Communications to SIMAI Congress,
ISSN 1827-9015, Vol. 1 (2006)
DOI: 10.1685/CSC06065

Composition Operators on Spaces of Holomorphic Functions

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Let D be a domain in $\mathbb{C}^n$, our aim is to investigate the behaviour of composition operators on linear subspaces of $\mathrm{Hol}(D)$, subspaces which in general can be endowed with topologies that can be different from the subspace topology. In particular, we study composition on generalized Bergman spaces and give results on the dynamical behaviour (*i.e.* cyclicity, hypercyclicity, compactness). Special attention will be paid to underline the analogies and differences between the case of bounded and unbounded domains in $\mathbb{C}$ and $\mathbb{C}^n$.

Let Ψ be a holomorphic self-map of the domain D. We call *composition operator* of *symbol* Ψ the linear operator C_Ψ defined by

$$\mathrm{Hol}(D) \ni f \mapsto C_\Psi(f) = f \circ \Psi \in \mathrm{Hol}(D).$$

If H is a linear subspace of $\mathrm{Hol}(D)$ such that $C_\Psi(H) \subset H$ we say that C_Ψ (or Ψ for short) *acts* on H. Under very mild assumptions on the topology of the subspace H (it is enough that the Banach topology is stronger than the topology of pointwise convergence on D), the operator $C_\Psi : H \to H$ turns out to be continuous.

In particular we focus our attention on the dynamical behaviour of bounded linear operators on H. Let T be a bounded linear operator on the Banach space H. We say that T is *cyclic* if there exists $h \in H$ (called a *cyclic vector*) such that the span of the orbit $\mathrm{Orb}(T,h) = \{T^n(h) \ : \ n \in \mathbb{N}\}$ is dense in H, *i.e.*, $\{p(T)(h) \ : \ p(t) \in \mathbb{C}[t]\}$ is dense in H. We say that T is *hypercyclic* if there exists $h \in H$ (called a *hypercyclic vector*) such that the orbit $\mathrm{Orb}(T,h) = \{T^n(h) \ : \ n \in \mathbb{N}\}$ is dense in H. In dynamical systems this last property is also called *topological transitivity* and it is a well know result that it can hold only if H has infinite dimension.

We denote by $\mathbb{B}_n$ the unit ball in $\mathbb{C}^n$ for the standard Hermitian norm, when $n = 1$ we denote $\mathbb{B}_1$ as Δ, too. We consider the Hardy space $H^2(\Delta)$: it is a Hilbert space whose Hilbert topology is stronger than the topology of uniform convergence on compacta of Δ. More generally we can take into account the $p-th$ Hardy space $H^p(\Delta)$ which turns out to be a Banach space. The first result from which the theory of composition operators originated is the so-called Littlewood's Subordination Principle.

Theorem 1. *Suppose Ψ is a holomorphic self-map of Δ with $\Psi(0) = 0$. Then for each $f \in H^2(\Delta)$ we have $C_\Psi(f) \in H^2(\Delta)$ and $||C_\Psi(f)|| \leq ||f||$.*

A trivial corollary gives that for any holomorphic self-map Ψ of Δ the composition operator C_Ψ acts on $H^2(\Delta)$. An accurate study of the main features (cyclicity, hypercyclicity, compactness) of composition operators on Hardy spaces is contained in [6] and [2].

We can also consider the Bergman space $A^2(\Delta)$ on Δ: as in the Hardy space case, the Hilbert topology on $A^2(\Delta)$ is stronger than the topology of uniform convergence on compacta of Δ; moreover composition operators act continuously on $A^2(\Delta)$.

Now we give a very short account of the generalizations of these topics to the several complex variables setting.

The Hardy space $H^2(\mathbb{B}_n)$ in the unit ball of $\mathbb{C}^n$ is again a Hilbert space whose Hilbert topology is stronger than the topology of uniform convergence on compact subsets of $\mathbb{B}_n$. For a more detailed study of the Hardy space in the several complex variable case see [5].

Anyway, if $n > 1$, there is no result like Littlewood's Subordination Principle. Indeed, following results by Hörmander and MacCluer, Wogen described a wide class of holomorphic self-maps Ψ of $\mathbb{B}_n$ such that the composition operator C_Ψ is unbounded on $H^2(\mathbb{B}_n)$ (see [7]).

Some positive results on the boundedness of composition operators on Hardy spaces in several complex variables were obtained by Cowen and MacCluer in [3] where they prove that linear fractional maps of $\mathbb{B}_n$ into itself induce bounded composition operators on $H^2(\mathbb{B}_n)$.

Further results were obtained by Bisi and Bracci in [1] where they give a classification of the cyclic and hypercyclic behaviour of composition operators induced by linear fractional maps according to their fixed point sets in the closure of the unit ball. In particular, they prove that if Ψ is a linear fractional map which has more than two fixed points in $\overline{\mathbb{B}_n}$ then C_Ψ is not cyclic and that a linear fractional map Ψ with exactly two boundary fixed points induces a hypercyclic composition operator if and only if df is invertible at one–and hence any–point of $\overline{\mathbb{B}_n}$.

