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Dyson formulas for financial and physical evolutions in $\mathcal{S}'_n$

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Abstract.

In this paper we apply the theory of superpositions to the resolution of some evolution abstract differential equations in the space of tempered distribution. Applications to physical and financial problems motivates the study.

Keywords: Abstract Evolution Equation; tempered distribtion; abstract Schrödinger equation; abstract heat equation.

Introduction

We start from the abstract Schrödinger equation

$$i\hbar u'(t) = H(t)u(t),$$

where: $H(t)$ is a linear operator on some topological vector space (X, σ), for every real time t; u is a σ-differentiable curve in that space; and u' is the derivative of u with respect to the topology σ. When X is an Hilbert space, physicists solve formally this equation using propagators, more precisely, for every time t and for every initial condition $(t_0, u_0) \in \mathbb{R} \times X$, they find a unique differentiable solution u starting from the initial condition defined by

$$u(t) = S(t_0, t)u_0,$$

for every time t, where the operator $S(t_0, t)$ is given by the *Dyson formula*

$$S(t_0, t) = \exp\left(\frac{1}{i\hbar}\int_{t_0}^{t} H(\tau)d\tau\right).$$

Unfortunately, it is not true that in Hilbert spaces the abstract Schrödinger equations has always a solution (because the operators $H(t)$, that are of interest in Quantum Mechanics, are always unbounded-linear operators and not-everywhere defined) and moreover in those spaces the Dyson formula has not a precise mathematical sense. In this paper we solve the problem in the space of tempered distributions (where the operators of Quantum Mechanics are always continuous and everywhere defined) we will find a unique solution, for every initial condition, in the propagator-form desired by physicists,

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and we shall give a meaning to the Dyson formula in this context. We solve in this space the abstract heat evolution equation too, in view of financial applications.

1. Integral of continuous function from $\mathbb{R}^m$ to $\mathcal{S}'_n$

Let us define the integral of function from $\mathbb{R}^m$ to $\mathcal{S}'_n$.

Definition. *Let u be a function from $\mathbb{R}^m$ into the distributions space $\mathcal{S}'_n$ and let ϕ be a test function in $\mathcal{S}_n$. We define image of ϕ by u the function*

$$u(\phi) : \mathbb{R}^m \to \mathbb{C} : u(\phi)(p) = u(p)(\phi).$$

The images by u of the test functions give informations on the entire function u, as, for instance, states the following.

Theorem. *Let u be a function from $\mathbb{R}^m$ into the space $\mathcal{S}'_n$. Then, u is continuous with respect to the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$, if and only if the function*

$$u(\phi) : \mathbb{R}^m \to \mathbb{C} : u(\phi)(t) = u(t)(\phi),$$

is continuous, for every test function ϕ in $\mathcal{S}_n$.

Proof. In fact, u is continuous at t_0, with respect to the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$, if and only if, for every test function ϕ (recall that $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ is induced by the family of seminorms $|\langle \cdot, \phi \rangle|$), the semi-distance $|\langle u(t) - u(t_0), \phi \rangle|$ vanishes, as $t \to t_0$; on the other hand

$$\lim_{t \to t_0} |(u(t) - u(t_0))(\phi)| = \lim_{t \to t_0} |u(\phi)(t) - u(\phi)(t_0)|,$$

so u is $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-continuous at t_0 if and only if for every test function ϕ, the complex function $u(\phi)$ is continuous at t_0. ■

The preceding theorem allow us to associate with the function u, in an extremely natural way, an operator from $\mathcal{S}_n$ into the space of continuous complex functions $C^0(\mathbb{R}^m, \mathbb{C})$, which in the following we shall denote by C^0_m.

Definition. *Let u be a function from $\mathbb{R}^m$ into the space $\mathcal{S}'_n$, continuous in the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$. The operator $\widehat{u} : \mathcal{S}_n \to C^0_m$ defined by $\widehat{u}(\phi) = u(\phi)$, for every test function ϕ, is called the operator induced by u.*

The operator associated with a continuous function in the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ is automatically continuous in the natural topologies of the two spaces $\mathcal{S}_n$ and C^0_m.

