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On the numerical controllability of the Ginzburg-Landau equation[1]

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The Ginzburg-Landau three dimensional system is considered in a bounded domain Ω of $\mathbb{R}^2$ with distributed control. The aim of the paper is the numerical investigation of the asymptotic controllability.

Let ω a nonempty subset of Ω and χ_ω its characteristic function, we assume $\mathbf{m}^0 \in H^1(\Omega;\mathbb{R}^d)$ $(d = 1, 2, 3)$ and consider the general Ginzburg-Landau equation with non homogeneous Dirichlet boundary condition

$$\text{(0.1)} \qquad \begin{cases} \mathbf{m}_t - \Delta\mathbf{m} + \dfrac{|\mathbf{m}|^2 - 1}{\varepsilon}\mathbf{m} = \mathbf{f}\chi_\omega & \text{in } Q = \Omega \times (0,T), \\ \mathbf{m} = \mathbf{m}^0 & \text{on } \Sigma = \partial\Omega \times (0,T), \\ \mathbf{m}(\cdot,0) = \mathbf{m}^0 & \text{in } \Omega, \end{cases}$$

where $\mathbf{f} \in L^2(\Omega \times (0,T);\mathbb{R}^d)$ is the unknown control function to be determined in order to drive the solution to a suitable state at the time T.
It is known that for every $\mathbf{f} \in L^2(\Omega \times (0,T);\mathbb{R}^d)$ and for every $\mathbf{m}^0 \in H^1(\Omega;\mathbb{R}^d)$ there exists an unique solution $\mathbf{m} \in H^{2,1}(\Omega \times (0,T);\mathbb{R}^d)$ to (0.1), where

$$H^{2,1}(\Omega \times (0,T)) = \{y \in L^2(0,T;H^2(\Omega)), y_t \in L^2(\Omega \times (0,T))\}.$$

The Ginzburg-Landau equation can model some phenomena in materials science as, for example, the phase transitions (d=1), the dynamics of vortices in superconductors (d=2) or the evolution of singularities for certain nematic liquid crystals (d=3). Let $\mathbf{m}^*$ be the solution to the stationary problem

$$\text{(0.2)} \qquad \begin{cases} \Delta\mathbf{m}^* - \dfrac{|\mathbf{m}^*|^2 - 1}{\varepsilon}\mathbf{m}^* = 0 & \text{in } \Omega, \\ \mathbf{m}^* = \mathbf{m}^0 & \text{on } \partial\Omega. \end{cases}$$

We can consider the following problems of controllability to $\mathbf{m}^*$.

Exact Controllability Problem: given $\mathbf{m}^0 \in H^1(\Omega;\mathbb{R}^d)$ and $T > 0$, to find the control function $\mathbf{f} \in L^2(\Omega \times (0,T);\mathbb{R}^d)$ such that the solution to the problem (0.1) satisfies the condition $\mathbf{m}(T) = \mathbf{m}^*$.
The system (0.1) is said to be *locally controllable* at time T to the stationary solution $\mathbf{m}^*$ if there exists a control function $\mathbf{f}$ and a positive constant γ such that for any initial data $\mathbf{m}^0$ satisfying $\|\mathbf{m}^0 - \mathbf{m}^*\|_{H_0^1(\Omega;\mathbb{R}^d)} < \gamma$, the solution $\mathbf{m}$ of system (0.1) satisfies $\mathbf{m}(T) = \mathbf{m}^*$ a.e. in Ω.

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Approximate Controllability Problem : given $\mathbf{m}^0$ in $H^1(\Omega;\mathbb{R}^d)$, $T>0$ and fixed $\sigma>0$, find the control function $\mathbf{f}\in L^2(\Omega\times(0,T);\mathbb{R}^d)$ such that the solution to the problem (0.1) satisfies $\|\mathbf{m}(T)-\mathbf{m}^*\|_{L^2(\Omega;\mathbb{R}^d)}\leq\sigma$.

