{u}(t,\boldsymbol{x}+d\boldsymbol{x}) \sim \boldsymbol{u}(t,\boldsymbol{x}) + \boldsymbol{L}(t,\boldsymbol{x})d\boldsymbol{x}(t,\boldsymbol{x}_0)d\boldsymbol{x}_0,$$

where $\boldsymbol{J}$ is the deformation tensor, $J_{ij}=\frac{\partial x_i}{\partial x_{0j}}$, J is the Jacobian of the transformation $\boldsymbol{x}_0 \to \boldsymbol{x}(t,\boldsymbol{x}_0)$, $d\tau = Jd\tau_0$, $\frac{dJ}{dt} = J\mathrm{div}\,\boldsymbol{u}, \boldsymbol{L} = \nabla\boldsymbol{u}, \boldsymbol{L}_{ij} = \frac{\partial u_i}{\partial x_j}, \boldsymbol{D} = (\boldsymbol{L}+\boldsymbol{L}^T)/2$ is the Euler tensor of velocities of deformation. It is also denoted by $\boldsymbol{a}$ or by $\boldsymbol{A}$. The diagonal components D_{ii} are the elongations, while D_{ij}, $i \neq j$, measure the angular deformations. We put $\boldsymbol{W} = (\boldsymbol{L}-\boldsymbol{L}^T)/2,\ \boldsymbol{W}\boldsymbol{u} = \boldsymbol{\omega}\times\boldsymbol{u},\ \boldsymbol{\omega} = \frac{1}{2}rot\,\boldsymbol{u} = \frac{1}{2}\nabla\times\boldsymbol{u},$ where $\boldsymbol{W}$ is the tensor of velocities of rotation, and $\boldsymbol{\omega}$ is the vortex (rotor, tourbillion). The third relation shows that locally, around a point $\boldsymbol{x}$, the velocity field in a deformable medium is the sum

$$\boldsymbol{u}(t,\boldsymbol{x}+d\boldsymbol{x}) = \boldsymbol{u}(t,\boldsymbol{x}) + \frac{1}{2}grad(d\boldsymbol{x}\cdot\boldsymbol{D}d\boldsymbol{x}) + \boldsymbol{\omega}\times d\boldsymbol{x},$$

where $\frac{1}{2}grad(d\boldsymbol{x}\cdot\boldsymbol{D}d\boldsymbol{x})$ is the velocity of deformation and $\boldsymbol{\omega}\times d\boldsymbol{x}$ is the instantaneous rotation of angular velocity $\boldsymbol{\omega}$.

The local deformations depend on the infinitesimal time *dt*.

For deformations depending on a longer time, the above expressions are completed with the contribution due to higher order terms in dt^2, dt^3 etc.

$$\boldsymbol{x}(t+dt,\boldsymbol{x}_0) \sim \boldsymbol{x}(t,\boldsymbol{x}_0) + \boldsymbol{u}(t,\boldsymbol{x})dt + \frac{1}{2}\boldsymbol{a}(t,\boldsymbol{x})dt^2 + \boldsymbol{O}(dt^3), \qquad dt \to 0,$$

$$d\boldsymbol{x}(t,\boldsymbol{x}_0+d\boldsymbol{x}_0)dt = d\boldsymbol{x}(t,\boldsymbol{x}_0)dt + \boldsymbol{L}(t,\boldsymbol{x})d\boldsymbol{x}(t,\boldsymbol{x}_0)dt + \frac{1}{2}(grad\,\boldsymbol{a})d\boldsymbol{x}(t,\boldsymbol{x}_0)dt^2 + O(dt^3),\ dt \to 0.$$

Here $\boldsymbol{A}_{n+1} = \frac{d\boldsymbol{A}_n}{dt} + \boldsymbol{L}^T\boldsymbol{A}_n + \boldsymbol{A}_n\boldsymbol{L}$ are the Rivlin-Ericksen tensors. In particular

$\boldsymbol{A}_1 = 2\boldsymbol{D}, \boldsymbol{A}_2 = grad\,\boldsymbol{a} + (grad\,\boldsymbol{a})^T + 2\boldsymbol{L}^T\boldsymbol{L}$.

