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A new class of non-separable symmetric wavelets for image processing

Guido Andaloro[1], Mariantonia Cotronei[2], Luigia Puccio[1]
[1] *Dipartimento di Matematica*
Università degli Studi di Messina, Italy
gandaloro@unime.it
gina@dipmat.unime.it

[2] *DIMET*
Università degli Studi "Mediterranea" di Reggio Calabria, Italy
mariantonia.cotronei@unirc.it

Abstract

It is known that wavelet analysis is a powerful mathematical tool for image processing. For such type of applications, symmetry of the wavelet filters is claimed to produce less visual artifacts than non-linear phase wavelets. On the other hand, the filters themselves can be separable or non-separable. While separable filters offer the advantage of low-complexity processing, their non-separable counterparts have more degrees of freedom and hence allow better designs. In this talk we discuss about new classes of non-separable wavelet filters with different types of symmetry. A scheme for their construction is given and some applications to edge detection over geometrical images and over industrial data are shown.

Keywords: `Symmetry, wavelets, edge detection.`

1. Introduction.

It is well known that some of the most desired properties in a *wavelet system* are *short support*, *high number of vanishing moments* and *symmetry*. In particular, symmetry plays a very important role, since it is claimed to produce few visual artifacts and to minimize phase distortion. Often, symmetry about a point (linear phase symmetry) of a wavelet system is not enough to handle the symmetrical boundaries in the extended data and axis symmetry (symmetry about all the axis super-planes) is required in order to facilitate and handle the symmetrical boundary conditions.

This paper deals with the construction and application of new classes of biorthogonal *non-separable* wavelets, symmetric with respect to any symmetry group, with arbitrary number of vanishing moments and compactly

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supported. Biorthogonal systems have proved to be very useful in image processing problems and their design is much more affordable in comparison to their orthogonal counterpart. In addition the biorthogonality requirement is completely compatible with symmetry. Our approach differs from the ones proposed by Bin Han et al. in [4,7–9], namely the *convolution method* and the *CBC algorithm*.In fact, we make use of the parameterization offered by the *lifting scheme*, [10,12] in order to show how to transfer the symmetry properties of the primal function to the whole system and to add desired approximation order/vanishing moment conditions.

The paper is organized as follows. In Section 2, we give the fundamental concepts and tools used in wavelet theory and signal processing. In Section 3, we recall the main results for the construction of symmetric wavelet filters with respect to any symmetry group. For a complete treatment, the reader is referred to [2,3]. In Section 4, we propose an algorithm, called SWF algorithm, for the construction of a symmetric biorthogonal filter bank with desired number of vanishing moments and minimal support. Furthermore, we present in detail some specific constructions of families of symmetric bivariate filter banks with respect to the coordinate axes and with respect to hexagonal symmetry. Finally, in Section 5, we show some of the results obtained from a wide experimentation in a typical context of 2-D signal processing, namely edge detection. We test the filters obtained by means of the previous algorithm on some geometrical and standard test images as well as on real data originated by industrial applications. It turns out that the symmetry property of our filters allows to get better results in comparison to the most commonly used filters in literature.

2. Filters and wavelets

We denote by $\ell(\mathbb{Z}^d)$ the linear space of all real valued sequences defined on $\mathbb{Z}^d$. A *d-variate signal* x is an element of $\ell(\mathbb{Z}^d)$, namely it is represented by the sequence: $x = \left\{x_\alpha \in \mathbb{R} : \alpha \in \mathbb{Z}^d\right\}$. A *filter* is a linear time invariant operator which transforms an input signal x in an output signal y. The action of a filter h is described in terms of *convolution* with its impulse response sequence $h := \left\{h_\alpha \in \mathbb{R} : \alpha \in \mathbb{Z}^d\right\} \in \ell(\mathbb{Z}^d)$, that is

$$(1) \qquad y_\beta = \sum_{\alpha\in\mathbb{Z}^d} h_\alpha x_{\beta-\alpha} = \sum_{\alpha\in\mathbb{Z}^d} h_{\beta-\alpha} x_\alpha, \quad \beta \in \mathbb{Z}^d.$$

