Communications to SIMAI Congress,
ISSN 1827-9015, Vol. 2 (2007)

DOI: 10.1685/CSC06035

MIXED DIRICHLET-ROBIN PROBLEMS IN IRREGULAR DOMAINS

RAFFAELA CAPITANELLI

Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate,
Università degli Studi di Roma "La Sapienza"
Via A. Scarpa 16, 00161 Roma, Italy.
E-mail: raffaela.capitanelli@uniroma1.it

We study mixed Dirichlet-Robin problems for the Laplacian in irregular domains. Our interest in these problems is motivated by some recent applications in physics, electrochemistry, heterogeneous catalysis and physiology.

Keywords: Mixed Dirichlet-Robin problems, Fractals, Irregular interfaces.

1. Introduction

The Laplacian transport to and across irregular and fractal interfaces are often encountered in nature or in technical processes: properties of rough electrodes in electrochemistry, steady-state diffusion towards irregular membranes in physiological processes, the Eley-Rideal mechanism in heterogeneous catalysis in porous catalysts, and in NMR relaxation in porous media (see Refs. 4, 5, 18 and the references therein). To characterize the mathematical formulation of these type of problems, we consider the current flowing through an electrochemical cell as shown in Fig. 1, where the working electrode presents an irregular boundary. The current density $\vec{J}$ is proportional to the electrostatic field $\vec{\nabla}\, V$, by classical equations of the type $\vec{J} = -\sigma\, \vec{\nabla}\, V$, where σ is the electrolyte conductivity. The conservation of this current throughout the bulk yields the Laplace equation for the potential V $\mathbf{div}(-\sigma\, \vec{\nabla}\, V) = 0$, that is, $\Delta V = 0$. The working electrode presents a finite faradaic resistance r to the current flow. If we assume that the outside of the irregular boundary is at zero potential, current conservation at the boundary leads to the following relation ($\vec{\nu}$ is the outward normal vector to the interface) $\vec{J}\,\vec{\nu} = \frac{V}{r}$, that is, $\frac{\partial V}{\partial \nu} = -\frac{V}{\Lambda}$, where $\Lambda = \sigma r$. A constant potential V_0 is applied on the counter electrode.

Licensed under the Creative Commons Attribution Noncommercial No Derivatives

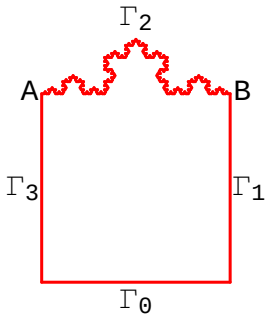


Fig. 1. An electrochemical cell.

The previous problem can be formally stated as

$$\begin{cases} \Delta V = 0 & \text{in } \Omega \\ V = V_0 & \text{on } \Gamma_0 \\ \frac{\partial V}{\partial \nu} = 0 & \text{on } \Gamma_1 \\ \frac{\partial V}{\partial \nu} + \frac{V}{\Lambda} = 0 & \text{on } \Gamma_2 \\ \frac{\partial V}{\partial \nu} = 0 & \text{on } \Gamma_3 \end{cases} \tag{1}$$

where $\Gamma_0 = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, y = -1\}$, $\Gamma_1 = \{(x,y) \in \mathbb{R}^2 : x = 1, -1 < y < 0\}$, $\Gamma_3 = \{(x,y) \in \mathbb{R}^2 : x = 0, -1 < y < 0\}$, and Γ_2 is either $\overset{\circ}{K}{}_n^{(l)} = K_n^{(l)} \setminus \{A,B\}$ or $\overset{\circ}{K}{}^{(l)} = K^{(l)} \setminus \{A,B\}$ (where $K_n^{(l)}$ denotes the prefractal curve approximating the Koch curve type fractal $K^{(l)}$, $A = (0,0)$ and $B = (1,0)$).

The layout of the paper is as follows. In the second section, we recall the definitions and the properties of the Koch curve type fractals. In the third section, we recall traces and embeddings theorems, and Green's formulae either on polygonal curve or on d-sets in the form which best fits our aims. In section 4, we state a variational principle for the problem (1) either in the prefractal case or in the fractal one. In the last section, we prove regularity results in terms of ordinary fractional Sobolev spaces and weighted Sobolev spaces when the interfaces are the prefractal curves approximating the Koch curve type fractals.