For $\boldsymbol{x}' = \boldsymbol{x}'(t,s,\boldsymbol{x}),\quad (\overset{t}{\boldsymbol{F}})_{ij} = \frac{\partial x_i'}{\partial x_j}$ is the tensor of relative deformation

and $\overset{t}{\boldsymbol{C}}(s,\boldsymbol{x}) = \overset{t}{\boldsymbol{F}}{}^T(s,\boldsymbol{x})\overset{t}{\boldsymbol{F}}(s,\boldsymbol{x})$ is the Cauchy-Green tensor of relative torsion. It describes the history of deformation. The symbol *"t"* over the quantities indicates the actual time.

Define now other materials.

The **simple materials** (e.g. Newtonian and nonNewtonian fluids) have a perfect memory but take into account only local effects (in space), i.e. are **geometrically linear.** Thus

$$(21)\ \ C(t,x_i) = \overset{\infty}{\underset{s=0}{\{}} [\overset{t}{\rho}(s,\chi_A), \overset{t}{\rho}_{,B}(s,\chi_A), \overset{t}{\chi}_i(s,\chi_A), \overset{t}{F}_{iB}(s,\chi_A), \overset{t}{T}(s,\chi_A), \overset{t}{T}_{,B}(s,\chi_A)],$$

where $\overset{t}{\Psi}(s) = \Psi(t-s), \Psi,_B = \dfrac{\partial \Psi}{\partial X_B}, \Psi = \Psi(t,\chi), F_{iB} = \dfrac{\partial \chi_i}{\partial X_B}$,

where $\boldsymbol{\chi}$ stands for $\boldsymbol{X}$ from the previous sections and *t* over the quantities indicates the actual time. Let us fiind simpler forms of (21) taking into account five principles. Thus, the continuity equation shows that $\overset{t}{\rho}$ and $\overset{t}{\rho}_{,B}$ occur no longer. Applying the **objectivity principle** to (21) and some particular choices for $a_{ij}(t)$ and $o'_i(t)$ it follows that $\boldsymbol{C}$ does not depend on $\boldsymbol{u}$ and the skewsymmetric part of $\nabla \boldsymbol{u}$, i.e. is an **isotropic functions**. **The fluid in the sense of Noll** is a material for which the isotropy group is equal to the unimodular group. An unimodular transformation $\boldsymbol{P}$ is a proper rotation which preserves the density. All fluids are isotropic materials. Here, the components of $\boldsymbol{P}$ are $P_{AB} = \dfrac{\partial \chi_A}{\partial \chi_B}$. Some other particular choices of $a_{ij}(t)$ and $o'_i(t)$ and the resulting property that the vectors C_i and tensors C_{kl} are objective imply

$$(22)\ \ T_{ij} = -p\delta_{ij} + 2\mu D_{ij} + c_3(\boldsymbol{D}^2)_{ij} + c_4 \frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j} + c_5 \frac{\partial T}{\partial x_k} + D_{k(i}\frac{\partial T}{\partial x_{j)}} + c_6 \frac{\partial T}{\partial x_k}(\boldsymbol{D}^2)_{k(i}\frac{\partial T}{\partial x_{j)}},$$

where the symmetry $T_{ij} = T_{ji}$ was accounted for and "$_{(\,)}$" put to indices denotes the symmetric part of the corresponding tensors,

$$(23)\ \ \varepsilon = \varepsilon(\rho, t, D_{ii}, (\boldsymbol{D}^2)_{ii}, (\boldsymbol{D}^3)_{ii}, \frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j}, \frac{\partial T}{\partial x_i} D_{ij}\frac{\partial T}{\partial x_j}, \frac{\partial T}{\partial x_i}(\boldsymbol{D}^2)_{ij}\frac{\partial T}{\partial x_j}),$$

$$(24)\ \ q_i = -\lambda \frac{\partial T}{\partial x_i} + c_1 D_{ij}\frac{\partial T}{\partial x_j} + c_2(\boldsymbol{D}^2)_{ij}\frac{\partial T}{\partial x_j}.$$