Theorem. *Let u be a function from $\mathbb{R}^m$ into the space $\mathcal{S}'_n$ continuous in the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ and let a be a Radon-measure on $\mathbb{R}^m$ with compact support. Then, the composition $a \circ \widehat{u}$ is $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-continuous for every a in $C^{0\prime}_m$ and, consequently, the operator $\widehat{u}$ is continuous in the topologies $\sigma(\mathcal{S}_n, \mathcal{S}'_n)$ and $\sigma(C^0_m, C^{0\prime}_m)$.*

Proof. Denote by $C^{0\prime}_m$ the dual of C^0_m with respect to its standard locally convex topology. The dual $C^{0\prime}_m$ is, by definition, the space of Radon-measures on $\mathbb{R}^m$ with compact support. Note, that the linear operator $\widehat{u}$ is continuous with respect to the topologies $\sigma(\mathcal{S}_n, \mathcal{S}'_n)$ and $\sigma(C^0_m, C^{0\prime}_m)$ if and only if for every a in $C^{0\prime}_m$, i.e., for every Radon-measure on $\mathbb{R}^m$ with compact support, the functional $a \circ \widehat{u}$ is $\sigma(\mathcal{S}_n, \mathcal{S}'_n)$-continuous (see, for example,

[5],[8]). Note, moreover, that this is true for a sequentially dense subset of $C_m^{0\prime}$, the linear subspace spanned by the delta-measures on $\mathbb{R}^m$, so by the Banach-Steinhaus theorem $a\circ\widehat{u}$ is $\sigma(\mathcal{S}_n',\mathcal{S}_n)$ -continuous, being $\mathcal{S}_n$ a Fréchet-space and thus a Baire-space (see Banach-Steinhaus on Laurent Schwartz's *Functional Analysis*). ■

Definition (integral of a function with respect to a measure). *In the conditions of the above theorem the composition $a\circ\widehat{u}$ is said the integral on $\mathbb{R}^m$ of the function u with respect to the measure a and it is denoted by $\int_{\mathbb{R}^m} uda$ or by $a(u)$.*

Our aim is, generalizing the above definition, to integrate u on every bounded Borel-subset of $\mathbb{R}^m$, with respect to a Radon-measure on $\mathbb{R}^m$ eventually not with compact support. To this end we have the following theorem.

Theorem. *Let a be a Radon-measure on $\mathbb{R}^m$, and let B be a bounded Borel-subset of $\mathbb{R}^m$. Consider the functional*

$$a_B : C_m^0 \to \mathbb{C} : f \mapsto \int_B f d\mu_a = \int_{\mathbb{R}^m} \chi_B f d\mu_a,$$

where μ_a is the Borel-measure associated canonically to a by the Riesz theorem. Then, a_B is a Radon-measure with compact support on $\mathbb{R}^m$.

Proof. We must prove that a_B is continuous in the standard topology of C_m^0 (that is the topology of compact convergence), it is the locally convex topology induced by the family of seminorms $(q_K)_{K\in\mathcal{K}(\mathbb{R}^m)}$ indexed by the set of compact subset of $\mathbb{R}^m$ and defined by $q_K(f) := \max_K |f|$. Remember that a linear functional T on C_m^0 is continuous in this topology if there exists a positive real number M and a seminorm q_K such that, for every function f in C_m^0 is $|T(f)| \le Mq_K(f)$. Indeed, we have

$$|a_B(f)| = \left|\int_B f d\mu_a\right| \le \sup_B |f| \cdot |\mu_a|(B) \le |\mu_a|(B) \cdot \max_{\overline{B}} |f|,$$

so, since the closure of B is compact, a_B is continuous with respect to the topology of compact convergence, and then it is a compact-support Radon-measure on $\mathbb{R}^m$. ■

We call the functional a_B of the preceding theorem restriction of a to B. We can give, so, the following definition.

Definition (integral on a bounded Borel set with respect to a measure). *In the above conditions the composition $a_B\circ\widehat{u}$ is said the integral on B of the function u with respect to the measure a, it is denoted by $\int_B uda$. In other terms we put*

$$\int_B uda := \int_{\mathbb{R}^m} uda_B.$$

Remark. Let B be a bounded Borel-subset of $\mathbb{R}^m$, and a_B be the restriction of a Radon-measure a to B. Then, a_B is the Radon-measure on $\mathbb{R}^m$ associated with the Borel-measure defined for every Borel-set E by $(\mu_a)_B(E) = \mu_a(E\cap B)$. In fact, we have