In case only a finite number of coefficients h_α are non-zero, the filter h is a *Finite Impulse Response* FIR filter. The *z-transform* or *symbol* of an FIR filter h is a Laurent polynomial $h(z)$ given by

$$h(z) = \sum_{\alpha \in \mathbb{Z}^d} h_\alpha z^\alpha, \quad z \in \mathbb{C}^d \setminus \{\mathbf{0}\} \tag{2}$$

where z^α has to be understood as $z^\alpha = \prod_{j=1}^d z_j^{\alpha_j}$, with $\alpha \in \mathbb{Z}^d$ and $|\alpha| = |\alpha_1| + ... + |\alpha_d|$.

A *filter bank* $\mathcal{F}$ is a set of filters, linked by sampling operators. $\mathcal{F}$ is called *perfect reconstruction* or *PR filter bank* if the output signal y coincides with the input signal x. Given filter bank $\mathcal{F} = \{h, \tilde{h}, g^k, \tilde{g}^k,\ k = 1, \ldots, M-1\}$, the perfect reconstruction conditions can be written as

$$h(\eta_1 z)\tilde{h}(\eta_2 z^{-1}) + \sum_{k=1}^{M-1} g^k(\eta_1 z)\tilde{g}^k(\eta_2 z) = 2^{2d}\delta_{\eta_1 - \eta_2} \tag{3}$$

with $\eta_1, \eta_2 \in \{\pm 1\}^d$ and $z \in \mathbb{C}^d$.

Now let $L_2(\mathbb{R}^d)$, with $d \geq 1$, be the Hilbert space of square-integrable d-variate functions. A function $\varphi \in L_2(\mathbb{R}^d)$ is said to be *$2I$-refinable* with *mask* $h \in \ell(\mathbb{Z}^d)$ if it is solution of the following *refinement equation*:

$$\varphi(x) = \sum_{\alpha \in \mathbb{Z}^d} h_\alpha \varphi(2x - \alpha), \quad x \in \mathbb{R}^d \tag{4}$$

If φ is *compactly* supported, then the sequence h is a *finitely* supported sequence on $\mathbb{Z}^d$, thus associated to an FIR filter.

In this paper, any mask is assumed to be a finitely supported real-valued sequence on $\mathbb{Z}^d$, satisfying the additional condition $\sum_{\alpha \in \mathbb{Z}^d} h_\alpha = 2^d$.

The concept of *stability* plays an important role in wavelet analysis (cf. [5]). In fact, any stable refinable function is also called a *scaling function* and generates a *multiresolution analysis* of $L_2(\mathbb{R}^d)$.

A scaling function $\varphi \in L_2(\mathbb{R}^d)$ is called *interpolatory* if

$$\varphi(\alpha) = \delta_\alpha, \quad \forall \alpha \in \mathbb{Z}^d,$$

where $\delta = \{\delta_\alpha\}$ denotes the Dirac sequence on $\mathbb{Z}^d$. In this case the mask satisfies the condition

$$h_{2\alpha} = \delta_\alpha, \quad \forall \alpha \in \mathbb{Z}^d,$$

which, in terms of symbol, reads as

$$\sum_{\epsilon \in \{\pm 1\}^d} h(\epsilon z) = 2^d.$$

Suppose φ and $\tilde{\varphi}$ are scaling functions with masks h and $\tilde{h}$ respectively such that the following *biorthogonality relation* $\langle\varphi(x), \tilde{\varphi}(x-\alpha)\rangle = \delta_\alpha$ holds for any $\alpha \in \mathbb{Z}^d$. Now let g^k and $\tilde{g}^k$, $k = 1, \ldots, M-1$, be finitely supported sequences and define

$$\psi^k(x) = \sum_{\alpha\in\mathbb{Z}^d} g^k_\alpha \varphi(2x-\alpha), \quad \tilde{\psi}^k(x) = \sum_{\alpha\in\mathbb{Z}^d} \tilde{g}^k_\alpha \tilde{\varphi}(2x-\alpha).$$