The constitutive relations (22) - (24) define the **viscous heat-conducting fluid**. Linearizing (22) - (24) it follows the Navier-Stokes-Fourier fluid. Hence it is **materially linear** and is defined by the constitutive relations

$$(25)\ \ T_{ij} = -p(\rho,T)\big|_E \delta_{ij} + \gamma(\rho,T) D_{kk}\delta_{ij} + 2\mu(\rho,T) D_{ij},\ \text{(Navier-Stokes law)}$$

$$(26)\ \ q_i = -\lambda(\rho,T)\big|_E \frac{\partial T}{\partial x_i},\ \text{(Fourier law)}$$

$$(27)\ \ \varepsilon = \varepsilon(\rho,T)\big|_E + c_7(\rho,T) D_{kk},$$

where γ is the second coefficient of viscosity, $\gamma+\frac{2}{3}\mu$ is the coefficient of bulk viscosity, λ is the coefficient of thermal conductivity and E indicates the equilibrium state.

Apart from objectivity (or material indifference) principle and isotropy, in deriving the simplest forms of the constitutive equations it is used the **entropy principle**, the **thermodynamic stability principle** and the **incompressibility**. Thus, it follows that $\lambda \geq 0, \mu \geq 0, \gamma+\frac{2}{3}\mu \geq 0, c_7 \equiv 0$.

Other rigorous definitions for large classes of fluids can be found in the papers of the American school of Serrin, Truesdell, Noll, Eringen, devoted to rheology.

The **transport coefficients** γ, μ, λ are determined by experiments or from statistical mechanics. We suppose them as being constant.

3.4 Equations of state and boundary conditions. Incompressible Navier-Stokes and Euler models

The equation $p = p(\rho, T)\big|_E$ is referred to as the **equation of classical thermodynamical state** (e.g. Gay-Lussac equation, Boyle-Mariotte equation, van der Waals equation, the law of perfect gases). It may be written equivalently as $\rho = \rho(p, T)$. As a consequence ε and all transport coefficients become functions of p and T. A material is said **incompressible** when these quantities do not depend on p. Then, the entropy principle implies that ρ does not depend on T either, hence $\rho = \text{const}$. Then the continuity equation shows that $\operatorname{div}\boldsymbol{u} = 0$, i.e. $\boldsymbol{u}$ is a solenoidal fluid. The equation $\varepsilon = \varepsilon(\rho, T)\big|_E$ or $\varepsilon = \varepsilon(p, T)\big|_E$ is called the equation of **energy state**. In the incompressible case it is $\varepsilon = \varepsilon(T)$ and it has various forms for gases or liquids.

The **equation of entropic** state must be also written if we want to determine even the entropy. In the electromagnetic case or for micropolar fluids some extra equations of state must be written too.

In general, $\varepsilon(p,t), \rho(p,t), \eta(p,t)$ etc. are given by experiments. However, they and the transport coefficients must obey the entropy and thermodynamic stability principle, inducing some restrictions on them.

Summarizing, for incompressible Navier-Stokes fluids (25) – (27) become

$$(28)\ T_{ij} = -p(T)\big|_E \delta_{ij} + 2\mu(T) D_{ij},\ q_i = -\lambda(T)\frac{\partial T}{\partial x_i},\ \varepsilon = \varepsilon(T)\big|_E,$$

and introducing (28) in (1), (10) we obtain the **incompressible Navier-Stokes equations**

(29) $\operatorname{div}\boldsymbol{u}=0,\ \dfrac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=-\dfrac{1}{\rho}\nabla p+\nu\Delta\boldsymbol{u}+\boldsymbol{F}.$

Remark that in this case the energy equation decouples from the motion equations. For $\nu=0$, equation (29) becomes the **incompressible Euler equations**

(30) $\operatorname{div}\boldsymbol{u}=0,\dfrac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=-\dfrac{1}{\rho}\nabla p+\boldsymbol{F}.$

In order to obtain a model, with the equations we must associate some conditions.