$$T(f) := \int_{\mathbb{R}^m} f\, d(\mu_a)_B = \int_B f\, d\mu_a = a_B(f).$$

2. Integral of continuous functions from $\mathbb{R}^m$ into ${}^{\mathcal{S}}End(\mathcal{S}'_n)$

In this section we give the definition of integral of an operator-valued function H, namely of functions from $\mathbb{R}^m$ into ${}^{\mathcal{S}}End(\mathcal{S}'_n)$ (the space of $\mathcal{S}$-endomorphisms on the space $\mathcal{S}'_n$, see [1] or [3]) in a Radon-measure on $\mathbb{R}^m$. Analogously to the preceding section it is possible to prove the following proposition.

Proposition. *A function $H : \mathbb{R}^m \to {}^{\mathcal{S}}End(\mathcal{S}'_n)$ is continuous with respect to the pointwise topology induced by the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ if and only if the function $H(u) : \mathbb{R}^m \to \mathcal{S}'_n$ defined by $H(u)(p) := H(p)(u)$, for every p, is continuous in the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$, for every tempered distribution u.*

Definition. *Let $H : \mathbb{R}^m \to {}^{\mathcal{S}}End(\mathcal{S}'_n)$ be a continuous function. We define integral of H in a Radon-measure a the operator*

$$a(H) = \int_{\mathbb{R}^m} H da : \mathcal{S}'_n \to \mathcal{S}'_n : u \mapsto \int_{\mathbb{R}} H(u)\, da.$$

The integral of H in the Lebesgue-measureis denoted by $\int_{\mathbb{R}^m} H$.

Theorem. *Let $H : \mathbb{R}^m \to {}^{\mathcal{S}}End(\mathcal{S}'_n)$ be a continuous function. Then, the integral of H in a Radon-measure a, is continuous in the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ and consequently $\mathcal{S}$-linear.*

Proof. Recall that an linear operator T from a seminormed space (E, p) to another seminormed space (F, q) is continuous if for every seminorm q_j of the family q there is a seminorm p_i of the family p and a positive real M such that $q_j(T(x)) \leq M p_i(x)$. Let ϕ be a test function, we have, then, to prove that there are another test function ψ and a positive real M such that

$$\left| \left\langle \phi, \int_{\mathbb{R}^m} H(u) da \right\rangle \right| \leq M \left| \langle \psi, u \rangle \right|.$$

Concluding we have, if K is the compact support of a, that there exist an m-index p_0 (by the Weierstrass theorem) and a test function ψ with a real M (by continuity of the operator H_{p_0}) such that

$$\begin{aligned} \left| \left\langle \phi, \int_{\mathbb{R}^m} H(u) da \right\rangle \right| = \left| \int_{\mathbb{R}^m} H(u)(\phi)\, da \right| \leq \\ \leq |a|(K) \sup_{p \in K} |H(u)(\phi)| = \\ = |a|(K) \left| H_{p_0}(u)(\phi) \right| = \\ = |a|(K) \left| \langle \phi, H_{p_0}(u) \rangle \right| \leq \\ \leq (|a|(K) M) \left| \langle \psi, u \rangle \right|. \end{aligned}$$

So the integral of H, by a Radon-measure with compact support, is a continuous operator in the weak* topology, and consequently (D. Carfi, *Topological characterizations of $\mathcal{S}$-linearity*, preprint) is $\mathcal{S}$-linear. ■

The last definition concerns integral on bounded Borel-set.

Definition (integral on a bounded Borel set with respect to a measure). *Let a be a Radon-measure on $\mathbb{R}^m$, B be a bounded Borel-subset of $\mathbb{R}^m$, a_B be the restriction of a to B and let $H : \mathbb{R}^m \to {}^{\mathcal{S}}End(\mathcal{S}'_n)$ be a continuous function. We define*

$$\int_B H da := \int_{\mathbb{R}^m} H da_B.$$

The integral of H in the Lebesgue measure is denoted by $\int_B H$.