DOI: 10.1685/CSC06035

2. The Koch curve type fractals

Let us give some definitions and notations which will be used later. We consider the family $\Psi^{(l)} = \{\psi_1^{(l)}, \ldots, \psi_4^{(l)}\}$ of contractive similitudes $\psi_i^{(l)} : \mathbb{C} \to \mathbb{C}$, $i = 1, \ldots, 4$, with contraction factor l^{-1}, $2 < l \leqslant 4$,

$$\psi_1^{(l)}(z) = \frac{z}{l}, \qquad \psi_2^{(l)}(z) = \frac{z}{l}e^{i\theta(l)} + \frac{1}{l},$$
$$\psi_3^{(l)}(z) = \frac{z}{l}e^{-i\theta(l)} + \frac{1}{2} + i\sqrt{\frac{1}{l} - \frac{1}{4}}, \qquad \psi_4^{(l)}(z) = \frac{z-1}{l} + 1,$$

where

$$\theta(l) = \arcsin\left(\frac{\sqrt{l(4-l)}}{2}\right). \tag{2}$$

By the general theory of self-similar fractals (see Ref. 8), there exists a unique closed bounded set $K^{(l)}$, which is *invariant* with respect to $\Psi^{(l)}$, that is,

$$K^{(l)} = \cup_{i=1}^{4} \psi_i^{(l)}(K^{(l)}). \tag{3}$$

We remark that for $l = 3$ we obtain the usual Koch curve.
Moreover, there exists a unique Borel regular measure $\mu^{(l)}$, with $supp\,\mu^{(l)} = K^{(l)}$, *invariant* with respect to $\Psi^{(l)}$, which coincides with the normalized $d_f(l)$-dimensional Hausdorff measure on $K^{(l)}$, where $d_f(l) = \frac{\ln 4}{\ln l}$,

$$\mu^{(l)} = (H^{d_f(l)}(K^{(l)}))^{-1} H^{d_f(l)}|_{K^{(l)}}. \tag{4}$$

The measure $\mu^{(l)}$ has the property that there exists two positive constants C_1, C_2, such that,

$$C_1 r^{d_f(l)} \leq \mu^{(l)}(B(P,r) \cap K^{(l)}) \leq C_2 r^{d_f(l)}\ , \quad \forall\, P \in K^{(l)}, \tag{5}$$

where $B(P,r)$ denotes the Euclidean ball with center in P and radius $0 < r \leq 1$. According to Jonsson and Wallin (see Ref. 10), we say that $K^{(l)}$ is a *d–set,* with $d = d_f(l)$.

Let K_0 be the line segment of unit length having as endpoints $A = (0,0)$ and $B = (1,0)$. We set, for each n in $\mathbb{N}$, $K_1^{(l)} = \bigcup_{i=1}^{4} \psi_i^{(l)}(K_0)$, $K_{n+1}^{(l)} = \bigcup_{i=1}^{4} \psi_i^{(l)}(K_n^{(l)})$; $\boldsymbol{K_n^{(l)}}$ is the so-called ***n*-th prefractal curve**. We have that $K_{n+1}^{(l)} = \bigcup_{M \in F_n^{(l)}} \bigcup_{i=1}^{4} \psi_i^{(l)}(M)$, where $F_n^{(l)} = \{M : M \text{ is a segment of } K_n^{(l)}\}$ denotes the set of segments of the n-th iterate $K_n^{(l)}$.



Fig. 2. The prefractal curves $K_n^{(l)}$ for $n = 4$ and $l = 3.8$, $l = 3$, and $l = 2.2$ respectively.

3. Functional spaces

In this section, we recall the definitions of some functional spaces which will be used in the following (we refer to Refs. 6 and 17 for a more complete discussion). Let D be an arbitrary open set of $\mathbb{R}^2$, we denote by $\boldsymbol{C_0^0(D)}$ and by $\boldsymbol{C_0^\infty(D)}$ the usual spaces of continuous or smooth functions with compact support on D, respectively. Let $\boldsymbol{L^2(\mu,\cdot)}$ be the Lebesgue space with respect to a measure μ on subsets of $\mathbb{R}^2$, which will be specified everytime. Moreover, let $\boldsymbol{H^s(D)}$ and $\boldsymbol{H_0^s(D)}$, where $s \in \mathbb{R}_+$, be the Sobolev spaces defined in Ref. 17. We consider the **trace Sobolev spaces** $H^s(\overset{\circ}{K}{}_n^{(l)})$ **on the prefractal curve** $\boldsymbol{K_n^{(l)}}$ for each $s \in \mathbb{R}_+$ defined according to Brezzi and Gilardi (see Definition 2.27 in Ref. 2), (see also Ref. 14). From now on, we shall denote $H^s(\overset{\circ}{K}_n)$ simply by $\boldsymbol{H^s(K_n)}$.
For f in $H^s(D)$, we define γ_0 as a **trace operator**

$$\gamma_0 f(x) := \lim_{r\to 0} \frac{1}{|B(x,r)\cap D|} \int_{B(x,r)\cap D} f(y)\,dy \tag{6}$$

at every point $x \in \overline{D}$ where the limit exists(see Ref. 1).