The problem of controllability to steady states can be reduced to a null controllability problem. Indeed introducing the function $\mathbf{y}=\mathbf{m}-\mathbf{m}^*$ one has

$$\mathbf{y}_t-\Delta\mathbf{y}+\mathbf{G}(\mathbf{m}^*,\mathbf{y})=\chi_\omega\mathbf{f}\qquad\text{in }\Omega\times(0,T) \tag{0.3}$$

where

$$\mathbf{G}(\mathbf{m}^*,\mathbf{y})=\frac{|\mathbf{y}|^2+|\mathbf{m}^*|^2+2\mathbf{y}\cdot\mathbf{m}^*-1}{\varepsilon}\mathbf{y}+\frac{|\mathbf{y}|^2+2\mathbf{y}\cdot\mathbf{m}^*}{\varepsilon}\mathbf{m}^* \tag{0.4}$$

with the associated boundary and initial conditions

$$\mathbf{y}=0\quad\text{on }\partial\Omega\times(0,T),\qquad\mathbf{y}(\cdot,0)=\mathbf{y}^0\quad\text{in }\Omega. \tag{0.5}$$

Moreover $\mathbf{G}(\mathbf{m}^*,\mathbf{y})$ satisfies the condition $\mathbf{G}(\mathbf{m}^*,\mathbf{y})\cdot\mathbf{y}\geq-\varepsilon^{-1}|\mathbf{y}|^2$.

The controllability of the parabolic equations in the linear and nonlinear case are widely studied in the last years from a theoretical point of view, we quote among others the pioneer papers by J.L. Lions [7],[8], and the succeeding papers [2], [5]. For the qualitative study of local controllability problem for Ginzburg-Landau equation, we can look at the results established in [3] where the local controllability to the stationary solutions of the phase-field model is proved by two control forces localized on the same subdomain, or also in [1] where the local controllability to the trajectories of phase-field models is obtained by one control force acting on a single equation of the system.

As a common praxis the controllability of the nonlinear problem is reduced to a sequence of controllability linear problems. The desired result for the nonlinear system follows then from a fixed point argument. So we introduce, for $l=0,1..$, the systems

$$\mathbf{y}_t^{l+1}-\Delta\mathbf{y}^{l+1}+\mathbf{A}(\mathbf{m}^*,\mathbf{y}^l)\mathbf{y}^{l+1}=\chi_\omega\mathbf{f}^{l+1}\qquad\text{in }\Omega\times(0,T] \tag{0.6}$$

where $\mathbf{A}=A_{ij}$, with $i,j=1,...,d$, defined by

$$A_{ij}(\mathbf{m}^*,\mathbf{z})=\varepsilon^{-1}(|\mathbf{z}+\mathbf{m}^*|^2-1)\delta_{ij}+\varepsilon^{-1}(z_j+2m_j^*)m_i^*,\qquad i,j=1,...,d \tag{0.7}$$

and the associated boundary and initial conditions

$$\mathbf{y}^{l+1}=0\text{ on}\quad\partial\Omega\times(0,T],\qquad\mathbf{y}^{l+1}(\cdot,0)=\mathbf{y}^0\text{ in }\Omega. \tag{0.8}$$

For each fixed l and $\mathbf{z}$ we solve the null controllability of the linear system (0.6), (0.7), (0.8) by means of a duality argument, firstly introduced by J.L. Lions (see for example [8]) in the framework of the approximate controllability, which reduces the control problem to the minimization of the functional

$$\mathcal{J}(\mathbf{p}^0)=\frac{1}{2}\iint_{\omega\times(0,T)}|\mathbf{p}|^2d\mathbf{x}dt+\sigma\|\mathbf{p}^0\|_{L^2(\Omega;\mathbb{R}^d)}+\int_\Omega\mathbf{p}(0)\mathbf{y}^0d\mathbf{x}. \tag{0.9}$$

where $\mathbf{p}$ is the solution of the associated backward system

$$\begin{cases}-\mathbf{p}_t-\Delta\mathbf{p}+\mathbf{A}(\mathbf{m}^*,\mathbf{z})\,\mathbf{p}=0 & \text{in }\Omega\times(0,T),\\ \mathbf{p}=0 & \text{on }\partial\Omega\times(0,T),\\ \mathbf{p}(\cdot,T)=\mathbf{p}^0 & \text{in }\Omega.\end{cases} \tag{0.10}$$