3. Curves of $\mathcal{S}$-linear operators

From this section to the end we shall use intensively the $\mathcal{S}$-linear Algebra, that will be assumed known. For the fundamentals of $\mathcal{S}$-linear Algebra and for the functional calculus of $\mathcal{S}$-diagonalizable endomorphisms see [1], [3]. We observe only that if A is an $\mathcal{S}$-diagonalizable operator and if α is one of its $\mathcal{S}$-eigenbasis for the space, then there is only a function a associating to each n-tuple p the unique complex eigenvalues a_p of A on the eigenvector α_p. We call a the system of eigenvalues of A on the $\mathcal{S}$-basis α. We remark that this function a is necessarily a smooth function of class $\mathcal{O}_M$. In fact, from $A\alpha_p = a_p\alpha_p$ we have $A(\alpha)\hat{}(\phi)(p) = a(p)\alpha\hat{}(\phi)(p)$; the functions $A(\alpha)\hat{}(\phi)$ and $\alpha\hat{}(\phi)$ are, by definition of $\mathcal{S}$-family, of class $\mathcal{S}$; since α is an $\mathcal{S}$-eigenbasis we have that the operator $\alpha\hat{}$ is surjective and injective. Let us prove that a is smooth at every p_0, let ϕ be a test function such that $\alpha\hat{}(\phi)(p_0)$ is different from 0 (it certainly exists because $\alpha\hat{}$ is surjective), it follows that there is a neighborhood of p_0 in which $\alpha\hat{}(\phi)$ is different from 0, for each p in this neighborhood we have

$$a(p) = \frac{A(\alpha)\hat{}(\phi)(p)}{\alpha\hat{}(\phi)(p)},$$

then a is smooth at p_0. Moreover, since $A(\alpha)\hat{}(\phi) = a\alpha\hat{}(\phi)$ and since $\alpha\hat{}$ is surjective, the product of the function a with all the functions of class $\mathcal{S}$ is yet of class $\mathcal{S}$, and then a is of class $\mathcal{O}_M$.

Now, let $A : \mathbb{R} \to {}^{\mathcal{S}}End(\mathcal{S}'_n)$ be a curve of $\mathcal{S}$-diagonalizable endomorphisms with a common $\mathcal{S}$-eigenbasis α, the family $a = (a(t))_{t\in\mathbb{R}}$ of the systems of eigenvalues of the operators $A(t)$ on α is called the system of eigenvalues of the curve A. For every p, we define the function $a_p : \mathbb{R} \to \mathbb{C}$ associating to t the value of the function $a(t)$ on the n-tuple p: $a_p(t) := a(t)(p)$.

If A is a continuous curve of $\mathcal{S}$-diagonalizable endomorphisms with a common $\mathcal{S}$-eigenbasis α, the functions a_p are continuous. In fact, from the equality $A(t)(\alpha_p) = a_p(t)\alpha_p$, if ϕ does not belong to the kernel of α_p, we deduce

$$a_p(t) = \frac{A(\alpha_p)(t)(\phi)}{\alpha_p(\phi)},$$

for every t, so a_p is proportional to the continuous function $A(\alpha_p)\hat{}(\phi)$ and then it is continuous.

Theorem. *Let A be a differentiable curve of $\mathcal{S}$-diagonalizable endomorphisms on $\mathcal{S}_n$, in the pointwise topology induced by $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$. Assume that the endomorphisms of the curve have a common $\mathcal{S}$-eigenbasis for the space and real eigenvalues. Then the curve of states $B : t \mapsto \exp(iA(t))$ is differentiable in the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$, and*

$$B'(t) = iA'(t)\exp iA(t),$$

where we used the multiplicative notation for the composition of endomorphisms. As a consequence, for every state ψ the curve u defined by $u(t) := \exp(iA(t))(\psi)$ satisfies the differential equality $u'(t) = iA'(t)(u(t))$.

Proof. Let t_0 be a real time, we have

$$B'(t_0)(\psi) = {}^{\sigma(\mathcal{S}'_n,\mathcal{S}_n)}\lim_{t\to t_0} \frac{\exp iA(t)(\psi) - \exp iA(t_0)(\psi)}{t-t_0},$$

if the right-hand limit exists. We prove the pointwise existence of $B'(t_0)$ for an eigenbasis α of A. To this aim $a_p : \mathbb{R} \to \mathbb{C}$ will be, for every n-tuple p the unique function such that $A(t)(\alpha_p) = a_p(t)\alpha_p$, it is continuous by the above argumentations; we have

$$\begin{aligned}
B'(t_0)(\alpha_p) &= \lim_{t\to t_0} \frac{\exp iA(t)(\alpha_p) - \exp iA(t_0)(\alpha_p)}{t-t_0} = \\
&= \lim_{t\to t_0} \frac{e^{ia_p(t)}\alpha_p - e^{ia_p(t_0)}\alpha_p}{t-t_0} = \\
&= \left(\lim_{t\to t_0} \frac{e^{ia_p(t)} - e^{ia_p(t_0)}}{t-t_0}\right)\alpha_p = \\
&= ia'_p(t_0)\left(e^{ia_p(t_0)}\alpha_p\right) = \\
&= ia'_p(t_0)\exp iA(t_0)(\alpha_p) = \\
&= iA'(t_0)(\exp iA(t_0)(\alpha_p)).
\end{aligned}$$