We recall the following trace Theorem on boundaries of polygonal domain (for a more general discussion, see Refs. 2, 6 and 17).

Theorem 3.1. *Let D be a bounded open subset of $\mathbb{R}^2$ whose boundary ∂D is a polygonal curve. For each $s > 1/2$, $H^{s-\frac{1}{2}}(\Gamma)$ is the trace space on $\Gamma \subset \partial D$ of $H^s(D)$ in the following sense:*

(i) *γ_0 is a continuous linear operator from $H^s(D)$ to $H^{s-\frac{1}{2}}(\Gamma)$;*
(ii) *there exists a continuous linear operator Ext from $H^{s-\frac{1}{2}}(\Gamma)$ to $H^s(D)$, such that $(\gamma_0 \circ Ext)$ is the identity operator in $H^{s-\frac{1}{2}}(\Gamma)$.*

We recall the definition of the **Lions-Magenes trace spaces** $H_{0,0}^{\frac{1}{2}}$ and the Green's formula on Lipschitz domains suitable for the study of the mixed

DOI: 10.1685/CSC06035

problems considered in this paper (see Ref. 15).

Theorem 3.2. *Let D be a bounded connected open subset of $\mathbb{R}^2$ whose boundary ∂D is Lipschitz. Let Γ be a connected open set of ∂D. Then, the following Green's formula*

$$\int_D \nabla u\, \nabla v\, dx\, dy + \int_D v\, \Delta u dx dy = < \frac{\partial u}{\partial \nu}, \gamma_0 v >_{((H^{\frac{1}{2}}_{0,0}(\partial D \setminus \overline{\Gamma}))', H^{\frac{1}{2}}_{0,0}(\partial D \setminus \overline{\Gamma}))} \quad (7)$$

holds, whatever $u \in H^1(D)$ such that $\Delta u \in L^2(D)$ and $v \in H^1(D)$ such that $\gamma_0 v = 0$ on Γ.

Here we denote

$$H^{\frac{1}{2}}_{0,0}(\partial D \setminus \overline{\Gamma}) := \left\{\theta \in L^2(\partial D \setminus \overline{\Gamma}) : \exists\, v \in H^1(D) : \gamma_0 v = 0 \text{ on } \Gamma,\ \gamma_0 v = \theta \text{ on } \partial D \setminus \overline{\Gamma}\right\},$$

equipped with the quotient norm

$$\|\theta\|_{H^{\frac{1}{2}}_{0,0}(\partial D \setminus \overline{\Gamma})} := \inf \left\{ \|v\|_{H^1(D)} : v \in H^1(D), \gamma_0 v = 0 \text{ on } \Gamma,\ \gamma_0 v = \theta \text{ on } \partial D \setminus \overline{\Gamma} \right\}.$$

($(X)'$ denotes the dual of X).

In order to consider traces on fractals, we use **Besov spaces** $B^{p,q}_{\alpha}$ on d-sets (for the definition, see, for instance, Refs. 10 and 19) and the following trace Theorem on d-sets (see Refs. 10, 19 and 20) and Green's formula on d-sets (see Ref. 12) specialized to our case.

Theorem 3.3. *Let D be a bounded open subset of $\mathbb{R}^2$. For each $s > 1 - \frac{d_f(l)}{2}$, $B^{2,2}_{s-1+\frac{d_f(l)}{2}}(K^{(l)})$ is the trace space to $K^{(l)}$ ($K^{(l)} \subset \overline{D}$) of $H^s(D)$ in the following sense:*

(i) *γ_0 is a continuous linear operator from $H^s(D)$ to $B^{2,2}_{s-1+\frac{d_f(l)}{2}}(K^{(l)})$,*

(ii) *there exists a continuous linear operator Ext from $B^{2,2}_{s-1+\frac{d_f(l)}{2}}(K^{(l)})$ to $H^s(D)$, such that $(\gamma_0 \circ Ext)$ is the identity operator in $B^{2,2}_{s-1+\frac{d_f(l)}{2}}(K^{(l)})$.*

Theorem 3.4. *Let D be a bounded connected open subset of $\mathbb{R}^2$, with boundary ∂D. Let Γ be a connected open set of ∂D and $\partial D \setminus \overline{\Gamma} = K^{(l)}$.*

Then, the following Green's formula

$$\int_D \nabla u\, \nabla v\, dx\, dy + \int_D v\, \Delta u dx dy = < \frac{\partial u}{\partial \nu}, \gamma_0 v >_{((B^{2,2}_{\beta,0}(\partial D \setminus \overline{\Gamma}))', B^{2,2}_{\beta,0}(\partial D \setminus \overline{\Gamma}))} \quad (8)$$

holds, whatever $u \in H^1(D)$ such that $\Delta u \in L^2(D)$ and $v \in H^1(D)$ such that $\gamma_0 v = 0$ on Γ.

