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GOTZMANN IDEALS AND APPLICATIONS TO GRAPHS II

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Abstract.

Let k be an infinite field and $S = k[x_1, \ldots, x_n]$ the polynomial ring over k with each $\deg x_i = 1$ and $\mathfrak{m} = (x_1, \ldots, x_n)$ the graded maximal ideal of S. A graded ideal I generated in degree d is called a Gotzmann ideal if the number of generators of $\mathfrak{m}I$ is the smallest possible, namely, equal to the number of generators of $(\mathfrak{m}I)^{lex}$. A graph G is called Gotzmann if the edge ideal $I(G)$ is a Gotzmann ideal. We determine some classes of Gotzmann graphs and we characterize all Cohen-Macaulay graphs which are Gotzmann and principal Borel.

Keywords: Gotzmann ideal, Gotzmann graph, principal Borel ideals.

1. Introduction

Let k be an infinite field and $S = k[x_1, \ldots, x_n]$ the polynomial ring over k with each $\deg x_i = 1$ and $\mathfrak{m} = (x_1, \ldots, x_n)$ the graded maximal ideal of S. If I is a graded ideal of S, then we write $I_{\langle k \rangle}$ for the ideal generated by all homogeneous polynomials of degree k belonging to I. Given a graded ideal $I \subset S$, there exists a unique lexsegment ideal I^{lex} such that I and I^{lex} have the same Hilbert function. A graded ideal I generated in degree d is called a Gotzmann ideal if the number of generators of $\mathfrak{m}I$ is the smallest possible, namely, equal to the number of generators of $(\mathfrak{m}I)^{lex}$. By Gotzmann's persistence theorem[9] a graded ideal I generated in degree d is a Gotzmann ideal if, and only if, I and $(I^{lex})_{\langle d \rangle}$ have the same Hilbert function. Thus, in particular, a Gotzmann ideal has a linear resolution with the same Betti numbers as the corresponding lexsegment ideal. Knowing which monomial ideals are Gotzmann is still an open problem. The general classification of Gotzmann ideals is unknown at the moment. By definition, all lexsegment ideals are Gotzmann. In the theory of monomial ideals, there is the following hierarchy of ideals: lexsegment monomial ideals $\Rightarrow$ strongly stable monomial ideals $\Rightarrow$ stable monomial ideals. When the base field k is of characteristic 0, the Borel-fixed ideals coincide with the

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strongly stable ideals. Strongly stable ideals play a special role in the study of Hilbert functions. The reason is that one can often apply Gröbner basis techniques to reduce the general case to the study of Hilbert functions of strongly stable ideals (see e.g. Refs. 1,2,3,4,7,8,12,14 and 18). Among the Borel ideals the principal ones are the most simple. Let u be a monomial of S, then $\langle u\rangle$ denotes the smallest Borel ideal which contains u. The ideal $\langle u\rangle$ is called principal Borel with Borel generator u. In Ref. 5 all principal Borel ideals with Borel generators up to degree 4 which are Gotzmann are characterized. From a combinatorial point of view, it is particularly interesting to consider ideals generated by squarefree monomials, since they appear as the defining ideals of Stanley-Reisner rings. Let $G = ([n], E(G))$ be a finite graph on the vertex set $[n] = \{1, 2, \ldots, n\}$ with the edge set $E(G)$ without loops, multiple edges and isolated vertices. Write $I(G)$ for the ideal generated by all squarefree monomials x_ix_j with $\{i, j\} \in E(G)$. A graph G is called Gotzmann if the edge ideal $I(G)$ is Gotzmann. The graph G is called Cohen-Macaulay over a field K, if $K[x_1, \ldots, x_n]/I(G)$ is a Cohen-Macaulay ring, and is called Cohen-Macaulay if it is Cohen-Macaulay over any field. We will give some properties of Gotzmann graphs when the edge ideal is a principal Borel ideal. We also characterize all Cohen-Macaulay graphs whose edge ideal is principal Borel and Gotzmann. We will show some classes of graphs which are Gotzmann.