Now, let z be a tempered distribution, for the difference-quotient we have

$$\begin{aligned}
Q_B(t_0,t)(z) &= \frac{\exp A(t)(z) - \exp A(t_0)(z)}{t-t_0} = \\
&= \frac{1}{t-t_0}\left[\exp A(t)\left(\int_{\mathbb{R}^n}(z)_\alpha\alpha\right) - \exp A(t_0)\left(\int_{\mathbb{R}^n}(z)_\alpha\alpha\right)\right] = \\
&= \frac{1}{t-t_0}\left[\int_{\mathbb{R}^n}(z)_\alpha \exp A(t)(\alpha) - \int_{\mathbb{R}^n}(z)_\alpha \exp A(t_0)(\alpha)\right] = \\
&= \int_{\mathbb{R}^n}(z)_\alpha \frac{\exp A(t)(\alpha) - \exp A(t_0)(\alpha)}{t-t_0},
\end{aligned}$$

passing to limit

$$\begin{aligned}
B'(t_0)(z) &= \lim_{t\to t_0}\int_{\mathbb{R}^n}(z)_\alpha \frac{\exp iA(t)(\alpha) - \exp iA(t_0)(\alpha)}{t-t_0} = \\
&= \int_{\mathbb{R}^n}(z)_\alpha iA'(t_0)\circ\exp iA(t_0)(\alpha) = \\
&= iA'(t_0)\circ\exp iA(t_0)\int_{\mathbb{R}^n}(z)_\alpha\alpha = \\
&= iA'(t_0)\circ\exp iA(t_0)(z),
\end{aligned}$$

so $B'(t) = iA'(t)\exp iA(t)$. Applying the preceding result, we have

$$u'(t) = iA'(t)(\exp iA(t)(\psi)) = iA'(t)(u(t)),$$

and the theorem is proved. ■

The preceding theorem can be generalized.

Theorem. *Let A be a curve of $\mathcal{S}$ -diagonalizable operators with a same $\mathcal{S}$-basis of the space, namely α, and let, for every n-tuple p, a_p the complex function defined on the real line such that $A(t)(\alpha_p) = a_p(t)\alpha_p$, for every real time t. Then, the curve A is differentiable with respect to the pointwise topology τ induced by the weak topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ on the space ${}^{\mathcal{S}}End(\mathcal{S}'_n)$ at a real time t_0 if and only if a_p is differentiable in t_0 for every n-tuple p. Moreover, in these conditions the operator $A'(t_0)$ is $\mathcal{S}$-diagonalizable with the same $\mathcal{S}$-basis α and its system of eigenvalues is the system of derivatives $p \mapsto a'_p(t_0)$.*

Proof. For every n-tuple p, assume the function a_p be differentiable at a time t_0, we have to prove that the pointwise-limit

$$^{\tau}\lim_{t\to t_0} \frac{A(t) - A(t_0)}{t - t_0}$$

there exists, i.e., that for every u in $\mathcal{S}'_n$ the $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-limit

$$^{\sigma(\mathcal{S}'_n,\mathcal{S}_n)}\lim_{t\to t_0} \left(\frac{A(t) - A(t_0)}{t - t_0}(u)\right)$$

there exists; we begin with the basis α:

$$\begin{aligned}
A'(t_0)(\alpha_p) &= {}^{\sigma(\mathcal{S}'_n,\mathcal{S}_n)}\lim_{t\to t_0} \left(\frac{A(t) - A(t_0)}{t - t_0}(\alpha_p)\right) = \\
&= {}^{\sigma(\mathcal{S}'_n,\mathcal{S}_n)}\lim_{t\to t_0} \frac{A(t)(\alpha_p) - A(t_0)(\alpha_p)}{t - t_0} = \\
&= \lim_{t\to t_0} \frac{a_p(t)\alpha_p - a_p(t_0)\alpha_p}{t - t_0} = \\
&= \left(\lim_{t\to t_0} \frac{a_p(t) - a_p(t_0)}{t - t_0}\right)\alpha_p = \\
&= a'_p(t_0)\alpha_p.
\end{aligned}$$

