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An optimal control approach for 1D-2D shallow water equations coupling

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1 Introduction

In this paper we consider the problem of the coupling between 1D and 2D shallow water systems. To this aim we introduce a domain decomposition technique with overlap (see [5]) to solve a 1D-2D shallow water system. The proposed method is based on the introduction of suitable boundary control functions and generalizes to the situation of heterogeneous differential problems in space the method, proposed by the first author, when the heterogeneity is confined at the differential level (see [1]).

2 The models and their discretization in time

First of all let us introduce the 2D model we are going to consider. We denote by Ω_2 an open limited regular set of $\mathbb{R}^2$ with boundary $\partial\Omega_2 \equiv \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_c$, $\mathbf{x} = (x_1, x_2)^T \equiv (x, y)^T$, $\mathbf{x} \in \bar{\Omega}_2$ (see Figure 4.1). We want to solve in Ω_2 the following *shallow water* system: for any $t > 0$, find $(\mathbf{u}, \xi)$ such that

$$\left\{ \begin{aligned} & \frac{\partial \mathbf{u}}{\partial t} + g\nabla\xi = 0\Omega_2, \\ & \frac{\partial \xi}{\partial t} + \nabla\cdot(h\mathbf{u}) = 0 \quad \text{in } \Omega_2, \end{aligned} \right. \tag{2.1}$$

where $\mathbf{u} = (u, v)^T$ represents the average velocity of the fluid along the vertical direction, ξ is the position of the free surface (the elevation) with respect to an horizontal reference plane, $h = h_0 + \xi$ is the total depth, $-h_0$ is the position of the bottom with respect to the same reference level (see Figure 4.1) and g is the gravity acceleration.

System (2.1) can be regarded as a model for an hydrostatic free surface fluid in which the convective and the diffusive terms in the momentum equation have been neglected.

As for the 1D model we consider the domain $\omega_1 \subset \mathbb{R}$ where following one-dimensional problem has to be solved (which is the one-dimensional counterpart of (2.1)): for any $t > 0$ find (v, η) such that

$$\left\{ \begin{aligned} & \frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = 0 \qquad \text{in } \omega_1, \\ & \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(ku) = 0 \quad \text{in } \omega_1, \end{aligned} \right. \tag{2.2}$$

where u is the velocity field, η is the elevation, $k = \eta + k_0$ is the total depth of the water, $-k_0$ is the position of the bottom with respect an horizontal reference level. Equations (2.2) described the motion of a fluid in a rectangular channel of width L.

From the mathematical view point both (2.1) and (2.2) are two strictly hyperbolic systems and they have to be completed with suitable initial and boundary conditions. In this paper we will consider only subcritical flows. We denote with Γ_{in} the inlet part of $\partial\Omega_2$, with Γ_{out} the outlet with Γ_c the closed part of $\partial\Omega_2$; on Γ_c we assume a slip condition.

Moreover we denote the boundary of ω_1 by $\partial\omega_1 = \gamma_{in} \cup \gamma_{out}$. The boundary conditions for the system (2.2) are: $\eta = \eta_d(t)$ at γ_{out} and $\eta = \lambda_1(t)$ at γ_{in} for any $t > 0$, where $\eta_d(t)$ and $\lambda_1(t)$ are two given functions (in the sequel λ_1 will be unknown).

Both the 2D and the 1D system are discretized in time using a first order semi implicit time advancing scheme (see [4]). Under suitable regularity assumption we can formally substitute the velocity (obtained from the momentum equations) into the continuity equation; in this way we end up with the following elliptic problems for the elevation $\xi^{(n+1)}$ and η^{n+1}

$$-\nabla \cdot (A\nabla\xi^{(n+1)}) + \sigma\xi^{(n+1)} = \sigma\xi^{(n)} - \frac{\partial F_1}{\partial x_1} - \frac{\partial F_2}{\partial x_2}, \tag{2.3}$$

and

$$-\frac{\partial}{\partial x}\left(a\frac{\partial\eta^{(n+1)}}{\partial x}\right) + \gamma\eta^{(n+1)} = \gamma\eta^{(n)} - \frac{\partial F_1^{1D}}{\partial x}, \tag{2.4}$$

with $\gamma = \sigma = 1/\Delta t$ while $A = (a_{ij})$ and a_{ij}, F_i depend on Δt and on the coefficients of system (2.1); similarly α_{1D}, a and F_1^{1D} depend on Δt and on the coefficients of system (2.2).

