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Monomial orders in the vast world of mathematics

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1 Introduction

Algebraic models of phenomena arising from different fields utilize the polynomial ring P in a finite number of variables, with coefficients in a field K (of characteristic zero). The study of phenomena does not always require to consider a total order on the multiplicative set of monomials. However, in the last years,this is the more attractive direction. Moreover, the classic lexicographic order has been substituted by other orders, sometimes more powerful. Examples of total orders different from the lexicographic order can be found in recent papers of algebraic combinatoric and commutative computational algebra. Algebras of Veronese type $K[L]$ (subrings of polynomial rings) are present in a lot of models like graphs, generalized graphs, transportation problems, economy. Segre products of type Veronese algebras are employed to study bipartite graphs, mixed bipartite generalized graphs, transportation problems and to study varieties of Veronese type and Segre varieties in algebraic geometry. The crucial points are the determination and the study of the toric ideal I of $K[L]$ that we would like to be generated from binomials of the lower possible degree(degree two). From the geometric point of view this means that our algebraic variety is an intersection of quadrics.The order on the monomials needs to determine a Gröbner basis G of I since the structure of G reflects strong properties of $K[L]$ (the binomials of degree two of G produce very good properties of the algebra $K[L]$, like Koszul, strongly Koszul).The interest to have a Gröbner basis of low degree answers to the request to have S-couples of the same degree of equations, in the not linear case . The linear case is in fact completely known (theory of circuits). Our problems are the following:
1. Research of monomial orders that produce Gröbner bases of the minimum possible degree for the ideal I.
2. Determination of the Gröbner basis of toric ideals, not necessarily of degree two.
We will examine monomial orders in a polynomial ring with multi indexed variables. They appear in the presentation of so called "subrings of mixed products". They come from ideals of mixed products studied in [4] and they can be associated to some special complete generalized bipartite graphs.

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More precisely, let P and Q be two polynomial rings, in one set of variables (respectively $\underline{X} = (X_1, \ldots, X_m)$, $\underline{Y} = (Y_1, \ldots, Y_m)$), let I_r be the ideal of P generated by all square-free monomials $X_{i_1} \cdots X_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq m$ and J_k the ideal of Q generated by all square-free monomials $Y_{j_1} \cdots Y_{j_s}$, $1 \leq j_1 < j_2 < \cdots < j_s \leq m$.
Given two integers k, r, s, t such that $k + r = s + t$, we consider the square-free monomial ideal of mixed products of the ring $R = K[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ given by $L = I_r J_k + I_s J_t$. If F is the minimal generating set of L consisting of square-free monomials, let $K[F]$ be the monomial mixed algebra,subring of $K[X_1, \ldots, X_m; Y_1, \ldots, Y_m]$. Let A be the ring such that $K[F] = A/J$. The monomial order we are going to utilize in A is the bi-sorted order. It is known in the case of one set of variables. This order is neither lexicographic, nor reverselexicographic, but it is useful to succeed in our research.
Project: The aim is to resolve a geometric conjecture for normality of toric varieties.
Result: The conjecture has a positive answer for $K[F]$.
For F given by L, we prove the affine toric variety $X_{\mathcal{A}}$ defined by J is normal. Consequently, we deduce the normality of the semigroup $\mathbb{N}\mathcal{A}$, where $\mathcal{A}$ is the finite subset of Z^d, underlying the ring $K[F]$. Moreover, the semigroup $\mathcal{A}$ admits an unimodular triangulation ([3], $cfr.$13). In fact in any case we determine a square-free quadratic Gröbner basis for J, that is sufficient for normality. The converse is not true. We have examples of projectively normal toric varieties without square-free initial ideal.

For $\mathcal{A} = \{(0,0,0,0,1), (1,0,0,0,1), (0,1,0,0,1), (0,0,1,0,1), (1,1,1,1,1),$
$(1,1,2,2,1), (1,2,2,3,1), (1,2,3,4,1), (1,2,3,5,1)\}$, the 4- dimensional projectively normal toric variety in $\mathbb{P}^{18}$ is such that no initial ideal of the toric ideal of $X_{\mathcal{A}}$ $I_{\mathcal{A}}$ is square-free. We refer to [3], chap.4 for the theory of the sorted order.