Now, let u be a tempered distribution, for the difference-quotient, we have

$$\begin{aligned}
\frac{A(t) - A(t_0)}{t - t_0}(u) &= \frac{A(t) - A(t_0)}{t - t_0}\left(\int_{\mathbb{R}^n} (u)_\alpha\, \alpha\right) = \\
&= \frac{1}{t - t_0}\left(A(t)\int_{\mathbb{R}^n} (u)_\alpha\, \alpha - A(t_0)\int_{\mathbb{R}^n} (u)_\alpha\, \alpha\right) = \\
&= \frac{1}{t - t_0}\left(\int_{\mathbb{R}^n} (u)_\alpha\ A(t)\alpha - \int_{\mathbb{R}^n} (u)_\alpha\ A(t_0)\alpha\right) = \\
&= \frac{1}{t - t_0}\int_{\mathbb{R}^n} (u)_\alpha\ (A(t) - A(t_0))\alpha = \\
&= \int_{\mathbb{R}^n} (u)_\alpha\ \left(\frac{A(t) - A(t_0)}{t - t_0}\right)\alpha,
\end{aligned}$$

so by $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-continuity of superpositions, the limit

$$^{\sigma(\mathcal{S}'_n,\mathcal{S}_n)}\lim_{t\to t_0} \left(\frac{A(t) - A(t_0)}{t - t_0}(u)\right)$$

exists and its value is the superposition

$$\int_{\mathbb{R}^n} (u)_\alpha \left(a'_p(t_0)\,\alpha_p\right)_{p\in\mathbb{R}^n}. \blacksquare$$

4. The Dyson Formula

First of all we generalize a fact of elementary Linear-Algebra, namely the following: if A and B are two diagonalizable endomorphisms on a finite-dimensional vector space with the same eigenbasis v, then every linear combination $aA+bB$ has the same eigenbasis and $\mathrm{ev}_{aA+bB}(v_i) = a\mathrm{ev}_A(v_i) + b\mathrm{ev}_B(v_i)$, for every vector v_i of v, where ev_L is the mapping associating to every eigenvector of the linear operator L the corresponding (unique) eigenvalues.

Theorem. *Let H be a continuous curve of $\mathcal{S}$-diagonalizable operators with the same $\mathcal{S}$-eigenbasis and let μ be a Radon-measure on $\mathbb{R}$. Let $\eta = (\eta_p)_{p\in\mathbb{R}^n}$ be the common eigenbasis of the operators of the curve, and let, for every n-tuple p, $E_p : \mathbb{R}\to\mathbb{C}$ be the unique function such that $H(t)(\eta_p) = E_p(t)\eta_p$, for every time t. Then, for every n-tuple p,*

$$\left(\int_{t_0}^{t} H d\mu\right)(\eta_p) = \left(\int_{t_0}^{t} E_p d\mu\right)(\eta_p).$$

Proof. It's straightforward,

$$\begin{aligned}\left(\int_{t_0}^{t} H d\mu\right)(\eta_p) &= \int_{t_0}^{t} H(\eta_p)\,d\mu = \\ &= \left(\int_{t_0}^{t} E_p\eta_p d\mu\right) = \\ &= \left(\int_{t_0}^{t} E_p d\mu\right)(\eta_p). \blacksquare\end{aligned}$$

Theorem. *Let H be a continuous curve of $\mathcal{S}$-diagonalizable operators with the same $\mathcal{S}$ -eigenbasis. Then, for every time t,*

$$\left(\int_{t_0}^{(\cdot)} H d\lambda\right)'(t) = H(t).$$

Proof. Let η be the common eigenbasis of the operators of the curve, and let $E_p : \mathbb{R}\to\mathbb{C}$ the unique function such that $H(t)(\eta_p) = E_p(t)\eta_p$, for every n-tuple p and for every time t. We have, for every n-tuple p,

$$\left(\int_{t_0}^{t} H d\lambda\right)(\eta_p) = \left(\int_{t_0}^{t} E_p d\lambda\right)(\eta_p).$$

