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Matrices arising from semigroup rings

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1 Introduction

Let K be a field (that will be $\mathbb{Q}$ or $\mathbb{R}$ in this paper) and let $R = K[x_1, x_2, \ldots, x_s]$ the polynomial ring in s variables $x_1, x_2, \ldots, x_s$. The introduction in R of a term order, that is a total order on the set Π of the monomials of R, is subordinated to the introduction of an order on the variables $x_1, x_2, \ldots, x_s$.
In the last years interesting results were obtained in the field of commutative computational algebra and algebraic geometry by considering orders of variables different from $x_1 > x_2 > \ldots > x_s$, for example there are results under the hypothesis $x_n > \ldots > x_1$. Since we have $s!$ orders of variables it is clear that we can choose orders linked to the resolution of our problem.

Definition 1.1. *Let $G \subset R$, $G = \{f_1, \ldots, f_n\}$ a set of polynomials. G is an universal Gröbner basis for $(f_1, \ldots, f_n)$ if G is a Gröbner basis with respect to all term orders on the monomials of R.*

In the Definition 1.1 it is obvious that G is a Gröbner basis with respect to all orders of the variables. Then it is natural to give the following

Definition 1.2. *Let $G \subset R$, $G = \{f_1, \ldots, f_n\}$ a set of polynomials. G is a universal lexicographical Gröbner basis for $(f_1, \ldots, f_n)$ if G is a Gröbner basis with respect to the lexicographical order on the monomials of R and with respect to the $s!$ orders of the variables $x_1, x_2, \ldots, x_s$.*

Definition 1.3. *Let $G \subset R$, $G = \{f_1, \ldots, f_n\}$ a set of polynomials. G is a universal reverse lexicographical Gröbner basis for $(f_1, \ldots, f_n)$ if G is a Gröbner basis with respect to the reverse lexicographical order on the monomials of R and with respect to the $s!$ orders of the variables $x_1, x_2, \ldots, x_s$.*

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The request to be an universal Gröbner basis is very strong.
In [5], orders on the variables,"via" the lexicographic order, called linear, are considered. They can be defined by special matrices and they constitute a proper subset of the set of all orders of cardinality $s!$.

For the definition, we need variables indexed by a set of integers $\{i_1, \ldots, i_k\}$, that is, the variables must be of the type $T_{i_1 \ldots i_k}$ that appear in the presentation of the toric variety called the square-free k-th Veronese embedding of $\mathbb{P}^{s-1}$. Our intention is to extend the linear order introduced in [4] to the variables that come from the Segre product of two k-square-free Veronese subrings, in particular to the Segre product of two polynomial rings in two sets of variables. We introduce also a so called "inverse linear order" on the variables $T_{i_1 \ldots i_k}$ via the reverse lexicographical order.
Problem : Let $<_1$ and $<_2$ be two linear orders (resp. inverse linear orders) on V_1 and V_2 respectively defined by the matrices $\mathcal{A}_1$ and $\mathcal{A}_2$, where V_1 is the set of all square free monomials of degree k in R , V_2 is the set of all square free monomials of degree l in $S = K[Y_1, ..., Y_t]$, such that they induce a lexicographical(resp. reverselex) Gröbner basis for the toric ideals of V_1 and V_2 generated by quadrics.
Do the linear order $<_1 *_S <_2$, defined by the matrix $\begin{pmatrix} \mathcal{A}_1 & \Omega_1 \\ \Omega_2 & \mathcal{A}_2 \end{pmatrix}$, induce a quadratic lexicographical (resp. reverselex) Gröbner basis for the toric ideal of the Segre product $K[V_1] * K[V_2]$?
Actually we have positive answer for $k = l = 1$.

2 Main Results

Given four integers s, t and k, l such that $s \geq k \geq 1$ and $t \geq l \geq 1$ we consider two "label sets"

$$\mathcal{I}_k^s = \{[i_1, ..., i_k], 1 \leq i_1 < ... < i_k \leq s\} \qquad \mathcal{I}_l^t = \{[j_1, ..., j_l], 1 \leq j_1 < ... < j_l \leq t\}$$

and the set of variables $T = \{T_{i_1 \ldots i_k, j_1 \ldots j_l} : (i_1, ..., i_k, j_1, ..., j_l) \in \mathcal{I}_k^s \times \mathcal{I}_l^t\}$.