3 The heterogeneous problem and the optimal control approach

For the sake of simplicity, in the exposition we refer to the simple situation represented in Figure 4.2: Ω_2 is a rectangle and ω_1 is the straight line $(\bar{x}_1, \bar{x}_3) \times L/2$. In this case the domain Ω may represents a two-dimensional straight channel with rectangular constant cross-section, while ω_1 is the one-dimensional approximation of this channel in Ω_1; finally Ω_{12} is the overlapping regions and Γ_{21} is the interface.

To exchange informations between the 1D and the 2D domain it is necessary to introduce an extension operator: given a one-dimensional quantity w defined in ω_1, we denote with $E_{2D}w \equiv \{w(x_1,t),\ \forall(x_1,x_2) \in \Omega_{12}\}$ the two-dimensional extension of w in the overlap region. Introducing the overlap region we assume implicitely that if (u,η) is a solution of (2.4) then $\mathbf{u} = (E_{2D}u, 0)^T$ and $\xi = E_{2D}\eta$ are a good approximation in Ω_{12} of the solution of the two dimensional shallow water system (2.1).

The proposed coupling method is based on the boundary control approach (see [3]). In fact, we search a "solution" in Ω_{12} that minimize in some sense the difference between the one-dimensional and the two-dimensional computed elevations.

In particular to adjust the 1D solution with the 2D one, let us consider the following minimization problem

$$\inf_{\lambda_1,\lambda_2} J_\alpha(\lambda_1,\lambda_2) = \inf_{\lambda_1,\lambda_2}, \frac{\alpha}{2}\int_{\Gamma_{21}} \lambda_2^2\, d\Gamma + \frac{\alpha}{2}\lambda_1^2 + J_0(\lambda_1,\lambda_2) \tag{3.1}$$

where $J_0(\lambda_1, \lambda_2) \equiv \frac{1}{2} \int\limits_{\Omega_{12}} (\xi^{(n+1)} - E_{2D}\eta^{(n+1)})^2 d\Omega$, and (λ_1, λ_2) are the so-called *boundary control functions* and α is a given nonnegative constant.

The optimal control problem ca be stated as follows: find $\xi^{(n+1)}$, $\eta^{(n+1)}$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ such that (2.3), (2.4) (along with their proper boundary and initial conditions) and (3.1) are satisfied.

The uniqueness of the solution of this control problem can be proved (see [2] for the proof and for a detailed description of the advocated procedure).

4 Numerical results

We present a numerical example in order to assess the effectiveness of the proposed method. To solve the optimality problem we use a GMRES algorithm that requires at each iteration the solution of a primal and a dual problem.

We consider the straight channel $(0, 12) \times (0, 1)$ with rectangular constant section divided in two parts: in the first we use the 2D model, while in the second the 1D model is considered as depicted in Figure 4.2. At the inlet we impose $\xi(\mathbf{x}, t^n) = 0.02 \sin(2\pi t^n/10)$. The depth is equal to 0.1 m and $\Delta t = 0.1$ s. Figure 4.4 shows the elevations computed varying the dimension of the overlap region (for a two-dimensional mesh with $h = 0.1$). The solutions are very similar; the reduced amplitude of the waves is due to the diffusion introduced by the numerical scheme and not to the coupling procedure. Figure 4.3 shows the behaviour of the number of iterations required by the GMRES algorithm to solve the control problem with respect the time and the overlap dimension. The number of iterations is approximately independent since the dimension of the discrete space control is approximately the same in these three cases. Table 4.1 shows the (averaged) number of iterations required by the GMRES method to solve the control problem using different mesh sizes h and different extensions of the overlap region (in this simple case the overlap region is a rectangle $\ell \times 1$). As expected, for a given mesh size the number of iterations increases when the size of the overlap region is reduced; similarly for a fixed size of the overlap region the number of iterations decreases when the mesh size is reduced.