The boundary conditions are different, for different types of fluids. For a viscous fluid we have the no-slip condition $\boldsymbol{u}\big|_{\partial\Omega}=\boldsymbol{u}_w$, while for inviscid fluids $(\boldsymbol{u}-\boldsymbol{u}_w)\cdot\boldsymbol{n}\big|_{\partial\Omega}=0,$ where $\boldsymbol{u}_w$ is the velocity of the wall $\partial\Omega$. If $\partial\Omega$ is at rest, for viscous fluid we have

(31) $\boldsymbol{u}\big|_{\partial\Omega}=0.$

The initial condition reads

(32) $\boldsymbol{u}\big|_{t=0}=\boldsymbol{u}_0,$

such that the **incompressible Navier-Stokes model** consists of equation (29), boundary condition (31) and initial condition (32).

Assuming that $\partial\Omega$ is at rest, for inviscid fluids we have

(33) $\boldsymbol{u}\cdot\boldsymbol{n}\big|_{\partial\Omega}=0,$

such that the **incompressible Euler model** consists of equations (30), boundary condition (33) and initial condition (32).

These models become stationary if $\boldsymbol{u}$ does not depend explicitely on t.
The Navier-Stokes-Fourier model is obtained similarly by replacing (25) – (27) into (1), (10)-(13). In this case, in general, the energy equation cannot be decoupled. In addition, boundary conditions for the temperature must be specified. They depend on the thermal conductivity of $\partial\Omega$ or if $\partial\Omega$ is kept at a constant temperature.

Boussinesq model is obtained from the Navier-Stokes-Fourier model by formally supposing that ρ is constant except for in $\boldsymbol{F}$.
The compressible Navier-Stokes and Euler models follow from the Navier-Stokes-Fourier model in the case when the temperature occurs no longer in the equations of motion, e.g. when γ and μ are constant and $\rho=\rho(p)$. In this case the energy equation is decoupled from the equations of motion. However, the complete characterization of the fluid flow requires the consideration of the energy (heat) equation too.
The **classical fluid mechanics** treated the Navier-Stokes and Euler models. If the temperature variation must be taken into account, the study of fluid flows naturally belongs to

thermodynamics of fluids and is based on models like Navier-Stokes-Fourier model, but suitably generalized if other effects, e.g. electromagnetic, chemical, are to be accounted for.

4. Models of asymptotic approximation for the incompressible Navier-Stokes model

Apart for the main models (Navier-Stokes and Navier-Stokes-Fourier) a lot of other important models in fluid dynamics are deduced from them as models of asymptotic approximation. We present a few of them in [7].

Incompressible stationary Navier-Stokes model is approximated as the Reinolds number $\mathrm{Re} \to 0$ by two linear models (in $\boldsymbol{u} = (u, v, w)$ and $\boldsymbol{u} - \boldsymbol{U}_\infty$, with $\boldsymbol{U}_\infty$ the velocity infinitely far, respectively):

Stokes model

$$grad\ p = \frac{1}{\mathrm{Re}} \Delta \boldsymbol{u},\ div\, \boldsymbol{u} = 0,\ u\big|_{\partial\Omega} = 0, \quad \text{near } \partial\Omega\,;$$

Oseen model

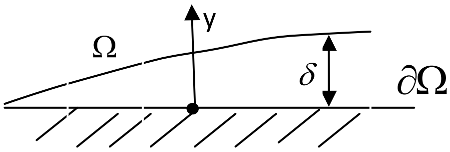
$$(\boldsymbol{U}_\infty \cdot grad)(\boldsymbol{u} - \boldsymbol{U}_\infty) + [(\boldsymbol{u} - \boldsymbol{U}_\infty) \cdot grad]\boldsymbol{U}_\infty = grad\ p + \frac{1}{\mathrm{Re}} \Delta(\boldsymbol{u} - \boldsymbol{U}_\infty),\ div\, \boldsymbol{u} = \boldsymbol{0},\ \lim_{|\boldsymbol{x}| \to \infty} \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{U}_\infty = 0.$$

This approximation is valid far from the body.