We let $g^0 = h$ and $\tilde{g}^0 = \tilde{h}$, so that $\psi^0 = \varphi$ and $\tilde{\psi}^0 = \tilde{\varphi}$. The set $\{\psi^k, \tilde{\psi}^k,\ k = 0, \ldots, M-1\}$ consists then of a *biorthogonal wavelet system* if

$$\int_{\mathbb{R}^d} \psi^k(x)\tilde{\psi}^{k'}(x-\alpha)dx = \delta_\alpha \delta_{k-k'}, \quad \alpha \in \mathbb{Z}^d,\ k, k' = 0, \ldots, M-1.$$

It is well known that the above relation holds if and only if the set of sequences g^k, $\tilde{g}^k \in \ell(\mathbb{Z}^d)$, $k = 0, \ldots, M-1$, is a *discrete biorthogonal system*,

$$\sum_{\beta\in\mathbb{Z}^d} g^k_{\beta-2\alpha} \tilde{g}^{k'}_\beta = M\delta_\alpha\delta_{k-k'}, \quad \alpha \in \mathbb{Z}^d,\ k, k' = 0, \ldots, M-1.$$

In terms of symbols the biorthogonality conditions can be written as

$$\sum_{\eta\in\{\pm1\}^d} g^k(\eta z)\tilde{g}^{k'}(\eta z^{-1}) = 2^{2d}\delta_{k-k'}, \quad k, k' = 0, \ldots, M-1.$$

which are equivalent to the PR conditions (3).

Finally, requiring sum rules of arbitrary order for a scaling filter is very important in application, since that it is connected to the vanishing moment property of a biorthogonal wavelet system. For a positive integer n, we say that a sequence h on $\mathbb{Z}^d$ satisfies the *sum rules* of order n if

$$\sum_{\alpha\in\mathbb{Z}^d} h_{2\alpha+\epsilon}(2\alpha+\epsilon)^\mu = \sum_{\alpha\in\mathbb{Z}^d} h_{2\alpha}(2\alpha)^\mu, \quad \forall\mu \in \mathbb{Z}^d_+,\ |\mu| < n.$$

3. Main results on symmetry

In this section we state the main results for the construction of symmetric wavelet filters. For a complete treatment and for the proofs, the reader is referred to [2,3].

We recall that a matrix $\Delta \in \mathbb{Z}^{d\times d}$ is unimodular if $|\det(\Delta)| = 1$ or, equivalently, there exists an inverse $\Delta^{-1} \in \mathbb{Z}^{d\times d}$. Let $\mathcal{A}$ be any finite set of n unimodular matrices. We denote by $\mathcal{G}$ the group generated by $\mathcal{A}$

$$\mathcal{G} := \left\{ \prod_{j=1}^{n} \Delta_j : \Delta_j \in \mathcal{A},\ n \in \mathbb{N} \right\}.$$

A mask $h \in l(\mathbb{Z}^d)$ is $\mathcal{G}$-symmetric with a center $\xi \in \mathbb{R}^d$ if

$$h_\alpha = h_{\xi - \Delta\xi + \Delta\alpha}, \quad \alpha \in \mathbb{Z}^d, \Delta \in \mathcal{G} \tag{5}$$

Given a refinable function $\varphi \in L_2(\mathbb{R}^d)$ and a symmetry group $\mathcal{G}$, we say that φ is $\mathcal{G}$-symmetric with a center $\xi \in \mathbb{R}^d$ if

$$\varphi(x) = \varphi(\xi - \Delta\xi + \Delta x), \quad x \in \mathbb{R}^d, \Delta \in \mathcal{G}. \tag{6}$$

Given a stable solution $\varphi \in L_2(\mathbb{R}^d)$ to the refinement equation (4), then φ is $\mathcal{G}$-symmetric with a center $\xi \in \mathbb{R}^d$ if and only if its mask is $\mathcal{G}$-symmetric with the same center ξ.