2 Bi-sorted monomials ideals in two sets of variables

We fix two integers r and s and we consider the set

$$T_{n,m} = \{(i_1, \ldots, i_n; j_1, \ldots, j_m)\} \subset \mathbb{N}^n \oplus \mathbb{N}^m,$$

$i_1 + \cdots + i_n = r$, $j_1 + \cdots + j_m = s$, $0 \leq i_1 \leq r_1, \ldots, 0 \leq i_n \leq r_n$ and $0 \leq j_1 \leq s_1, \ldots, 0 \leq j_m \leq s_m$.

There is a natural bijection among the elements of $T_{n,m}$ and increasing strings of length $r+s$ over the alphabet $A = \{1, 2, \ldots, n\}$ until r and over the alphabet $A' = \{1, 2, \ldots, m\}$ from $r+1$ until s and having at most r_t occurrences of the letter t until r and at most s_t occurrences of the letter t from $r+1$ until s. Under this bijection the vector $(i_1, \ldots, i_n; j_1, \ldots, j_m) \in T_{n,m}$ is mapped to the increasing string

$$u_1 u_2 \ldots u_r v_1 v_2 \ldots v_s = \underbrace{11\ldots1}_{i_1-times}\underbrace{22\ldots2}_{i_2-times}\ldots\underbrace{nn\ldots n}_{i_n-times};\underbrace{11\ldots1}_{j_1-times}\underbrace{22\ldots2}_{j_2-times}\ldots\underbrace{mm\ldots m}_{j_m-times}$$

We write $X_{u_1 u_2 \ldots u_r v_1 v_2 \ldots v_s}$ for the corresponding variable in our polynomial ring $K[\underline{X}]$. Let sort $(\cdot;\cdot)$ denote the operation which takes any string over the two alphabets A and A' and sorts it into increasing order.

Proposition 2.1. *The toric ideal defined by the set $T_{n,m}$ equals:*

$$I_{T_{n,m}} = (X_{\underline{u_1};\underline{v_1}} X_{\underline{u_2};\underline{v_2}} \cdots X_{\underline{u_\rho};\underline{v_\rho}} - X_{\underline{u'_1};\underline{v'_1}} X_{\underline{u'_2};\underline{v'_2}} \cdots X_{\underline{u'_\rho};\underline{v'_\rho}}) :$$

$$sort(\underline{u_1 u_2} \cdots \underline{u_\rho}; \underline{v_1 v_2} \cdots \underline{v_\rho}) = sort(\underline{u'_1 u'_2} \cdots \underline{u'_\rho}; \underline{v'_1 v'_2} \cdots \underline{v'_\rho}).$$

Now, we consider the polynomial ring $K[\underline{X}]$.

DEFINITION 2.1. *A monomial* $X_{\underline{u};\underline{v}}X_{\underline{u'};\underline{v'}}\cdots X_{\underline{u^{(i)}};\underline{v^{(i)}}} = X_{u_1u_2\cdots u_r;v_1v_2\cdots v_r}\cdot X_{u'_1u'_2\cdots u'_r;v'_1v'_2\cdots v'_r}\cdots X_{u_1{}^{(i)}u_2{}^{(i)}\cdots u_r{}^{(i)};v_1{}^{(i)}v_2{}^{(i)}\cdots v_r{}^{(i)}}$ *of* $K[\underline{X}]$ *is said to be bi-sorted if*

1) $u_1 \leq u'_1 \leq \cdots \leq u_1{}^{(i)} \leq u_2 \leq u'_2 \leq \cdots \leq u_2{}^{(i)} \cdots u_r \leq u'_r \leq \cdots \leq u_r{}^{(i)}$

2) $v_1 \leq v'_1 \leq \cdots \leq v_1{}^{(i)} \leq v_2 \leq v'_2 \leq \cdots \leq v_2{}^{(i)} \cdots v_r \leq v'_r \leq \cdots \leq v_r{}^{(i)}$

For monomials which are not bi-sorted we define the inversion numbers pair (a, b) as the numbers of inversions in the two strings 1) and 2) respectively. We define inversion in the string 1) or 2) a pair of indices (i, j) such that $i < j$ and $u_i > u_j$ in 1) and $v_i > v_j$ in the string 2).