Here we recall the definition of the space $\boldsymbol{B^{2,2}_{\beta,0}}$, which is the "fractal" analogue of the Lions-Magenes space $H^{\frac{1}{2}}_{0,0}$,

$$\boldsymbol{B^{2,2}_{\beta,0}(\partial D \setminus \overline{\Gamma})} := \left\{\theta \in L^2(\partial D\setminus\overline{\Gamma}) : \exists\, v \in H^1(D) : \gamma_0 v = 0 \text{ on } \Gamma,\ \gamma_0 v = \theta \text{ on } \partial D \setminus \overline{\Gamma}\right\},$$

equipped with the quotient norm

$$\|\theta\|_{B^{2,2}_{\beta,0}(\partial D\setminus\overline{\Gamma})} := \inf\left\{\|v\|_{H^1(D)} : v \in H^1(D), \gamma_0 v = 0 \text{ on } \Gamma,\ \gamma_0 v = \theta \text{ on } \partial D \setminus \overline{\Gamma}\right\}.$$

For a characterization of the dual of Besov spaces, see also Refs. 11 and 20.

4. Variational formulation

When the interface Γ_2 is the prefractal curve $K_n^{(l)}$ approximating the Koch curve type fractal $K^{(l)}$, we consider the following - formally stated - problem (which is equivalent to (1))

$$\begin{cases} -\Delta u = f & \text{in } \Omega_n^{(l)} \\ u = 0 & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \nu} + c_n u = d_n & \text{on } K_n^{(l)} \setminus \{A, B\} \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_3 \end{cases} \tag{9}$$

where f is a given function in $L^2(\Omega_n^{(l)})$, $c_n \geqslant 0$ and d_n are constant for each n in $\mathbb{N}$. In the present paper, the constants c_n and d_n will be left indetermined since we work with n fixed; in the asymptotic analysis, on the contrary, these constants will be chosen in a suitable way

We give the weak formulation of the previous problem (9) and, for every n in $\mathbb{N}$, we prove the existence and the uniqueness of its weak solution u_n.

Theorem 4.1. *For any $f \in L^2(\Omega_n^{(l)})$, there exists one and only one solution u_n of the following problem*

$$\begin{cases} \text{find } u_n \in V(\Omega_n^{(l)}) := \{u_n \in H^1(\Omega_n^{(l)}) :\ \gamma_0\, u_n = 0 \text{ on } \Gamma_0\} \quad \text{s.t.} \\ \int_{\Omega_n^{(l)}} \nabla u_n \,\nabla v\, dxdy + c_n \int_{K_n^{(l)}} \gamma_0 u_n\, \gamma_0 v\, ds = \int_{\Omega_n^{(l)}} f\, v\, dxdy + d_n \int_{K_n^{(l)}} \gamma_0 v\, ds \\ \forall\, v \in V(\Omega_n^{(l)}). \end{cases} \tag{10}$$

Moreover, u_n is obtained by

$$\min_{v\in V(\Omega_n^{(l)})} \left\{ \int_{\Omega_n^{(l)}} |\nabla v|^2\, dxdy + c_n \int_{K_n^{(l)}} |\gamma_0 v|^2\, ds - 2\int_{\Omega_n^{(l)}} f\, v\, dxdy - 2d_n \int_{K_n^{(l)}} \gamma_0 v\, ds \right\}.$$

DOI: 10.1685/CSC06035

Proof. The thesis follows by applying Lax-Milgram Theorem to the bilinear form $a_n(u_n, v) = \int_{\Omega_n^{(l)}} \nabla u_n \nabla v \, dx\, dy + c_n \int_{K_n^{(l)}} \gamma_0 u_n \gamma_0 v \, ds$, defined in $V(\Omega_n^{(l)}) \times V(\Omega_n^{(l)})$ (ds denotes the one-dimensional measure on $K_n^{(l)}$). This form is continuous and coercive in $V(\Omega_n^{(l)})$ by using the Poincarè inequality and Theorem 3.1. Moreover, the linear functional $F(v) = \int_{\Omega_n^{(l)}} f\, v \, dxdy + d_n \int_{K_n} \gamma_0 v \, ds$ is bounded in $V(\Omega_n^{(l)})$. □