2. Main results

We will characterize all Gotzmann graphs whose edge ideal is a principal Borel ideal.

Proposition 2.1. *Let $I(G) = \langle u\rangle$ be a principal Borel ideal. The following conditions are equivalent:*

(a) $I(G)$ is a Gotzmann ideal.
(b) $I(G)$ is a lexsegment ideal.

If the equivalent conditions hold, then $u = x_1x_{i_2}$ or $u = x_{i_1}x_n$.

Proof. ¿From Proposition 3.1 in Ref. 5, in the squarefree case the assertion follows. □

Remark 2.1. In general (b) implies (a), but not the converse. In fact if $I(G)$ is not a principal Borel ideal the following example

$$I(G) = (x_1x_2, x_1x_4, x_2x_3, x_3x_4) \subset k[x_1, x_2, x_3, x_4]$$

shows that $I(G)$ is Gotzmann, but it is not lexsegment.

We need some preliminaries. Let G be a graph and $(\{r_1, r_2\}, \ldots, \{r_s, r_{s+1}\})$ a cycle in G, where $r_{s+1} = r_1$. An edge $\{i, j\} \in E(G)$ is called a *chord* of the cycle $(\{r_1, r_2\}, \ldots, \{r_s, r_{s+1}\})$ if $i, j \in \{r_1, r_2, \ldots, r_s\}$ and $\{i, j\} \neq \{r_i, r_{i+1}\}$ for all i. We call G *chordal* if every cycle in G of lenght > 3 has a chord. A *stable subset* or *clique* of G is a subset F of $[n]$ such that $\{i, j\} \in E(G)$ for all $i, j \in F$ with $i \neq j$. We write $\Delta(G)$ for the simplicial complex on $[n]$ whose faces are the stable subsets of G. A graph G is called *shifted* if, for any edge $\{i, j\} \in E(G)$ and for any integers $i' \leq i$ and $j' \leq j$ one has $\{i', j'\} \in E(G)$. The shifting operation of graphs is an operation which associate a shifted graph $\Delta(G)$ with a graph G. Shifting operations were first considered by Erdös, Ko and Rado.[6] Their shifting operation $G \longrightarrow \Delta^c(G)$ is called combinatorial shifting. Instead, algebraic shifting

was introduced by Kalai.[15] The main variants of algebraic shifting are exterior algebraic shifting $G \longrightarrow \Delta^e(G)$ and symmetric algebraic shifting $G \longrightarrow \Delta^s(G)$, see Kalai Ref. 16 and Herzog Ref. 10. Shifting operations are defined for simplicial complexes, however, we here only consider finite graphs. The graded Betti numbers $\beta_{ij}(I)$ of a homogeneos ideal $I \subset S$ are the integers $\beta_{ij}(I) = dim_k(Tor_i^S(I,k)_j)$. In other words, $\beta_{ij}(I)$ appear in the minimal graded free resolution

$$0 \longrightarrow \bigoplus_j S(-j)^{\beta_{hj}} \longrightarrow \ldots \longrightarrow \bigoplus_j S(-j)^{\beta_{1j}} \longrightarrow \bigoplus_j S(-j)^{\beta_{0j}} \longrightarrow I \longrightarrow 0$$

of I over S.

Corollary 2.1. *If $I(G) = \langle u \rangle$ is Gotmann then*

(a) G is a chordal graph and $\Delta^e(G) = \Delta^s(G)$
(b) $\beta_{ii+2}(I(G)) = \beta_{ii+2}(I(\Delta^e(G))) = \beta_{ii+2}(I(\Delta^s(G)))$ for all $i \geq 0$.