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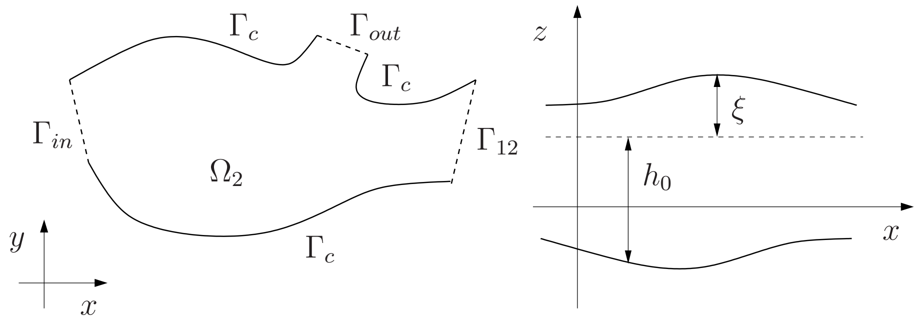

Figure 4.1: Notations in the two-dimensional case

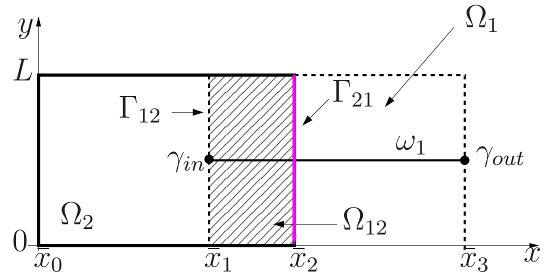

Figure 4.2: The 2D-1D reference domain with the overlap region $\Omega_{12} = \Omega_1 \cap \Omega_2$ (shaded). In this case $\omega_1 = \{(\bar{x}_1, \bar{x}_3) \times L/2\}$, $\gamma_{in} = \{(\bar{x}_1, L/2)\}$ and $\gamma_{out} = \{(\bar{x}_3, L/2)\}$

	$h = 0.1$	$h = 0.05$	$h = 0.025$
$\ell = 3$	3.98	3.83	3.79
$\ell = 1$	6.16	5.41	4.68
$\ell = 0.3$	8.05	6.92	6.05
$\ell = 0.15$	14.15	8.75	7.15

Table 4.1: The number of GMRES iterations as a function of the extension ℓ of the overlap region and the space-discretization step h

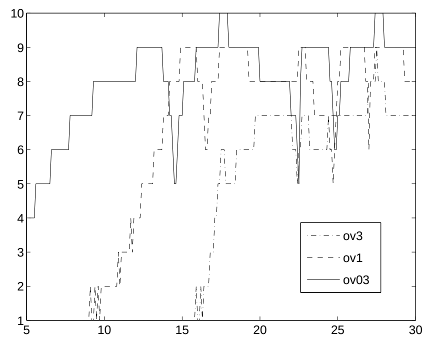

Figure 4.3: The number of GMRES iterations with respect the time

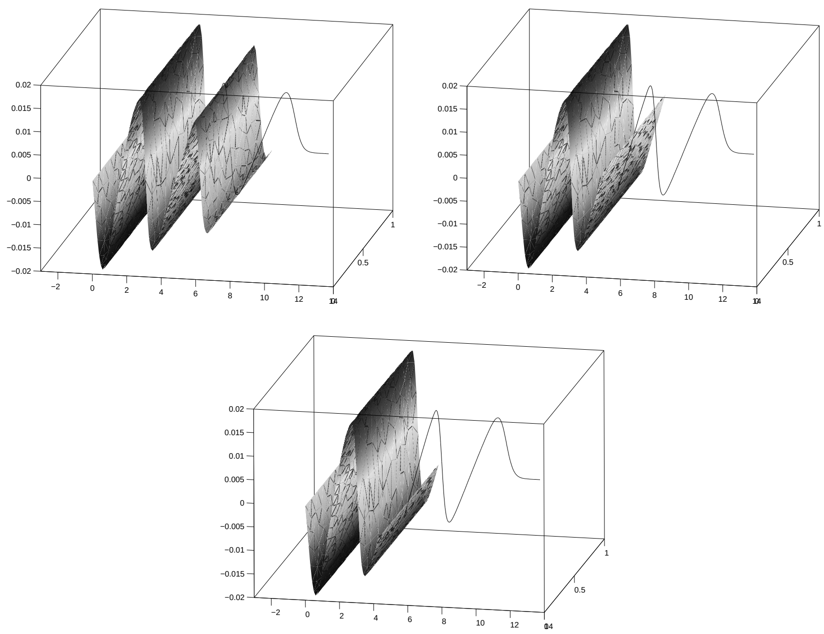

Figure 4.4: The solutions obtained with the 2D-1D coupling reducing the overlap region: from top to bottom, from left to right, 3 meters, 1 meter and 0.3 meters