For the construction of multivariate wavelet filters, we need the generalization of the *lifting scheme* proposed by Sweldens [12] to the multivariate setting. Given an initial set of finite biorthogonal filters $\{h, \tilde{h}, g^k, \tilde{g}^k,\ k = 1, \ldots, M-1\}$, then a new set of finite biorthogonal filters $\{h, \tilde{h}_{new}, g^k_{new}, \tilde{g}^k,\ k = 1, \ldots, M-1\}$ can be found as

$$g^k_{new}(z) = g^k(z) + h(z)s^k(z^2), \quad \tilde{h}_{new}(z) = \tilde{h}(z) - \sum_{k=1}^{M-1} \tilde{g}^k(z)s^k(z^{-2}) \tag{7}$$

with $z \in \mathbb{C}^d \backslash \{\mathbf{0}\}$ and where $s^k(z)$ are Laurent polynomials, for $k = 1, \ldots, M-1$. Similarly we can give the multivariate *dual lifting scheme*. Given an initial set of finite biorthogonal filters $\{h, \tilde{h}, g^k, \tilde{g}^k,\ k = 1, \ldots, M-1\}$, then a new set of finite biorthogonal filters $\{h_{new}, \tilde{h}, g^k, \tilde{g}^k_{new},\ k = 1, \ldots, M-1\}$ can be found, in terms of symbols, as

$$\tilde{g}^k_{new}(z) = \tilde{g}^k(z) - \tilde{h}(z)t^k(z^{-2}), \quad h_{new}(z) = h(z) + \sum_{k=1}^{M-1} g^k(z)t^k(z^2) \tag{8}$$

where $t^k(z)$ are Laurent polynomials, for $k = 1, \ldots, M-1$.

We can construct a symmetric dual scaling filter with the symmetry assumption made only on the scaling filter. In fact, given a symmetry group $\mathcal{G}$ and a set of biorthogonal filters $\{h, g^k, \tilde{h}, \tilde{g}^k,\ k = 1, \ldots, M-1\}$ with $\tilde{h}$ $\mathcal{G}$-symmetric around $\xi \in \mathbb{R}^d$, there always exists a dual filter $\tilde{h}^{new}$ $\mathcal{G}$-symmetric around the point $\xi \in \mathbb{R}^d$.

A similar result holds for the symmetry of the wavelet filters. Given a symmetry group $\mathcal{G}$ and a set of biorthogonal filters $\{h, g^k, \tilde{h}, \tilde{g}^k,\ k = 1, \ldots, M-1\}$ with g^k $\mathcal{G}$-symmetric around $\mu^k \in \mathbb{R}^d$, there always exists a wavelet filter g^k_{new} $\mathcal{G}$-symmetric around the point $\mu^k \in \mathbb{R}^d$. The proofs of these two theorem lead to the following systems:

$$\sum_{k=1}^{M-1}\sum_{\beta\in\Xi^K}(\tilde{g}^k_{\Delta^{-1}(\alpha-\xi)+\xi+2\beta}-\tilde{g}^k_{\alpha+2\beta})s^k_\beta=0. \tag{9}$$

$$\sum_{\beta\in\Xi^k}(h_{\Delta^{-1}(\alpha-\mu^k)+\mu^k-2\beta}-h_{\alpha-2\beta})s^k_\beta=0. \tag{10}$$

which are the basics for the SWF algorithm described in Section 4.