Let us compute the difference-quotient at a time t_1

$$\begin{aligned}\frac{1}{t-t_1}\left(\int_{t_0}^{t} H - \int_{t_0}^{t_1} H\right)(\eta_p) &= \frac{1}{t-t_1}\left(\left(\int_{t_0}^{t} H\right)(\eta_p) - \left(\int_{t_0}^{t_1} H\right)(\eta_p)\right) = \\ &= \frac{1}{t-t_1}\left(\int_{t_0}^{t} E_p - \int_{t_0}^{t_1} E_p\right)\eta_p,\end{aligned}$$

passing to limit we have

$$\left(\int_{t_0}^{(\cdot)} H d\lambda\right)' (t_1)\,(\eta_p) = E_p(t_1)\,(\eta_p) = H\,(t_1)\,(\eta_p)\,.$$

For every u, we have

$$\frac{1}{t-t_1}\left(\int_{t_0}^{t} H - \int_{t_0}^{t_1} H\right)(u) = \frac{1}{t-t_1}\left(\int_{t_0}^{t} H - \int_{t_0}^{t_1} H\right)\left(\int (u)_\eta\,\eta\right) = \\ = \int_{\mathbb{R}^n} (u)_\eta \frac{1}{t-t_1}\left(\int_{t_0}^{t} H - \int_{t_0}^{t_1} H\right)(\eta)\,,$$

when $t \to t_1$, by the previous step, the right hand side tend to

$$\int_{\mathbb{R}^n} (u)_\eta\, H(t_1)\eta = H(t_1)(u),$$

and the theorem is proved. ■

We finally can conclude with the main result.

Theorem. *Let H be a continuous curve of $\mathcal{S}$-diagonalizable operators with a same $\mathcal{S}$-eigenbasis for the space $\mathcal{S}'_n$, let (t_0, u_0) be an initial condition in $\mathbb{R}\times\mathcal{S}'_n$ and let u be the curve in $\mathcal{S}'_n$ defined by*

$$u : t \mapsto \exp\left(\int_{t_0}^{t} H d\lambda\right)(u_0)\,,$$

for every real time t. Then u is $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-differentiable and it is such that

$$u'\,(t) = H\,(t)\,(u\,(t))\,,$$

for every real time t, and $u(t_0) = u_0$. Moreover, for every pair of times s and t we have

$$u(t) = S(s,t)u(s),$$

where $S : \mathbb{R}^2 \to {}^{\mathcal{S}}\mathrm{End}\,(\mathcal{S}'_n)$ is the propagator defined by

$$S(s,t) = \exp\left(\int_{s}^{t} H d\lambda\right).$$

Proof. Applying the preceding theorem we obtain

$$u'\,(t) = \left(\int_{t_0}^{(\cdot)} H d\lambda\right)'(t)\left(\exp\left(\int_{t_0}^{t} H d\lambda\right)(u_0)\right) = \\ = \left(\int_{t_0}^{(\cdot)} H d\lambda\right)'(t)\,(u\,(t)) = \\ = H\,(t)\,(u\,(t))\,.$$

Concerning the propagator we have

$$u(t) = S(t_0,s)S(s,t)u(t_0) = S(s,t)S(t_0,s)u(t_0) = S(s,t)u(s).\ \blacksquare$$

Analogously, for the Abstract Heat equation, fundamental in the study of financial evolution, we have the following.

Theorem (on the abstract Heat equation). *Let* $A : \mathbb{R}^{\geq} \to {}^{\mathcal{S}}\mathrm{End}\left(\mathcal{S}'_n\right)$ *be a continuous curve of* $\mathcal{S}$*-diagonalizable operators with a common* $\mathcal{S}$*-eigenbasis for* $\mathcal{S}'_n$ *and with real and positive eigenvalues. Then, for every initial condition* $(0, u_0) \in \mathbb{R}^{\geq} \times \mathcal{S}'_n$*, the curve* $\psi : \mathbb{R}^{\geq} \to \mathcal{S}'_n$*, given by*

$$\psi(t) = \exp\left(-\int_{t_0}^{t} A d\lambda\right)(u_0),$$

for every $t \geq 0$*, is such that* $\psi'(t) = -A(\psi(t))$*, and* $\psi(0) = \psi_0$*.*

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