Proof. If $I(G)$ is Gotzmann from Proposition 2.1 $I(G)$ is a lexsegment ideal and has a linear resolution and by Lemma 3.2 in Ref. 17, G is chordal and $\Delta^e(G) = \Delta^s(G)$. Moreover $\beta_{ii+2}(I(G)) = \beta_{ii+2}(I(\Delta^e(G))) = \beta_{ii+2}(I(\Delta^s(G)))$ for all $i \geq 0$. □

We will characterize all Cohen-Macaulay graphs whose edge ideal is principal Borel and Gotzmann.

Theorem 2.1. *Let k be a field, let G be a graph on the vertex set $[n]$ and let $I(G) = \langle u \rangle$ be a principal Borel ideal which is Gotzmann. Let $F_1, \ldots, F_m$ be the facets of $\Delta(G)$ which admit a free vertex. The following conditions are equivalent:*

(a) G is Cohen-Macaulay;
(b) G is Cohen-Macaulay over k;
(c) G is unmixed;
(d) $[n]$ is the disjoint union of $F_1, \ldots, F_m$.

Proof. The assertion follows from Corollary 2.4 and from Theorem 2.1 in Ref. 11. □

We will show some classes of graphs which are Gotzmann. In order to do this we need some preparations. The *complement* $\overline{G}$ of a graph G has vertex set $V(G)$ and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Let G^* be the graph obtained by the complement $\overline{G}$ by removing the set $\{x_1, \ldots, x_r\}$ of all its isolated vertexes:

$$G^* = \overline{G} \setminus \{x_1, \ldots, x_r\}.$$

The shadow of a set B of squarefree monomials is the set of the monomials $\mathrm{Shad}(B) = \{x_i b : b \in B, i \notin \mathrm{supp}(b)\}$, where $\mathrm{supp}(b) = \{i : x_i \text{ divides } b\}$. We denote by M_d^s the set of all squarefree monomials of degree d. Let $B \subset M_d^s$. Then there is a unique lexsegment in M_d^s denoted by B^{lex} such that $\left|B^{lex}\right| = |B|$. The following important result holds

Theorem 2.2. *For any subset $B \subset M_d^s$, one has $\left|\mathrm{Shad}(B^{lex})\right| \leq |\mathrm{Shad}(B)|$. In other words, lexsegments have the smallest possible shadow.*

Proof. For the proof of this theorem we refer the reader to Ref. 13. □

Definition 2.1. A subset $B \subset M_d^s$ is called Gotzmann if $|\mathrm{Shad}(B^{lex})| = |\mathrm{Shad}(B)|$.

After these preparations we can prove the following:

Theorem 2.3. *Let $S = k[x_1, \ldots, x_n]$. Let G be a connected graph on the vertex set $V(G) = \{x_1, \ldots, x_n\}$ and $I(G) \subset S$ its edge ideal. Then G is Gotzmann in the following cases:*

(a) G is complete;
(b) G^* is complete;
(c) $|E(G^*)| \leq 2$;
(d) $|E(G)| \leq 2$;
(e) $G = \mathcal{K}_{a,b}$ is a bipartite complete graph with $a \leq 2$ and $b \leq 3$.

Proof.

(a) If G is complete, then $I(G)$ is an initial lexsegment ideal. Then it follows from the definition that $I(G)$ is a Gotzmann ideal and the assert follows.