The minimization of the functional $\mathcal{J}$ is the crucial point of the controllability problem. The numerical simulation shown below is carried out applying the Conjugate Gradient Method for finding the function $\mathbf{p}^0$ such that

$$\mathcal{J}(\mathbf{p}^0) = \min_{\mathbf{q}^0 \in L^2(\Omega;\mathbb{R}^d)} \mathcal{J}(\mathbf{q}^0)$$

The rate of convergence of the CGM depends in particular on the the subset ω and requires many iterations when the volume of ω decreases. Numerical experiments with more efficient minimization methods (in particular variants of the quasi Newton method) will be the contest of a forthcoming paper. The numerical complexity of the minimization problem arises in the computation of the gradient of the functional $\mathcal{J}$ defined by the variational equation

$$\int_\Omega \mathbf{q}^0 \, [\int_0^T \mathbf{g}^s(T) ds] d\mathbf{x} + \frac{\sigma}{\|\mathbf{p}^0\|_{L^2(\Omega;\mathbb{R}^d)}} \int_\Omega \mathbf{q}^0 \, \mathbf{p}^0 d\mathbf{x} + \int_\Omega q^0 \, \mathbf{w}(T) d\mathbf{x} = 0 \tag{0.11}$$

where $\mathbf{w}$ is the solution to (0.6), (0.7), (0.8) with $\mathbf{f} = 0$ and $\mathbf{g}^s$ solves the problem

$$\begin{cases} \mathbf{g}_t - \Delta \mathbf{g} + \mathbf{A}(\mathbf{m}^*, \mathbf{z})\, \mathbf{g} = 0 & \text{in } \Omega \times (s, T), \\ \mathbf{g} = 0 & \text{on } \partial\Omega \times (s, T), \\ \mathbf{g}(\mathbf{x}, s) = \mathbf{p}(\mathbf{x}, s)\chi_\omega & \text{in } \Omega. \end{cases} \tag{0.12}$$

Finally we want to point out that for a suitable choice of the static solution $\mathbf{m}^*$, as for the case of the trivial solution $\mathbf{m}^* = (0, 0, 1)$ used in the next section for the numerical experiments, the linearized system reduces to a cascade system with d equations which can be independently studied (see also [6]). Indeed in this case the matrix $\mathbf{A}$ defined in (0.7) reduces to a triangular matrix.

The computation algorithm we propose requires the numerical implementation of the minimization method (CGM) and the fixed point iterative procedure. In all the tests here reported we assume $\Omega = (0, 1) \times (0, 1)$, $d = 3$ and $\mathbf{m}^* = (0, 0, 1)$. The parabolic equations are approximated by an Euler explicit scheme for the derivative in time and a finite difference scheme of five points for the approximation in the space using a uniform grid with steps δt for the time variable and h for the two space variables. The iterative minimization method arrests when the $L^2(\Omega)$-norm of the gradient of the functional $\mathcal{J}(\mathbf{p}^0)$ is less than 10^{-5}. In order to solve the nonlinear case we have used a standard fixed point method with the stopping criterium $\sup_{\Omega\times[0,T]} |\mathbf{y}^{l+1} - \mathbf{y}^l| \leq 10^{-4}$.

We have considered the following initial data both of them in $H_0^1(\Omega; S^2)$

$$y_1^0 = y_2^0 = \sin(x\pi)\sin(y\pi) \cdot 0.1, \quad y_3^0 = -1 + \sqrt{1 - (y_1^0)^2 - (y_2^0)^2} \tag{0.13}$$

or

$$y_1^0 = \frac{x}{r}\sin f(r), \quad y_2^0 = \frac{y}{r}\sin f(r), \quad y_3^0 = \cos f(r) - 1, \tag{0.14}$$

where $r = \sqrt{(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2}$ and $f(r) = 8\pi r(r - 1)$ for $r \leq 1/2$ and $f(r) = 0$ otherwise.