We conclude the section stating a result on the existence of the dual filters satisfying certain sum rules conditions. Given an interpolatory mask h, a set of biorthogonal filters $\{h, g^k, \tilde{h}, \tilde{g}^k,\ k=1,\ldots,M-1\}$ associated to h and a positive integer n, there exists a dual filter $\tilde{h}^{new}$ of h satisfying sum rules of order n if s^k is solution of the following systems

$$\sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{M-1}\sum_{\beta\in\mathbb{Z}^d}(\tilde{g}^k_{2\alpha+\epsilon+2\beta}(2\alpha+\epsilon)^\mu-\tilde{g}^k_{2\alpha+2\beta}(2\alpha)^\mu)s^k_\beta=0 \tag{11}$$

for $1<|\mu|<n$ and $\epsilon\in\{0,1\}^d$ and

$$\sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{M-1}\sum_{\beta\in\mathbb{Z}^d}(\tilde{g}^k_{2\alpha+\epsilon+2\beta}-\tilde{g}^k_{2\alpha+2\beta})s^k_\beta=-2^d \tag{12}$$

for $\mu=(0,0)$ and with $\epsilon\in\{0,1\}^d\setminus\{\mathbf{0}\}$.

4. SWF algorithm

The results obtained in the previous section allow to propose an algorithm for the construction of a symmetric wavelet system with desired number of vanishing moment and minimal support, called SWF algorithm, where SWF stands for symmetric wavelet filters.

Given

- *a filter h satisfying:*

$$\sum_{\alpha\in\mathbb{Z}^d}h(\alpha)=2^d,\quad h(0)=1,\quad h(2\alpha)=0,\quad \forall\alpha\in\mathbb{Z}^d\setminus\{0\};$$

- *a symmetry group $\mathcal{G}$;*
- *a positive integer n;*

then

1. *construct a filter bank associated to h (a trivial set of biorthogonal filters can always be found, as shown in [1]);*
2. *solve the systems (9), (11), (12);*
3. *choose the solutions s^k, $k = 1, ..., M-1$, with minimal support;*
4. *construct the new filters $\tilde{h}_{new}$, g^k_{new}, $k = 1, ..., M-1$ by means of the multivariate lifting scheme (7).*

The resulting dual filter $\tilde{h}_{new}$ is symmetric with respect to the symmetry group $\mathcal{G}$, satisfies the sum rules of order n and is optimal, in the sense that it is minimally supported.

The above algorithm allows us to obtain a $\mathcal{G}$-symmetric dual mask of the given interpolatory mask h. We can add or substitute in step 2 the system (9) with the system (10) to also have the symmetry of the wavelet filters. Now we use the results given above to construct families of bivariate filter banks symmetric with respect to the coordinate axes and with respect to hexagonal symmetry.

4.1. *Square symmetric filters - SSF*

Let $h \in l(\mathbb{Z}^2)$ be the following interpolatory filter

$$h = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & \mathbf{1} & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

and let $\mathcal{G}_1$ be the square symmetry group

$$\mathcal{G}_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

which gives the symmetry with respect to the coordinate axes. Let $n = 2$.

Since h is interpolatory, it can be associated to the following filter bank:

$$g^1(z_1, z_2) = z_1, \quad g^2(z_1, z_2) = z_2, \quad g^3(z_1, z_2) = z_1 z_2,$$

$$\tilde{h}(z_1, z_2) = 4, \quad \tilde{g}^1(z_1, z_2) = -2 + 4z_1 - 2z_1^2,$$

$$\tilde{g}^2(z_1, z_2) = -2 + 4z_2 - 2z_2^2, \quad \tilde{g}^3(z_1, z_2) = -1 + 4z_1 z_2 - z_1^2 z_2^2 - z_1^2 - z_2^2.$$