(b) Now suppose that G^* is complete graph. Let $W = V(G^*) = \{x_{i_1}, \ldots, x_{i_r}\}$, with $i_1 < i_2 < \ldots < i_r$. We define the permutation $\pi : V \to V$ such that $\pi(W) = \pi(x_{i_1}, \ldots, x_{i_r}) = (x_{n-r+1}, \ldots, x_n)$ and $\pi(V \setminus W) = (x_1, \ldots, x_{n-r})$. We have the following isomorphisms: $G \cong G'$, $G^* \cong (G^*)'$, $\mathcal{K}_n \cong \mathcal{K}'_n$ where G', $(G^*)'$ and $\mathcal{K}'_n$ are the graphs obtained by applying π on the set of vertices of G, G^* and $\mathcal{K}_n$, respectively. Then $(G^*)'$ is a complete graph on the vertices set $\pi(W) = (x_{n-r+1}, \ldots, x_n)$. Then $I((G^*)')$ is the set of all monomials of degree 2 in $k[x_{n-r+1}, \ldots, x_n]$. It follows from the definition of $(G^*)'$ that $I(G') \cap I((G^*)') = (0)$ and $I(G') + I((G^*)') = I(\mathcal{K}'_n)$ and so $I(\mathcal{K}'_n)$ is the ideal generated by the disjoint union of the generating sets of $I(G')$ and $I((G^*))'$. Then the generating set of $I(G')$ is given by the difference between the generating set of $I(\mathcal{K}'_n)$ and the generating set of $I((G^*)')$. Moreover $I(\mathcal{K}'_n)$ is the ideal generated by all monomials of degree 2 in $k[x_1, \ldots, x_n]$ and if we review $I((G^*)')$ in the polynomial ring $k[x_1, \ldots, x_n]$, it is an ideal generated by the final lexsegment $L^f(x_{n-r+1}x_{n-r+2})$, i.e., by the set of all monomials of degree 2, less or equal than $x_{n-r+1}x_{n-r+2}$ in the lexicographic order. Then $I(G')$ is generated by the difference between the set of all monomials of degree 2 and a final lexsegment. Then $I(G')$ is an initial lexsegment ideal. It follows that $G \cong G'$ is Gotzmann.

(c) Now we suppose $|E(G^*)| \leq 2$. If $|E(G^*)| = 0$ then G is a complete graph and from case (a) G is Gotzmann. If $|E(G^*)| = 1$ then G^* is a complete graph and from (*b*) G is Gotzmann. Now we assume $|E(G^*)| = 2$. Let $E(G^*) = \{z, w\}$, $V(G^*) = \{x_{i_1}, x_{i_2}, x_{i_3}\} \subset V(G) = \{x_1, \ldots, x_n\}$. According with 2.1 we will show that $|\mathrm{Shad}(B)| = |\mathrm{Shad}(B^{lex})|$. It follows from Theorem 2.2 that if $|\mathrm{Shad}(B)| \leq |\mathrm{Shad}(B^{lex})|$, then $|\mathrm{Shad}(B)| = |\mathrm{Shad}(B^{lex})|$. We prove $|\mathrm{Shad}(B)| \leq |\mathrm{Shad}(B^{lex})|$. Note that if $|E(G^*)| = 2$ then $B^{lex} = M_2^s \setminus \{x_{n-2}x_n, x_{n-1}x_n\}$. Suppose that there exists a monomial $m \in \mathrm{Shad}(B)$, $m \notin \mathrm{Shad}(B^{lex})$. Since $m \in \mathrm{Shad}(B)$ then $m = x_i p$ with p monomial, $p \in B$ and $1 \leq i \leq n$, $i \notin \mathrm{supp}(p)$. Since $m \notin \mathrm{Shad}(B^{lex})$ then $p \notin B^{lex}$. Then $p = x_{n-2}x_n$ or $p = x_{n-1}x_n$. If $p = x_{n-2}x_n$ then $m = x_i x_{n-2} x_n$, with $i \neq n-2, n$. Since $m \notin \mathrm{Shad}(B^{lex})$ then $x_i x_{n-2} \notin B^{lex}$. But this implies $i = n$ and this is of contradiction. If $p = x_{n-1}x_n$ then $m = x_i x_{n-1} x_n$, with $i \neq n-1, n$. Since $m \notin \mathrm{Shad}(B^{lex})$ then

$x_i x_{n-1} \notin B^{lex}$. But this implies $i = n$ and this is a contradiction. Then in case (c) G is Gotzmann.