Let $\mathcal{A}$ be a $(k+l) \times (k+l)$ matrix of the type $\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \Omega_1 \\ \Omega_2 & \mathcal{A}_2 \end{pmatrix}$, where $\mathcal{A}_1$ is a matrix of order k, $\mathcal{A}_2$ is a matrix of order l of the form:

$$\mathcal{A}_1 = \begin{pmatrix} \pm e_{i_1} \\ \vdots \\ \pm e_{i_k} \end{pmatrix} \qquad \textit{and} \qquad \mathcal{A}_2 = \begin{pmatrix} \pm e'_{j_1} \\ \vdots \\ \pm e'_{j_l} \end{pmatrix}$$

and Ω_1 is the $l \times k$ null matrix and Ω_2 is the $k \times l$ null matrix.
The sets $\{e_{i_1}, ..., e_{i_k}\}$ and $\{e'_{j_1}, ..., e'_{j_l}\}$ are a permutation of the set of canonical vectors $\{e_1, ..., e_k\}$ in $\mathbb{R}^k$ and $\mathbb{R}^l$. The set of all matrices of type $\mathcal{A}_1$ (or $\mathcal{A}_2$) is a group of order $2^k k!$ (resp. $2^l l!$) and a subgroup of the orthogonal group $\mathcal{O}_k(\mathbb{R})$ (resp.$\mathcal{O}_l(\mathbb{R})$).

Definition 2.1. *The matrix $\mathcal{A}$ defines a linear ordering in $\mathbb{Z}^{k+l}$ given by*

$$(x_1, ..., x_k; x'_1, ..., x'_l) >_{lex} (y_1, ..., y_k; y'_1, ..., y'_l)$$

if and only if

$$(x_1, ..., x_k; x_1^{'}, ..., x_l')\mathcal{A} >_{lex} (y_1, ..., y_k; y_1^{'}, ..., y_l')\mathcal{A}.$$

Definition 2.2. *The matrix $\mathcal{A}$ defines an inverse linear ordering in $\mathbb{Z}^{k+l}$ given by*

$$(x_1, ..., x_k; x_1^{'}, ..., x_l') >_{revlex} (y_1, ..., y_k; y_1^{'}, ..., y_l')$$

if and only if

$$(x_1, ..., x_k; x_1^{'}, ..., x_l')\mathcal{A} >_{revlex} (y_1, ..., y_k; y_1^{'}, ..., y_l')\mathcal{A}.$$

From now on let $k = l = 1$. Then $\mathcal{A} = \begin{pmatrix} \pm e_1 & 0 \\ 0 & \pm e_1' \end{pmatrix}$ and the set of matrices is

$$A = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Given two polynomial rings $K[X_1, ..., X_s]$, $K[X_1, ..., X_t]$ and the Segre product

$$K[X_1, ..., X_s] \otimes_S K[Y_1, ..., Y_t] \cong K[X_1Y_1, ..., X_sY_t],$$

we consider the map $\widehat{\pi} : K[T_{1,1}, T_{1,2}, ..., T_{s,t}] \to K[X_1Y_1, ..., X_sY_t]$.

The new set of variables is $T_{1,1}, T_{1,2}, ..., T_{s,t}$.

For this set, the matrices in A define four linear orders in $\mathbb{Z}^{k+l}$:

$$L_1 : T_{s,t} > T_{s,t-1} > ... > T_{s,1} > ... > T_{1,t} > ... > T_{1,1}$$
$$L_2 : T_{1,t} > T_{1,t-1} > ... > T_{1,1} > ... > T_{s,t} > ... > T_{s,1}$$
$$L_3 : T_{s,1} > T_{s,t-1} > ... > T_{s,t} > ... > T_{1,1} > ... > T_{1,t}$$
$$L_4 : T_{1,1} > T_{1,2} > ... > T_{1,t} > ... > T_{s,1} > ... > T_{s,t}$$

For $s = t = 2$, $\operatorname{Ker}\hat{\pi}$ admits an universal Gröbner basis. In particular it is a quadratic Gröbner basis with respect to the lexicographical (or reverse lexicographical) order on the monomials, and with the linear (or inverse linear) orders on the variables.