We now consider the problem when the interface Γ_2 is the Koch curve type fractal $K^{(l)}$: it can be formally stated as

$$\begin{cases} -\Delta u = f & \text{in } \Omega^{(l)} \\ u = 0 & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \nu} + cu = d & \text{on } K^{(l)} \setminus \{A, B\} \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_3 \end{cases} \tag{11}$$

where f is a given function in $L^2(\Omega^{(l)})$, $c \geqslant 0$ and d are constant.

Proceeding as above we can prove existence and uniqueness of the weak solution.

Theorem 4.2. *For any $f \in L^2(\Omega^{(l)})$, there exists one and only one solution u of the following problem*

$$\begin{cases} \textit{find } u \in V(\Omega^{(l)}) := \{u \in H^1(\Omega^{(l)}) : \ \gamma_0 u = 0 \textit{ on } \Gamma_0\} \quad \textit{s.t.} \\ \int_{\Omega^{(l)}} \nabla u \nabla v \, dxdy + c \int_{K^{(l)}} u\, v \, d\mu^{(l)} = \int_{\Omega^{(l)}} f\, v \, dxdy + d \int_{K^{(l)}} v \, d\mu^{(l)} \\ \forall\, v \in V(\Omega^{(l)}). \end{cases} \tag{12}$$

Moreover, u is obtained by

$$\min_{v \in V(\Omega^{(l)})} \left\{ \int_{\Omega^{(l)}} |\nabla v|^2 \, dxdy + c \int_{K^{(l)}} |\gamma_0 v|^2 \, d\mu^{(l)} - 2 \int_{\Omega^{(l)}} f\, v \, dxdy - 2d \int_{K^{(l)}} \gamma_0 v \, d\mu^{(l)} \right\}$$

Proof. The thesis now follows by applying Lax-Milgram Theorem to the bilinear form $a(u, v) = \int_{\Omega^{(l)}} \nabla u \nabla v \, dxdy + c \int_{K^{(l)}} \gamma_0 u \gamma_0 v \, d\mu^{(l)}$, where $\mu^{(l)}$ denotes the normalized $d_f(l)$-dimensional Hausdorff measure on $K^{(l)}$ as in (4) and γ_0 is the trace operator defined in Theorem 3.3.

This form is continuous and coercive in $V(\Omega^{(l)})$ by using the Poincarè inequality in $\Omega^{(l)}$ and Theorem 3.3. Moreover, the linear functional $F(v) = \int_{\Omega^{(l)}} f\, v \, dxdy + d_n \int_{K^{(l)}} v \, d\mu^{(l)}$ is bounded in $V(\Omega^{(l)})$. □

We show in which sense the variational solution of the problem (10) solves the mixed Dirichlet-Robin problem introduced formally above.

Theorem 4.3. *The weak solution u_n of problem (10) solves in the following sense*

$$\begin{cases} -\Delta u_n = f & \text{in } L^2(\Omega_n^{(l)}) \\ u_n = 0 & \text{in } H^{\frac{1}{2}}(\Gamma_0) \\ \frac{\partial u_n}{\partial \nu} = 0 & \text{in } (H^{\frac{1}{2}}_{0,0}(\Gamma_1))' \\ \frac{\partial u_n}{\partial \nu} + c_n u_n = d_n & \text{in } (H^{\frac{1}{2}}_{0,0}(K_n^{(l)}))' \\ \frac{\partial u_n}{\partial \nu} = 0 & \text{in } (H^{\frac{1}{2}}_{0,0}(\Gamma_3))'. \end{cases} \tag{13}$$

Proof. The proof is obtained by usual duality arguments. Let us choose $v \in \mathcal{D}(\Omega_n^{(l)})$ in (10): we obtain $-\Delta u_n = f$ in the sense of distribution of $\Omega_n^{(l)}$, and, then, in $L^2(\Omega_n^{(l)})$ ($f \in L^2(\Omega_n^{(l)})$). The homogeneous Dirichlet boundary condition on Γ_0 - in the sense of the traces of functions belonging to $H^1(\Omega^{(l)})$ - follows from the fact that $u_n \in V(\Omega^{(l)})$. Moreover, the other boundary conditions follow from the Green's formula for Lipschitz domains (Theorem 3.2). □

In the fractal case, we obtain that the variational solution of problem (12) solves the mixed problem in the following sense.