A **model is linear**, if written as $T\boldsymbol{v} = 0$, when the operator T is linear and $\boldsymbol{v}$ is the vector function the components of which are the unknown functions. In our case $\boldsymbol{v} = (\boldsymbol{u}, p)$.

The nondimensionalization as $\mathrm{Re} \to 0$ uses the assumption $[p] = \rho U_\infty^2 \mathrm{Re}^{-1}$.

The first approximation of the incompressible Navier-Stokes model as $\mathrm{Re} \to \infty$, where $\mathrm{Re} = \dfrac{U_\infty^2 L^{-1}}{\nu U_\infty L^{-2}} = \dfrac{U_\infty L}{\nu}$, $[\boldsymbol{v}] = U_\infty L$, is represented by the Euler and Prandtl models.



Far from $\partial\Omega$ the characteristic quantities are: for time, U_∞ / L; for space, L; for velocity, U_∞; for pressure, ρU_∞^2.

They lead to the Euler model (30).

Prandtl main hypotheses are:

1. the influence of viscosity is felt only in a thin layer, near $y = 0$, of thickness δ;

2. in that layer inertia forces are of the same order of magnitude as the viscous forces. This implies that $\delta^2 \sim \nu$, consequently δ is taken as a small parameter, or equivalently, $\varepsilon \sim \frac{1}{\sqrt{\mathrm{Re}}}$.

   Other hypotheses: $u >> \mathrm{v}$; $\frac{\partial u}{\partial y} >> \frac{\partial u}{\partial x}$

Near $\partial\Omega$ the characteristic quantities are: for time, U_∞ / L; for space, L for x and δ for y; U_∞ for velocity u; $U_\infty \frac{\delta}{L}$ for velocity v; and ρU_∞^2 for pressure.

The Navier-Stokes model in two dimensions reads

$$\underline{u\frac{\partial u}{\partial x}}+\underline{\mathrm{v}\frac{\partial u}{\partial y}}=\underline{-\frac{1}{\rho}\frac{\partial p}{\partial x}}+\nu(\frac{\partial^2 u}{\partial x^2}+\underline{\frac{\partial^2 u}{\partial y^2}}),\ u\frac{\partial \mathrm{v}}{\partial x}+\mathrm{v}\frac{\partial \mathrm{v}}{\partial y}=\underline{-\frac{1}{\rho}\frac{\partial p}{\partial y}}+\nu(\frac{\partial^2 \mathrm{v}}{\partial x^2}+\frac{\partial^2 \mathrm{v}}{\partial y^2}),\ \underline{\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}}=0,$$

where x, y are the coordinates of $\boldsymbol{x}$.

In the boundary layer the underlined terms are the highest order terms. Indeed, near $\partial\Omega$ the orders of the terms in the Navier-Stokes equations are

$$U_\infty^2 L^{-1};\ \frac{U_\infty\delta}{L}\frac{U_\infty}{\delta};\ U_\infty^2 L^{-1};\ \delta^2\left(\frac{U_\infty}{L^2};\frac{U_\infty}{\delta^2}\right);\ \ \frac{U_\infty^2\delta}{L};\ \frac{U_\infty^2\delta^2}{L^2\delta};\ \frac{U_\infty^2}{\delta};\delta^2\left(\frac{U_\infty\delta}{L^3};\frac{U_\infty\delta}{L\delta^2}\right);\frac{U_\infty}{L};\frac{U_\infty\delta}{L\delta},$$

respectively.