Solving the system (9) with respect to the symmetry group $\mathcal{G}_1$, and the systems (11) and (12) for $n = 2$, we obtain the following solutions

$$s^1 = \begin{bmatrix} 0 & d & f & f & d \\ 0 & e & c & c & e \\ 0 & b & \mathbf{a} & a & b \\ 0 & e & c & c & e \\ 0 & d & f & f & d \end{bmatrix}, \quad s^2 = \begin{bmatrix} g & h & l & h & g \\ p & n & m & n & p \\ p & n & \mathbf{m} & n & p \\ g & h & l & h & g \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad s^3 = \begin{bmatrix} 0 & q & r & r & q \\ 0 & z & t & t & z \\ 0 & z & \mathbf{t} & t & z \\ 0 & q & r & r & q \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with $a = -1/8 - 2f - 2c - b - 2e - 2d$, $m = -1/8 - l - 2p - 2h - 2n - 2g$ and $t = -1/16 - r - q$. If we choose the solutions with minimal support, that is $s^1 = -1/8 - 1/8z_1$, $s^2 = -1/8 - 1/8z_2$, $s^3 = -1/16 - 1/16z_1 - 1/16z_2 - 1/16z_1z_2$, we get the following $\mathcal{G}_1$-symmetric dual filter of h with sum rules of order $n = 2$

$$\tilde{h}_{new} = \begin{bmatrix} -1/16 & 0 & -3/8 & 0 & -1/16 \\ 0 & 1/4 & 1/2 & 1/4 & 0 \\ -3/8 & 1/2 & \mathbf{11/4} & 1/2 & -3/8 \\ 0 & 1/4 & 1/2 & 1/4 & 0 \\ -1/16 & 0 & -3/8 & 0 & -1/16 \end{bmatrix}$$

which is also symmetric with respect to the lines $z_1 = z_2$ and $z_1 = -z_2$. We can use the parameters s^1, s^2 and s^3 to build, by means of the equation (7), the new wavelet filters which complete the biorthogonal filter bank.

4.2. *Hexagonal symmetric filters - HSF*

Now let $n = 4$ and let $\mathcal{G}_2$ be the following hexagonal symmetry group

$$\mathcal{G}_2 = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \right.$$
$$\left. \pm \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \right\}.$$

Let $h \in l(\mathbb{Z}^2)$ be the interpolatory filter of the *hat function* [1]. Similarly to the previous example, we get the following $\mathcal{G}_2$-symmetric dual filter $\tilde{h}$ of h with sum rules of order $n = 4$:

$$\tilde{h} = \begin{bmatrix} 0 & 0 & 0 & 0 & 3/64 & 0 & 0 & 0 & 3/64 \\ 0 & 0 & 0 & 0 & -3/32 & 0 & 0 & -3/32 & 0 \\ 0 & 0 & 0 & 0 & -7/32 & -1/16 & -7/32 & 0 & 0 \\ 0 & 0 & 0 & -1/16 & 21/32 & 21/32 & -1/16 & 0 & 0 \\ 3/64 & -3/32 & -7/32 & 21/32 & \mathbf{65/32} & 21/32 & -7/32 & -3/32 & 3/64 \\ 0 & 0 & -1/16 & 21/32 & 21/32 & -1/16 & 0 & 0 & 0 \\ 0 & 0 & -7/32 & -1/16 & -7/32 & 0 & 0 & 0 & 0 \\ 0 & -3/32 & 0 & 0 & -3/32 & 0 & 0 & 0 & 0 \\ 3/64 & 0 & 0 & 0 & 3/64 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

5. Applications

In this section we want to show some applications of the filters constructed in the above examples in a typical context of 2-D signal processing, namely *edge detection*. We show the advantages of different symmetries over most widely used geometrical test images and over real data.