PROPOSITION 2.2. *We have the following facts:*

1) *Every variable is sorted*

2) *Every power of variable is sorted*

3) *If a monomial is not bi-sorted, then it contains a quadratic factor which is not bi-sorted.*

4) *For every binomial*

$$f = X_{\underline{u_1};\underline{v_1}}X_{\underline{u_2};\underline{v_2}}\cdots X_{\underline{u_\rho};\underline{v_\rho}} - X_{\underline{u'_1};\underline{v'_1}}X_{\underline{u'_2};\underline{v'_2}}\cdots X_{\underline{u'_\rho};\underline{v'_\rho}} \in I_A$$

such that

$$sort(\underline{u_1}\cdots\underline{u_\rho};\underline{v_1}\cdots\underline{v_\rho}) = sort(\underline{u'_1}\cdots\underline{u'_\rho};\underline{v'_1}\cdots\underline{v'_\rho}),$$

there exists a binomial

$$X_{\underline{w_1};\underline{z_1}}\cdots X_{\underline{w_\rho};\underline{z_\rho}} - X_{\underline{u'_1};\underline{v'_1}}\cdots X_{\underline{u'_\rho};\underline{v'_\rho}} \in I_A$$

and such that the first monomial is bi-sorted.

Let $f \in k[\underline{X}]$ be a polynomial. We say that f is marked if the initial term $in(f)$ of f is specified , where $in(f)$ can be any term of f. Given a set F of marked polynomials, we define the reduction relation module F in the sense of the theory of Gröbner bases.

We say that F is coherently marked if there exists a term order $<$ on $k[\underline{X}]$ such that $in_<(f) = in(f)$ for all $f \in F$. Then, if F is coherently marked, the reduction relation $\xrightarrow{F}$ is Noetherian.

THEOREM 2.3. *A finite set* $F \subset k[\underline{X}]$ *of polynomials marked is marked coherently if and only if the reduction relation module* F *is Noetherian, i.e., every sequence of reductions module* F *terminates.*

THEOREM 2.4. *Let* $F = \{X_{\underline{u_1};\underline{v_1}}X_{\underline{u_2};\underline{v_2}} - X_{\underline{u'_1};\underline{v'_1}}X_{\underline{u'_2}\underline{v'_2}} : sort(u_{11}u_{21}u_{12}u_{22}\cdots u_{1n}u_{2n}; v_{11}v_{21}v_{12}v_{22}\cdots v_{1m}v_{2m}) = u'_{11}u'_{21}u'_{12}u'_{22}\cdots u'_{1n}u'_{2n}; v'_{11}v'_{21}v'_{12}v'_{22}\cdots v'_{1m}v'_{2m}\}$ *be a set of marked binomials of* $K[\underline{X}]$*,with* $u_1 = (u_{11}, \cdots, u_{1n}), v_1 = (v_{11}, \cdots, v_{1m})$*, and so on.*

1) *Every monomial* $m \in k[\underline{X}]$ *is a normal form with respect to this reduction if and only if* m *is bi-sorted.*

2) *If a not bi-sorted monomial m_1 reduces to another monomial m_2 using F at least one of the inversion numbers a and b of m_2 is strictly less than the corresponding inversion number of m_1.*

3) *The reduction relation defined by F is Noetherian.*

THEOREM 2.5. *There exists a term order $\prec$ on $K[\underline{X}]$ such that the bi-sorted monomials are precisely the $\prec$-standard monomials modulo the ideal $I = (F)$. F is the reduced Gröbner basis of I and consequently $in_{\prec} I$ is generated by square-free quadratic monomials.*

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