In our case $\hat{\pi} : K[T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}] \to K[X_1Y_1, ..., X_2Y_2]$ and by the mapping:

$$T_{1,1} \to X_1Y_1, \qquad T_{1,2} \to X_1Y_2, \qquad T_{2,1} \to X_2Y_1, \qquad T_{2,2} \to X_2Y_2$$

$\operatorname{Ker}\hat{\pi}$ has only the binomial $f = T_{1,2}T_{2,1} - T_{1,1}T_{2,2}$ as generator and then $\{f\}$ is a minimal universal Gröbner basis for $ker\hat{\pi}$.

Theorem 2.3. *If we consider the linear orders L_i, $i = 1, ..., 4$ on the set of variables $T_{1,1}, T_{1,2}, ..., T_{s,t}$, then* $\operatorname{Ker}\hat{\pi}$ *has a lexicographical quadratic square-free Gröbner basis given by the set of binomials*

$$G_1 = \{T_{j,k}T_{i,l} - \underline{T_{j,l}T_{i,k}}, 1 \le i < j \le s, 1 \le k < l \le t\}$$
$$G_2 = \{\underline{T_{i,l}T_{j,k}} - T_{i,k}T_{j,l}, 1 \le i < j \le s, 1 \le k < l \le t\}$$
$$G_3 = \{\underline{T_{j,k}T_{i,l}} - T_{j,l}T_{i,k}, 1 \le i < j \le s, 1 \le k < l \le t\}$$
$$G_4 = \{T_{i,l}T_{j,k} - \underline{T_{i,k}T_{j,l}}, 1 \le i < j \le s, 1 \le k < l \le t\}$$

where in each binomial the initial term is underlined.

Methods: we study the map of semigroups $\pi : \mathbb{N}^{st} \to \mathbb{N}^s \oplus \mathbb{N}^t$ given by the configuration $\{e_i \oplus e'_j, 1 \leq i \leq s, 1 \leq j \leq t\} \subset \mathbb{N}^s \oplus \mathbb{N}^t$, $e_1, ..., e_s$ and $e'_1, ..., e'_t$ canonical vectors of $\mathbb{N}^s$ and $\mathbb{N}^t$. We identify $\mathbb{N}^{st}$ with the set of non negative integer $s \times t$ matrices (u_{ij}).
The analogous result is true for the inverse linear orders $L'_i,\ i = 1, \ldots, 4$, and for the reverse lexicographical order on monomials.

There exist other orders on the variables that are not linear in our sense, nevertheless they induce lexicographical quadratic Gröbner basis for the toric ideal of the Segre product of two polynomial rings.

Example 2.4. Let $s = t = 3,\ k = 1$, for the order on the variables

$$L: \qquad T_{1,1} > T_{1,2} > T_{2,2} > T_{2,1} > T_{1,3} > T_{2,3} > T_{3,2} > T_{3,3}$$

we have a lexicographical quadratic Gröbner basis for the toric ideal $Ker\widehat{\pi}$. But L is not a linear order according to definition 1.1.
In fact the result is true for any $s, t \geq 2$ and with order on the variables $T_{ij} > T_{kl}$ if $i + j < k + l$ or $i + j = k + l$ and $i < k$ ([4]).

Remark 2.5. For the reverse lexicographical order it's known that $ker\widehat{\pi}$ has an universal reverse lexicographical Groebner basis with respect to all permutations of variables $T_{i,k}$. However, we give a direct proof of this fact by the same technique used in theorem 2.3.

Before proving a general result, we consider the lexicographical case.
In [3], [5], we have the complete list of $k \times k$ matrices of $\mathcal{O}_k(\mathbb{R})$ that induce a quadratic lexicographic Groebner basis of the toric ideal of $K[V_1]$, for $k = 2, 3$ and $s = 6$,$k = 4$ and $s = 8$. The research can be done by Macaulay or COCOA programs.

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