The tests are carried out using the following parameter values: $h = 0.008$ and $\delta t = 10^{-1}h^2$. Two different examples are presented corresponding to the initial data (0.13)

and (0.14) respectively. In the example 1 we consider the local exact controllability, we take $\sigma = 0$ and the initial datum (0.13). In the example 2 we consider the initial datum (0.14) and for the tests concerning the approximate controllability we take $\sigma = 0.1$. In both the examples we assume $\omega = \Omega$ and $\varepsilon = 0.0002$.

Example 1. In this example we have considered the initial data (0.13), that verifies the smallness condition required by the theory to obtain the local exact controllability. In Figure 0.1 we report the plots of the numerical experiments obtained when the control is applied only in the first two components of the system, that is we take the third component of the control function equal to zero. Thanks to the choice of $\mathbf{m}^*$ we reduce to consider a cascade system. We show in figure 0.1 that one can control also the last component, and hence the whole system, by controls acting on the first ones. Since $\mathbf{m}^*$ is a stable steady state and $\mathbf{m}^0$, computed by (0.13), is a perturbation of $\mathbf{m}^*$ we obtain, according also to the theoretical controllability results, that the numerical simulation is not so surprising. More emphasis will be given to the next example concerning the control of large gradient solutions.

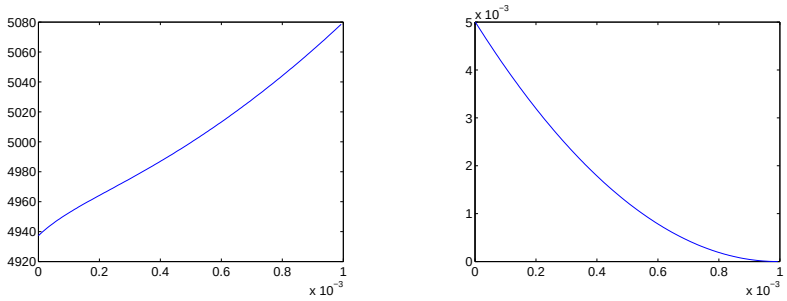


Figure 0.1: Profiles versus time of $\|\mathbf{f}\|_{L^2(\Omega)}$ with $f_3 = 0$ (on the left) and $\|\mathbf{y}\|_{L^2(\Omega)}$ (on the right).

Example 2. The experiments shown in this example are related to the controllability of blowing-up solutions. More precisely we consider the initial datum (0.14) which has been proved ([4]) to develop singularities, in the sense that the L^∞-norm of the gradient of the

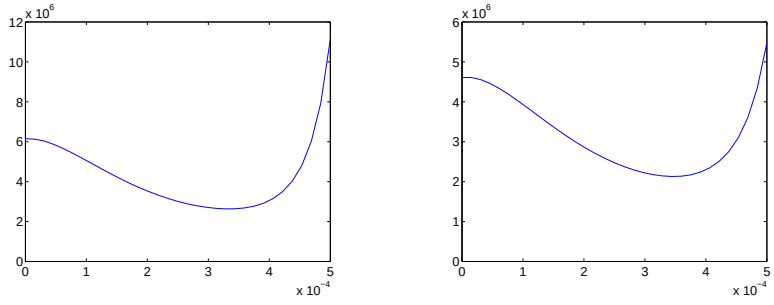


Figure 0.2: Profiles versus time of $\|\mathbf{f}\|_{L^2(\Omega)}$. Local exact controllability (on the left) and approximate controllability (on the right).

solution blows-up in a finite time, when $\varepsilon \to 0$. The procedure we have used to control the singularity lies in applying the control action before the blow up of the gradient of the solution occurs. In this example we have also asked for an approximate controllability property, since less control is needed to drive the solution to an approximate desired state and then, since we have a parabolic system, we can take advantage of the dissipation