(d) Now we prove that in case (d) G is Gotzmann. Since from hypothesis G is connected, if $|E(G)| \leq 1$ then $I(G)$ is obviously isomorphic to an initial lexsegment ideal. If $|E(G)| = 2$ then $V(G) = \{x_1, x_2, x_3\}$ and the graph G has two edges, with a common vertex. We can suppose that x_1 is the common vertex to the elements of $E(G)$. In fact if the common vertex is different from x_1 we can construct a graph isomorphic to G by defining a permutation on the set of vertices $V(G)$. Then $I(G)$ is the initial lexsegment ideal generated by $\{x_1x_2, x_1x_3\}$ in $k[x_1, x_2, x_3]$. Then G is Gotzmann.

(e) Now we consider the complete bipartite graphs $\mathcal{K}_{a,b}$ with $a \leq 2$, $b \leq 3$. Since $|E(\mathcal{K}_{1,1})| = 1$, $|E(\mathcal{K}_{1,2})| = 2$ then it follows from (d) that $\mathcal{K}_{1,1}$ and $\mathcal{K}_{1,2}$ are Gotzmann. With direct calculations it easy to see that $\mathcal{K}_{1,3}$,$\mathcal{K}_{2,2}$ and $\mathcal{K}_{2,3}$ are Gotzmann too. □

Example 2.1. The assertion (b) of Theorem 2.3 shows that if a graph is obtained by removing from a complete graph the edges of a complete subgraph, then it is Gotzmann. Let us consider the graph G on the vertex set $V = \{x_1, \ldots, x_5\}$. G is obtained by removing G^* from $\mathcal{K}_5$. We consider the permutation $\pi : V \to V$, such that $\pi(V(G^*)) = \pi(W) = \pi(x_2, x_3, x_4) = (x_3, x_4, x_5)$ and $\pi(V \setminus W) = \pi(x_1, x_5) = (x_2, x_1)$ and we obtain the situation of the following figure:

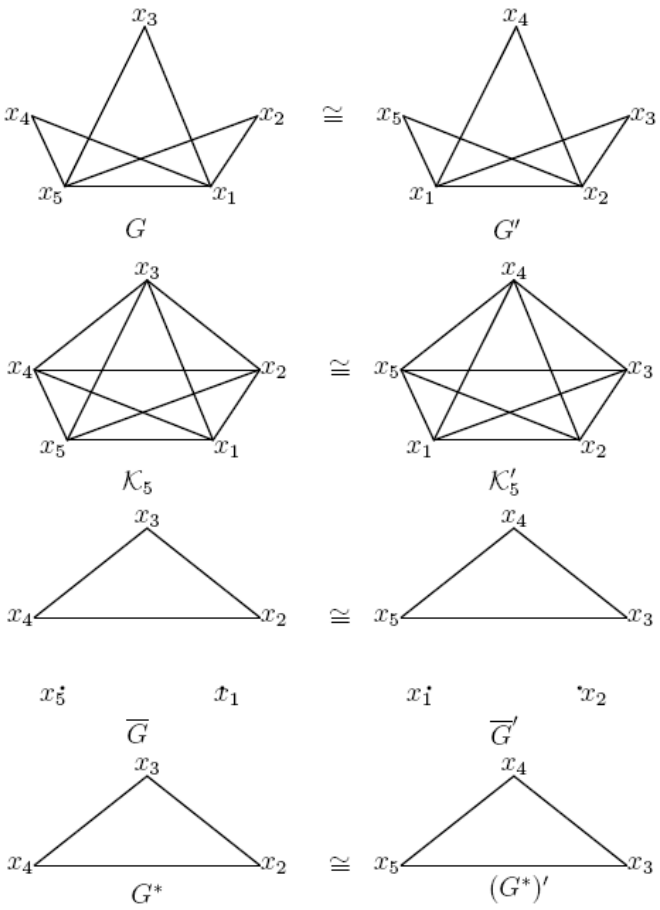


Fig. 1.

Remark 2.2. We can remark that the bipartite complete graph $K_{3,3}$ is not Gotzmann and a cycle G of lenght $l \leq 4$ is Gotzmann, but it is not Gotzmann if $l \geq 5$.

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