Therefore, disregarding second order terms from the Navier-Stokes model it follows the Prandtl model [16]

$$u_0\frac{\partial u_0}{\partial x}+\mathsf{v}_0\frac{\partial u_0}{\partial y}=-\frac{1}{\rho}\frac{\partial p_0}{\partial x}+\nu\frac{\partial^2 u_0}{\partial y^2},\ \frac{\partial p_0}{\partial y}=0,\ \frac{\partial u_0}{\partial x}+\frac{\partial \mathsf{v}_0}{\partial y}=0;\ u_0=\mathsf{v}_0=0 \text{ at } y=0$$

and the Euler model is

$$U_0\frac{\partial U_0}{\partial x}+V_0\frac{\partial U_0}{\partial y}=-\frac{1}{\rho}\frac{\partial P_0}{\partial x},\ U_0\frac{\partial V_0}{\partial x}+V_0\frac{\partial V_0}{\partial y}=-\frac{1}{\rho}\frac{\partial P_0}{\partial x},\ \frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial y}=0;$$

$$V_0=0 \text{ at } y=0; V_0=0, U_0=U_\infty \text{ at } \quad y=\infty$$

These Euler and Prandtl models were respectively obtained by using the following asymptotic expansions which take into account the quoted hypotheses [7]

$$u(x,y)\sim U_0(x,y)+\frac{1}{\sqrt{\mathrm{Re}}}U_1(x,y)+...,$$

$$\mathsf{v}(x,y)\sim V_0(x,y)+\frac{1}{\sqrt{\mathrm{Re}}}V_1(x,y)+...,$$

$$p(x,y) \sim P_0(x,y) + \frac{1}{\sqrt{\mathrm{Re}}} P_1(x,y) + \ldots,$$

$$u(x,y) \sim \hat{u}(x,\eta) \sim \hat{u}_0(x,\eta) + \frac{1}{\sqrt{\mathrm{Re}}} \hat{u}_1(x,\eta) + \ldots,$$

$$\hat{v}(x,\eta) \sim \frac{1}{\sqrt{\mathrm{Re}}} \hat{v}_1(x,\eta) + \ldots,$$

$$\hat{p}(x,\eta) \sim \hat{p}_0(x,\eta) + \frac{1}{\sqrt{\mathrm{Re}}} \hat{p}_1(x,\eta) + \ldots \ .$$

Then, the 1x1 asymptotic matching principle between the two models $U_0(x,0) = u_0(x,\infty)$, $0 = V_0(x,0) = \mathrm{v}_0(x,\infty)$, where $\eta = y/\delta$ (nondimensional), with $\hat{y} = y\sqrt{\mathrm{Re}}$ (dimensional), is applied.

We remark that in the Prandtl model we disregarded the hat over the quantities.

We took into account that the characteristic forces are of the order of magnitude of their leading (i.e. first) terms. In the Euler model they are different from those in the Prandtl model.

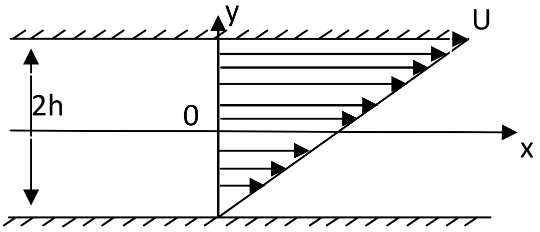
In the given main Navier - Stokes model some terms were of the magnitude $0 \cdot \infty$, such that the numerical computatios were not possible.

In the equations of a model of asymptotic approximation all terms are of the same order of magnitude, rendering the numerical computations feasible. This justifies the derivation of a lot of models of asymptotic approximation from the main models in fluid dynamics.

5. The plane Couette flow model

Let us derive a simple exact solution of the Navier – Stokes model and illustrate which are the fluid flow characteristics which can be calculated by means of this solution.

Plane Couette flow is defined as having $p = \text{const.}$, u(y), v=w=0. The quantities: $U^{-1}h,\ h,\ U$ are the characteristic time, length and velocity respectively. For an easier writing, let us use the symbols: u_1, u_2, u_3 and x_1, x_2 for u, v, w and x, y respectively. Then the Navier – Stokes model becomes (see figure)



$$\nu \frac{d^2 u_1}{dx_2^2} = 0, (x_1, x_2) \in \Omega,\ u_1(-h) = 0,\ u_1(h) = U$$

and has the solution $u_1 = \dfrac{U}{2h}(x_2 + h),\ u_2 = u_3 = 0.$

Using it, it follows

the vorticity field $rot\,\boldsymbol{u} = \dfrac{U}{2h}\boldsymbol{e}_3$,

the tensors $\boldsymbol{D}$ and $\boldsymbol{T}$ read $\boldsymbol{D} = \begin{pmatrix} 0 & U/(4h) & 0 \\ U/(4h) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\boldsymbol{T} = \begin{pmatrix} -p & \mu U/(2h) & 0 \\ \mu U/(2h) & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$.