Now we turn our attention to the unbounded case, in particular we consider the case of Bergman spaces on the punctured complex plane $\mathbb{C}^*$.

We denote by $d\omega(z)$ the $(1,1)$-form given by $i(dz\wedge d\bar{z})/2$ which is the form associated to the Lebesgue measure on $\mathbb{C}$. Let φ be a Lebesgue-measurable positive (*that is*, $\operatorname{infess}_L \varphi > 0$ on any compact $L \subset \mathbb{C}^*$) function on $\mathbb{C}^*$ such that $\varphi \in L^1_{\mathrm{loc}}(\mathbb{C}^*)$. The *(weighted) Bergman space associated to* φ is the complex vector space of holomorphic functions on $\mathbb{C}^*$ which are square-integrable with respect to the weight φ, that is

$$\mathcal{F}_\varphi = \left\{ f \in \mathrm{Hol}(\mathbb{C}^*, \mathbb{C}) \ : \ \int_{\mathbb{C}^*} \varphi(z) \big| f(z) \big|^2 d\omega(z) < \infty \right\}$$

and, again, the topology of $\mathcal{F}_\varphi$ as a Hilbert space is stronger than the topology of the uniform convergence on compact subsets of $\mathbb{C}^*$.

A first difference with the bounded case is the fact (due to Liouville's theorem) that the Bergman space associated to the constant weight 1 contains only the function which is identically zero, while on bounded domains all polynomials belong to the classical Bergman space (which has therefore infinite dimension).

The investigation on cyclicity of composition operators on Bergman spaces yields the following results which illustrate the distintive effect of dimension on the dynamics properties of composition operators.

Theorem 2. *Let $\mathcal{F}$ be a finite dimensional Bergman space with $\dim \mathcal{F} = d + 1 > 1$ and suppose that Ψ acts on $\mathcal{F}$. Then either $\Psi \in Aut\mathbb{C}^*$ or Ψ is constant. If C_Ψ is cyclic, then there exists $c \in \mathbb{C}^*$ such that $\Psi(z) = cz$ for any $z \in \mathbb{C}^*$. Moreover if c is not a root of 1, then C_Ψ is always cyclic; if c is a root of 1 of order q, then C_Ψ is cyclic if and only if $q > d$.*

This result is obtained by a thorough classification of finite dimensional Bergman spaces and of the holomorphic maps which act on them. A similar result holds also for one-

dimensional Bergman spaces which are different from the space of constant functions, the interested reader can find more details in [4]. The infinite dimensional case is examined in the following:

Proposition 3. *Let $\mathcal{F}$ be a weighted Bergman space of infinite dimension and suppose C_Ψ acts cyclically on it. Then Ψ is of the form $\Psi(z) = cz$ for a suitable constant $c \in \mathbb{C}^*$.*

A complete classification of holomorphic maps acting on infinite dimensional Bergman spaces is possible, up to now, only in the case in which the group of rotations in $\mathbb{C}^*$ acts on the Bergman space itself.

A Bergman space $\mathcal{F}_\varphi$ is said to be *rotationally invariant (r.i.)* if the composition operators whose symbols are the rotations map the space $\mathcal{F}_\varphi$ into itself. A weight φ is said to be *rotationally invariant (r.i.)* if $\varphi(z) = \varphi(e^{i\theta}z)$ for all $\theta \in \mathbb{R}$ and $z \in \mathbb{C}^*$.

Even if the weight is not r.i., we can modify it, without affecting the topology, and find a different Bergman space structure on the same topological space whose weight is r.i.. Indeed we have

Theorem 4. *Let $\mathcal{F}_\varphi$ be a r.i. Bergman space. Then there exists a r.i. weight ϕ such that $\mathcal{F}_\varphi = \mathcal{F}_\phi$ (as topological spaces). These spaces coincide as Hilbert spaces if and only if all monomials in $\mathcal{F}_\varphi$ are mutually orthogonal.*

The previous conclusion gives the possibility to classify all holomorphic maps acting on infinite dimensional r.i. Bergman spaces.

Theorem 5. *Suppose that $\mathcal{F}_\varphi$ is an infinite dimensional r.i. Bergman space and that the composition operator C_Ψ acts on it. Then either Ψ is constant or there exists $c \in \mathbb{C}^*$ such that $\Psi(z) = cz$ or $\Psi(z) = c/z$. Moreover if $\Psi(z) = c/z$, then all monomials of any degree belong to $\mathcal{F}_\varphi$.*

These last two results complete the investigation on cyclity of composition operators on r.i. Bergman spaces.

Proposition 6. *Suppose $\mathcal{F}_\varphi$ is a r.i. weighted Bergman space of infinite dimension. Given $c \in \mathbb{C}$ with $|c| = 1$, the composition operator associated to the symbol $\Psi(z) = cz$ is cyclic if and only if c is not a root of 1.*

Proposition 7. *Suppose $\mathcal{F}$ is a r.i. weighted Bergman space which contains monomials whose degree in bounded below by n_1 and unbounded above. Then for any $c \in \mathbb{C}$ with $0 < |c| < 1$ the composition operator associated to the symbol $\Psi(z) = cz$ is cyclic.*

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