Theorem 4.4. *The weak solution u of problem (12) solves in the following sense*

$$\begin{cases} -\Delta u = f & \text{in } L^2(\Omega^{(l)}) \\ u = 0 & \text{in } H^{\frac{1}{2}}(\Gamma_0) \\ \frac{\partial u}{\partial \nu} = 0 & \text{in } (H^{\frac{1}{2}}_{0,0}(\Gamma_1))' \\ \frac{\partial u}{\partial \nu} + cu = d & \text{in } (B^{2,2}_{\beta,0}(K^{(l)}))' \\ \frac{\partial u}{\partial \nu} = 0 & \text{in } (H^{\frac{1}{2}}_{0,0}(\Gamma_3))', \end{cases} \tag{14}$$

with $\beta = \frac{d_f(l)}{2}$.

Proof. As in the proof of the previous Proposition, let us choose $v \in \mathcal{D}(\Omega^{(l)})$ in (12): we obtain $-\Delta u = f$ in the sense of distribution of $\Omega^{(l)}$, and, then, in $L^2(\Omega^{(l)})$ ($f \in L^2(\Omega^{(l)})$). The homogeneous Dirichlet boundary condition on Γ_0 - in the sense of the traces of functions belonging to $H^1(\Omega^{(l)})$ - is fulfilled since $u \in V(\Omega^{(l)})$. The boundary conditions on Γ_1 and Γ_3 follow from the Green's formula for Lipschitz domains. Moreover, on the fractal

DOI: 10.1685/CSC06035

interface, the boundary condition holds in the sense of the dual of Besov spaces: this condition follows by Green's formula for d-sets (Theorem 3.4). □

5. Regularity results

In this section, we prove some regularity results for the weak solution of the problem (10) in terms of ordinary fractional Sobolev spaces and weighted Sobolev spaces. By the general theory of regularity, three elements are involved in determining the regularity of the solution of a variational problem, associated with a boundary value problem on an open set $D \subset \mathbb{R}^n$:

1) the regularity of the coefficients of the operator;

2) the geometry of the open set D and, in particular, the regularity of its boundary;

3) the type of boundary conditions imposed.

We remark that, since the coefficients are constants, the problem (9) is characterized by the irregular boundary and by the boundary mixed Dirichlet-Robin conditions. These two factors are crucial in determining some result of global regularity, that is, up to boundary of the open set D.

If we consider a boundary mixed Dirichlet-Robin problem on a regular domain, there are no problems when the Dirichlet condition is "disjoined" from the Robin one. For instance, we obtain H^2-regularity of the solution of the Dirichlet-Robin problem on $D = B(0,2) \setminus \overline{B(0,1)}$ with Dirichlet condition on $\partial B(0,1)$ and Robin condition on $\partial B(0,2)$ (see, for istance, Ref. 3).

The situation is completely different when, for example, the part of boundary where the Dirichlet condition holds meets the part of boundary where another type of condition is satisfied. For instance, if we consider $D = B(0,1) \cap \{0 < \theta < \pi\}$, $\Gamma_1 = \partial D \cap \{\theta = \pi\}$ and $\Gamma_0 = \partial D \setminus \overline{\Gamma_1}$ the function $u = \rho^{\frac{1}{2}} \sin(\frac{\theta}{2})$ solve the following mixed problem: $\Delta u = 0$, $u = g$ on Γ_0, $\frac{\partial u}{\partial \nu} = 0$ on Γ_1, where $g = \sin(\frac{\theta}{2})$ on $\partial D \cap \partial B(0,1)$ and $g = 0$ elsewhere on Γ_0. Even if the boundary is smooth near the point $(0,0)$, the solution u does not belong to H^s for $s \geqslant \frac{3}{2}$ (see Ref. 2).

On the other hand, the regularity of the boundary is involved in a decisive manner. For instance, let $D \subset \mathbb{R}^2$ be a bounded polygon: there exists a function u in $H_0^1(D)$ such that $\Delta u \in C^\infty(\overline{D})$ but u does not belong to $H^{1+\frac{\pi}{\omega}}$, where ω is the amplitude of the largest of the internal angles of the polygon. In particular, if the polygon is not convex, the solution of the Dirichlet problem cannot be in H^2 even if the datum f is very regular (see Refs. 6 and 7).

In the following, we give a result of regularity of the solution of the

problem (10) in terms of ordinary fractional Sobolev spaces.