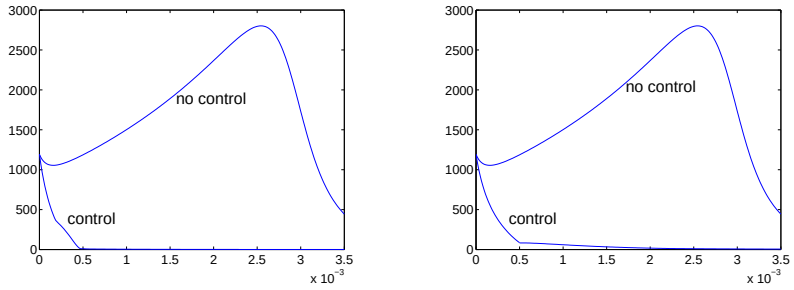


Figure 0.3: Profiles versus time of $\|\nabla \mathbf{m}\|_{L^\infty(\Omega)}$. Exact (on the left) and approximate (on the right) controllability.

property for driving exactly the solution to the stable state $\mathbf{m}^*$. We can see in the next graphics (Figures 0.2 and 0.3) the comparison between the results obtained both in the case of the local exact controllability to the stationary solutions and in the case of the approximate controllability. In the plots of Figure 0.4 we report the results obtaining in the case of approximate controllability with only one control ($f_1 = f_2 = 0$) on the third components of the system. Although it is not a natural way to control a cascade system the goal to drive the system to the steady state $\mathbf{m}^*$ is again reached. The shape of the $L_\infty(\Omega)$ gradient norm of the controlled state brings to the conjecture of a blow-up in advance when $\varepsilon \to 0$.

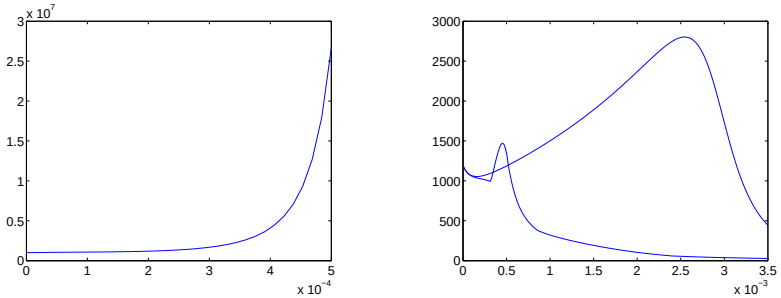


Figure 0.4: Profiles versus time of $\|f_3\|_{L^2(\Omega)}$ (on the left) and $\|\nabla \mathbf{m}\|_{L^\infty(\Omega)}$ (on the right) compared to non controlled behavior.

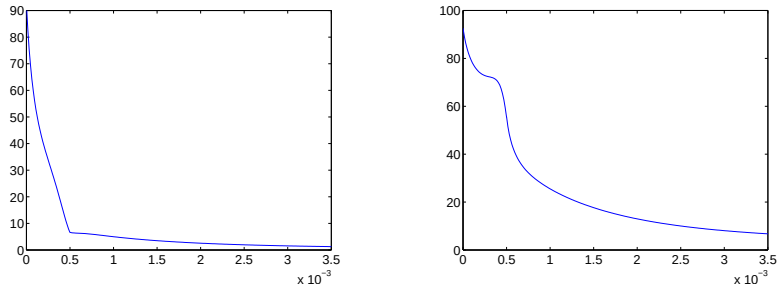


Figure 0.5: Profiles versus time of $\|\nabla \mathbf{m}\|_{L^2(\Omega)}$ when three control are applied (on the left) and only one control is applied (on the right).

Finally, in Figure 0.5 it is shown the behavior of $\|\nabla \mathbf{m}\|_{L^2(\Omega)}$, representative of the system energy, when we apply the control both in all the equations and in the last one

of the system, within the framework of the approximate controllability. The sudden decrease of the energy corresponding to the pick in the gradient observed in figure 0.4 on right, confirms the conjecture of a blow-up in advance.

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