Communications to SIMAI Congress,
ISSN 1827-9015, Vol. 1 (2006)

DOI: 10.1685/CSC06075

Dynamics of materials with a deformability threshold

Angiolo Farina, Antonio Fasano, Lorenzo Fusi
Dipartimento di Matematica "U.Dini"
Viale Morgagni 67/a - 50134 - Firenze
farina,fasano,fusi@math.unifi.it

K.R Rajagopal
Dept. of mechanical Engineering - Texas A&M University
College Station, Texas 77843, USA
krajagopal@mengr.tamu.edu

Some biological tissues exhibit a sharp reduction of deformability beyond some stress threshold, below which they may be considered elastic. The simplest constitutive equation used to describe the mechanical behavior of tendons and ligaments is a stress-strain relation in which the stress is a sharply increasing function of the strain (for instance exponential, [Fung, 1993]). The system can be thus described by classical hyperelasticity. A limit case of such a behavior is a material that beyond a certain stress is no longer deformable.

Here we investigate the mechanical response of a layer of material characterized by a no-deformation stress threshold below which the system is modeled as a neo-Hookean nonlinear elastic solid. In particular, we consider the case in which the bottom face of the layer is kept fixed while on the top a stress eventually exceeding threshold is applied. This problem reveals surprisingly complicated.

The corresponding mathematical model is a free boundary problem for the wave equation, in which the free boundary conditions may be of two different types, according to whether the stress is continuous or discontinuous across the interface. The first class, corresponding to a rather artificial set of initial data, produces a subsonic motion of the interface. The second, corresponding to the case in which the system is initially at rest and the applied load is or becomes larger than the threshold value, exhibits instead a supersonic interface. Both situations have been studied, obtaining an existence and uniqueness theorem for the former case and studying the latter case for some specific data.

1 The mathematical model

We consider a homogeneous slab of thickness h loaded on the top surface with a known shear stress $\widehat{\sigma}(t) > \tau_o$, where τ_o is the stress threshold. Let $x = X + f\ (y,t)$ be a pure shear motion, $f\ (y,t)$ being the unknown displacement. The elastic and fully stretched regions are divided by a sharp interface $0 \leq s(t) \leq h$. In the deformable region $0 < y < s(t)$

$$f_{tt} - c^2 f_{yy} = 0, \tag{1.1}$$

where $c^2 = \mu/\rho$ (μ being the elastic modulus and ρ being density). In the fully stretched part

$$f\,(y,t) = f\left(s^-,t\right) + \frac{\tau_o}{\mu}\,(y-s)\,, \qquad y \in [s,h]\,.$$

Licensed under the Creative Commons Attribution Noncommercial No Derivatives

The genearl free boundary in the elastic region is

$$
\left\{
\begin{array}{l}
f_{tt} - c^2 f_{yy} = 0, \\
f\left(0,t\right) = 0, \\
f_y\left(s\left(t\right),t\right) = \frac{\sigma\left(s^-,t\right)}{\mu}, \quad \text{with} \quad \sigma\left(s^-,t\right) \leq \tau_o, \\
\left[\left(h-s\right)\left(f_y - \frac{\tau_o}{\mu}\right)\ddot{s} + \left(\left(h-s\right)f_{yy} + \frac{\tau_o}{\mu} - f_y\right)\dot{s}^2 + \right. \\
\left. + 2\left(h-s\right)f_{yt}\dot{s} + \left(h-s\right)f_{tt} + c^2 f_y\right]_{y=s^-} = \frac{\widehat{\sigma}(t)}{\rho}, \\
f\left(y,0\right) = f_o(y), \\
f_t\left(y,0\right) = f_1\left(y\right), \\
s(0) = s_o, \\
\frac{\dot{s}^2(0)}{c^2}\left(\tau_o - \sigma\left(s_o^-,0\right)\right) = \left[\!\left[\sigma\right]\!\right]_{\mathcal{S}}|_{t=0}
\end{array}
\right. \tag{1.2}
$$

where $\sigma(y,t)$ denotes the Cauchy stress and f_o, f_1 the initial data. In (1.2) one condition is still missing since the system contains the extra unknown $\sigma\left(s^-,t\right)$. Depending on the initial and boundary data, the problem itself selects the additional information which is required to close the system. This depend on whether $\left[\!\left[\sigma\right]\!\right]_{\mathcal{S}} = 0$, that is $\sigma(s^-,t) = \tau_o$, or $\left[\!\left[\sigma\right]\!\right]_{\mathcal{S}} > 0$. We have the following proposition

Let (f,s,σ) be a solution of problem (1.2) and $\widehat{\sigma}\left(t\right) > \tau_o$, for some $t \geq 0$, then:

1. *If $|\dot{s}| < c$, then $\left[\!\left[\sigma\right]\!\right]_{\mathcal{S}} = 0$.*
2. *If $|\dot{s}| = c$, then $\sigma\left(s^+,t\right) = \tau_o$ (and thus $\left[\!\left[\sigma\right]\!\right]_{\mathcal{S}} \geq 0$).*
3. *If $|\dot{s}| > c$, then either $\sigma\left(s^-,t\right) = \tau_o = \sigma\left(s^+,t\right)$ (i.e. $\left[\!\left[\sigma\right]\!\right]_{\mathcal{S}} = 0$) or $\sigma\left(s^-,t\right) < \tau_o < \sigma\left(s^+,t\right)$.*
4. *If $\left[\!\left[\sigma\right]\!\right]_{\mathcal{S}} > 0$, then $\tau_o > \sigma\left(s^-,t\right)$ and $|\dot{s}| > c$.*

When the stress is continuous across $y = s(t)$, $(1.2)_3$ and $(1.2)_4$ reduce to

$$
f_y\left(s\left(t\right),t\right) = \frac{\tau_o}{\mu}, \tag{1.3}
$$

$$
\left[f_{ty}\left(s^-,t\right)\,\dot{s} + c^2 f_{yy}\left(s^-,t\right)\right]\left(h-s\right) = \frac{\widehat{\sigma} - \tau_o}{\rho}. \tag{1.4}
$$

If the initial data satisfies some compatibility conditions and

$$
W_1 \leq h f_o^{''}(y) - \frac{h}{c} f_1^{'}(y) \leq W_2 \tag{1.5}
$$

with W_1, W_2 positive constants we have $\left[\!\left[\sigma\right]\!\right]_s = 0$ and the following proposition

A time θ can be computed such that a unique solution (f,s) to problem (1.2) (with $(1.2)_3$, $(1.2)_4$ replaced by (1.3) and (1.4)), exists for $t \in [0,\theta)$, with the property $-c < \dot{s} < 0$.

It can also be proved that, when $\widehat{\sigma} > \tau_o$ is constant and

$$
h f_o^{''}(y) - \frac{h}{c} f_1^{'}(y) = const
$$

there exists a solution with a stationary interface and the system comes to a stop (it becomes fully stretched everywhere) at time $t = 2s_o/c$.

Let us now consider then the case when $f_o(y) \equiv f_1(y) \equiv 0$ (i.e. the system is initially at rest) and the applied stress $\widehat{\sigma}$ increases in time, i.e. $\widehat{\sigma}'(t) > 0$ from $\widehat{\sigma}(0) = 0$, and, at some time $t_o < s_o/c$, $\widehat{\sigma}(t_o) = \tau_o$. The dynamics is now characterized by a jump of σ across $\mathcal{S}$, with the interface traveling faster than the sound speed of the deformable medium. Of course now (1.5) is not fulfilled. We can prove the following *Let (f, σ, s) be a solution of problem (1.2) for $t > t_o$. Then $s(t) < h - c(t - t_o)$, i.e. we have a supersonic interface.* The problem for a general applied stress $\widehat{\sigma}(t)$ is exceedingly complex so we have focused on some particular cases. When the applied load is$\widehat{\sigma}(t) = (\tau_o t)/t_o$ for $t \geq 0$ then

$$s(t) = h - \frac{3c}{2}(t - t_o).$$

When $\widehat{\sigma}(t) = (\tau_o t)/t_o$ in $[0, t_o]$ and $\widehat{\sigma}(t) = \tau_o$ for $t > t_o$ then $s(t)$ concides with the characteristic

$$s(t) = h - c(t - t_o).$$

When $\widehat{\sigma}$ is constant and greater than τ_o for all times $t \geq 0$ then

$$s(t) = h - \gamma c(t - t_o).$$

where $\gamma = \sqrt{\widehat{\sigma}/\tau_o}$.

Finally we have shown that the dynamics of the material we have studied cannot be retrieved as a limit case from the piecewise linear constitutive equation

$$\sigma = \begin{cases} \mu\,\epsilon\,, & 0 \leq \epsilon \leq \frac{\tau_o}{\mu}, \\ \mu\lambda^2\left(\epsilon - \frac{\tau_o}{\mu}\right) + \tau_o, & \epsilon > \frac{\tau_o}{\mu}, \end{cases} \quad \text{with } \lambda^2 > 1, \tag{1.6}$$

which represent a material with two different elastic moduli. Using the more sophisticated constitutive relation

$$\sigma(\epsilon) = \begin{cases} \mu\,\epsilon\,, & 0 \leq \epsilon \leq \epsilon_1, \\ \mu\epsilon_1\left[1 + \frac{1}{\lambda^n\left(1-\frac{1}{\lambda}\right)}\frac{\frac{1}{\lambda} - \left(1 - \frac{\epsilon}{\epsilon_o}\right)}{\left(1-\frac{\epsilon}{\epsilon_o}\right)^n}\right] & \epsilon_1 \leq \epsilon \leq \epsilon_o, \end{cases} \tag{1.7}$$

allows to retrieve the fully stretched model in the limit $\lambda \to \infty$.

REFERENCES

1. Fung Y.C. Biomechanics: Mechanical Propertiesa of Living Tissue *Springer-Verlag* 1993