Let $S = \{(x_1, x_2, x_3) \in R^3 \mid x_1 \in (x_1', x_1''), x_2 = h, x_3 \in (x_3', x_3'')\}$ $\boldsymbol{n} = (0,\text{-}1,0)$. Then the force on S is

$$\boldsymbol{R} = \iint_S T_{ij} n_j \boldsymbol{e}_i dx_1 dx_3 = \underbrace{-\mu \frac{U}{2h}(x_1'' - x_1')(x_3'' - x_3')\boldsymbol{e}_1}_{\text{drag=resistence to advancement}} + \underbrace{p(x_1'' - x_1')(x_3'' - x_3')\boldsymbol{e}_2}_{\text{lift=portance}}.$$

Remark that the sense of the velocity and drag are opposite.

Let $S_1 = \{(x_1, x_2, x_3) \in R^3 \mid x_1 = x_{10}, x_2 \in (-h, h), x_3 \in (x_3', x_3'')\}$. Then

average velocity is $\boldsymbol{u}_m = \dfrac{1}{mes\, S_1} \iint_{S_1} \boldsymbol{u} d\sigma = \underbrace{\dfrac{U}{2}}_{u_m}\boldsymbol{e}_2$;

mass flux reads $Q_{S_1} = \iint_{S_1} \rho \boldsymbol{u} \cdot \boldsymbol{n}\, d\sigma = \rho(mes\, S_1) \cdot u_m$.

Define $V_c = \{(x_1, x_2, x_3) \in R^3 \mid x_1 \in (x_1', x_1''), x_2 \in (-h, h), x_3 \in (x_3', x_3'')\}$. Then the kinetic energy of V_c is

$$E_c(V_c) = \iiint_{V_c} \frac{\rho |\boldsymbol{u}|^2}{2} d\tau = \frac{4}{3}\frac{u_m^2}{2}(\text{mass of } V_c).$$

Inertia forces and pressure forces are equal to zero. It follows that the viscous forces are null too.

References

As a consequence of the important applications of fluid mechanics, an enormous number of papers were and are continuously devoted to it.

They treat various aspects of fluid motions or equilibria, namely theoretical, experimental or numerical. Herein we are interested in theoretical questions, which are physical or mathematical. The physical aspects are determinant in defining the fluid materials and their types of motions and, so, they are basic in the derivation of the mathematical models governing the fluid flows or equilibria. Once deduced the models, they must be studied by analytical or numerical methods. As the models differ very much from a class of fluids to other or from a type of motion to other, the required methods differ accordingly. For instance, we have studies for viscous, inviscid, compressible, incompressible fluids, in laminar, transitional or turbulent regime. Various types of solutions can be treated: classical, generalized, stationary, nonstationary. Due to the difficulty in studying the governing models, several approximations of them are mostly used. The asymptotic approximation is much involved, although less recognized. It help us to establish the relationships between the multitude of mathematical models of fluid mechanics. Also due to the difficult mathematical problems occurred, a lot of papers deal with qualitative studies of these models, e.g. existence, uniqueness, regularity, stability, bifurcation.

All these show why we cannot name one single book as a "bible" of fluid mechanics. Thus, in the list of references we quote a few titles of theoretical treatments realized in the framework of various schools and at various periods of time. Some of them are well-known, some others not, in spite of their very high quality.

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ACKNOWLEDGEMENT

The author thanks Prof. Liliana Restuccia for the invitation to Department of Mathematics of Messina University in order to deliver the above course on thermodynamics of fluids during the period May 13-23, 2005, and for working out these lectures. Without her moral support these notes would never have been written down. The course was supported by a Messina University grant n. 235.