Theorem 5.1. *Let u_n be the weak solution of problem (10), then*

$$u_n \in H^s(\Omega_n^{(l)}), \text{ with } \quad s < 1 + \frac{\pi}{\pi + \theta(l)}, \tag{15}$$

where $\theta(l) = \arcsin\left(\frac{\sqrt{l(4-l)}}{2}\right)$, *for* $2 < l < 4$.

Proof. From Proposition 3.1 of Ref. 16, we deduce that the variational solution of an elliptic problem with mixed Dirichlet-Robin conditions on a polygonal domain has the *Regularity-Decomposition property* with the same number of singular functions as in the case of an elliptic problem with mixed Dirichlet-Neumann conditions. Before recalling this property, we introduce some notations. We denote each segment of the polygonal boundary of $\Omega_n^{(l)}$ by $\overline{\Lambda_j}$, Λ_j open, where the index j ranges from 1 to $3 + 4^n$; these segments are numbered in such a way that Λ_{j+1} follows Λ_j according to the positive orientation. We denote by S_j the vertex of $\Omega_n^{(l)}$ which is the right endpoint of Λ_j and by ω_j the amplitude of the interior angle at S_j. The polar coordinates with origin at S_j will be denoted by r_j and θ_j. For every j, we introduce a truncation function $\eta_j \in \mathcal{D}(\overline{\Omega^{(l)}})$ which depends only on the distance r_j to S_j such that $\eta_j \equiv 1$ near S_j and η_j vanishes near all $\overline{\Lambda_k}$ but for $k = j$ and $k = j + 1$. Moreover, the union of the Λ_j with $j \in \mathbf{D}$ (resp. $j \in \mathbf{R}$) is going to be the part of the boundary where we consider a Dirichlet (resp. Robin) boundary condition.
The *Regularity-Decomposition property* explains the behaviour of the variational solution of these type of problems by showing that the solution can be written as a sum of a *regular* function that belongs to H^2 and a finite number of *singular* functions (not belonging to H^2) (see Theorem 2.4.3 in Ref. 7). More precisely, if u_n is the weak solution of problem (10), there exist unique numbers $c_{j,m}$ ($m \in \mathbb{N}, m \geqslant 1$), such that,

$$u_n - \sum_{1 \leqslant j \leqslant 3+4^n} \left\{ \sum_{0 < \lambda_{j,m} < 1} c_{j,m} S_{j,m} \right\} \in H^2(\Omega_n^{(l)}), \tag{16}$$

where

$$S_{j,m}(r_j, \theta_j) = \eta_j(r_j) r_j^{\lambda_{j,m}} \varphi_{j,m}(\theta_j),$$

with

- for $j \in \mathbf{D}$ and $j + 1 \in \mathbf{D}$, $\varphi_{j,m}(\theta) = \sqrt{\frac{2}{\omega_j}} \sin(\theta \lambda_{j,m})$, $\lambda_{j,m} = \frac{m\pi}{\omega_j}$,

DOI: 10.1685/CSC06035

- for $j \in \mathbf{R}$ and $j+1 \in \mathbf{D}$, $\varphi_{j,m}(\theta) = \sqrt{\frac{2}{\omega_j}} \sin(\theta\lambda_{j,m})$, $\lambda_{j,m} = \frac{(m-\frac{1}{2})\pi}{\omega_j}$,

- for $j \in \mathbf{D}$ and $j+1 \in \mathbf{R}$, $\varphi_{j,m}(\theta) = \sqrt{\frac{2}{\omega_j}} \sin((\omega_i - \theta)\lambda_{j,m})$, $\lambda_{j,m} = \frac{(m-\frac{1}{2})\pi}{\omega_j}$,

- for $j \in \mathbf{R}$ and $j+1 \in \mathbf{R}$, $\varphi_{j,m}(\theta) = \sqrt{\frac{2}{\omega_j}} \cos(\theta\lambda_{j,m})$, $\lambda_{j,m} = \frac{(m-1)\pi}{\omega_j}$

when $m \geqslant 2$, $\varphi_{j,1}(\theta) = \sqrt{\frac{1}{\omega_j}}$, $\quad \lambda_{j,1} = 0$ when $m = 1$.

From (16), we obtain a precise description of the behavior of the solution near the various corners. In fact, let V_j be an open neighborhood of S_j which does not contain any other corner. Then $u_n \in H^s(V_j)$ for every $s \leqslant 2$ such that

$$s < 1 + \inf_m \{\lambda_{j,m},\, 0 < \lambda_{j,m} < 1\}$$

(see Corollary 2.4.4 in Ref. 7). Globally, the previous estimate gives that $u_n \in H^s(\Omega_n^{(l)})$ for every $s \leqslant 2$ such that

$$s < 1 + \inf_{j,m} \{\lambda_{j,m},\, 0 < \lambda_{j,m} < 1\}$$

(see Remark 2.4.6 in Ref. 7). By considering the amplitude of the angles in $\Omega_n^{(l)}$ and the boundary conditions of problem (9), we obtain estimate (15). □

For numerical purposes, the regularity of u_n in terms of weighted Sobolev spaces (for the definition, see Ref. 6), is more useful than the one in terms of ordinary fractional Sobolev spaces. We note that the regularity of the weak solution u_n in terms of weighted Sobolev spaces follows from estimate (15) (see Refs. 6, 7 and 9).

Theorem 5.2. *Let u_n be the weak solution of the problem (10). Then*

$$u_n \in H^{2,\alpha}(\Omega_n^{(l)}), \text{ with } \quad \alpha > \frac{\theta(l)}{\pi + \theta(l)}, \tag{17}$$

where $\theta(l) = \arcsin\left(\frac{\sqrt{l(4-l)}}{2}\right)$, *for* $2 < l < 4$.

In particular, we remark that when $K^{(l)}$ is the usual Koch curve we obtain that $u_n \in H^s(\Omega_n^{(l)})$, with $s < \frac{7}{4}$ and $u_n \in H^{2,\alpha}(\Omega_n^{(l)})$, with $\alpha > 1/4$.

Acknowledgements

The author wishes to thank Prof. Maria Agostina Vivaldi for the several stimulating discussions during the preparation of this paper.

References

1. Adams D.R., Hedberg L.I. "Function spaces and Potential Theory", Springer-Verlag, Berlin, 1966.
2. Brezzi F., Gilardi G. "Finite Elements Mathematics", in *Finite Element Handbook*, Eds. Kardestuncer H., D.H.Norrie, MacGraw-Hill Book Co., New York, 1987.
3. Dautray R., Lions J.L. "Mathematical analysis and numerical methods for science and technology", Vol.2, Springer-Verlag, Berlin, 1984.
4. Filoche M., Sapoval B. *Transfer across random versus Deterministic Fractal Interfaces*, Phys. Rev. Lett. 84 (2000), 5776-5779.
5. D. S. Grebenkov D. S. , Filoche M., Sapoval B. *Mathematical Basis for a General Theory of Laplacian Transport towards Irregular Interfaces*, Phys. Rev. E 73, 021103, 2006.
6. Grisvard P. "Elliptic problems in nonsmooth domains", Pitman, Boston, 1985.
7. Grisvard P. "Singularities in boundary value problems", Paris, Masson, 1992.
8. Hutchinson J.E. *Fractals and selfsimilarity*, Indiana Univ. Math. J. 30 (1981), 713-747.
9. Jerison D., Kenig C.E. *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal. 130 (1995), 161–219.
10. Jonsson A., Wallin H. *Function spaces on subsets of* $\mathbb{R}^n$, Math. Rep. 2 (1984), no. 1, xiv+221.
11. Jonsson A., Wallin H. *The dual of Besov spaces on fractals*, Studia Math. 112 (1995), no. 3, 285–300.
12. Lancia M.R. *A transmission problem with a fractal interface*, Z. Anal. und Ihre Anwend. 21 (2002), 113-133.
13. Lancia M.R., Vivaldi M.A. *On the regularity of the solutions for transmission problems*, Adv. Math. Sc. Appl. 12 (2002), 455-466.
14. Lancia M.R., Vivaldi M.A. *Asymptotic convergence of transmission energy forms*, Adv. Math. Sc. Appl. 13 (2003), p. 315-341.
15. Lions J.L, Magenes E. "Non-Homogeneous Boundary Value Problems and Applications", Springer-Verlag, Berlin,1972.
16. Mghazli Z. *Regularity of an elliptic problem with mixed Dirichlet-Robin boundary conditions in a polygonal domain*, Calcolo 29 (1992), no. 3-4, 241–267 (1993).
17. Nečăs J. "Les méthodes directes en théorie des équationes elliptiques", Paris, Masson, 1967.
18. Sapoval B. *General formulation of laplacian transfer across irregular surfaces*, Phys. Rev. Lett. 73 (1994), 3314-3316.
19. Triebel H. "Fractals and spectra. Related to Fourier analysis and function spaces." Monographs in Mathematics, 91. Birkhäuser Verlag, Basel, 1997.
20. Wallin H. *The trace to the boundary of Sobolev spaces on a snowflake*, Manuscripta Math. 73 (1991), no. 2, 117–125.