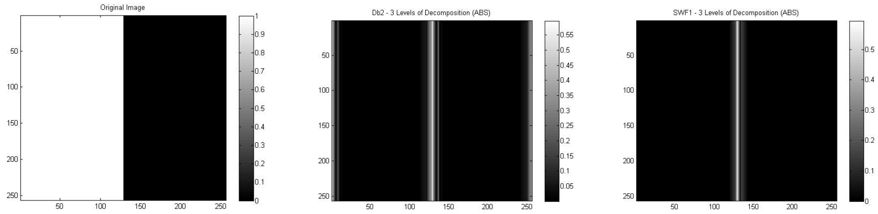

Fig. 1. Absolute values of the edge using 3 levels of decomposition. Original image on the left, Daubechies 2 filters on center and SSF filters on the right.

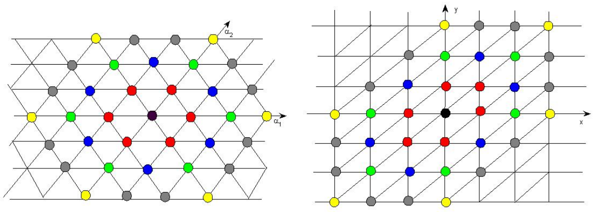

Fig. 2. Hexagonal symmetry in the hexagonal lattice on the left. Hexagonal (or three-direction) symmetry in the planar lattice on the right.

To show the edges of the test images, we consider different levels of decomposition depending on the features of the images, and we compare our filters with Daubechies 2 and Biorthogonal 2.2 filters, which are among the most commonly used filters in literature.

5.1. *Geometrical data*

Fig. 1 shows on the left a very simple geometrical image, a square matrix of 256×256 pixels with half coefficients equal to zero and half equal to one. In view of the features of the image, we apply the SSF filters (Section 4.1), which are perfectly symmetric with respect to the coordinate axes and with two vanishing moments for the primal and the dual wavelet filters.

We note that the edge detection with SSF filters (on the right) is visually better than the Daubechies 2 filters (on the center), because, with SSF filters, the edge is more sharp and the thickness of the edge is more fine. In addition, our filters produce less visual artifacts on the boundary and minimize phase distortion. In Section 4.2 we showed the construction of a set of bivariate biorthogonal wavelet filters (HSF) with symmetry respect to the hexagonal symmetry group $\mathcal{G}_2$. Before considering an application of this type of symmetry, we want to illustrate the action of the hexagonal symmetry in the hexagonal lattice and in the plane. The hexagonal lattice $\mathcal{K}$ in

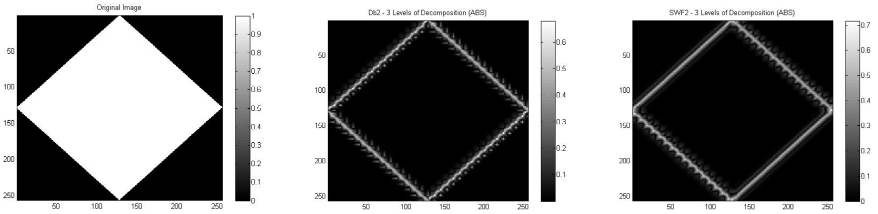

Fig. 3. Absolute values of the edge using 3 levels of decomposition. Original image on the left, Daubechies 2 filters in the center and HSF filters on the right.

$\mathbb{R}^2$ can be defined as the image of $\mathbb{Z}^2$ by the following linear transformation

$$\mathcal{K} = \Gamma \mathbb{Z}^2, \quad \Gamma = \begin{bmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix}$$

The hexagonal lattice allows the densest repartition of points in $\mathbb{R}^2$ with given minimal distance among them. Moreover, it is isotropic since it has three equivalent main directions, against, for instance, only two for the square lattice. Fig. 2 shows on the left the hexagonal symmetry in the hexagonal lattice, while on the right represents the same symmetry in the planar lattice.

To point out the importance of the symmetry of the filters according the geometry of the image to test, we use the image on the left of Fig. 3. We compare HSF filters with Daubechies 2 filters showing the absolute values of the edges. Fig. 3 shows that Daubechies 2 filters (in the center) acts over all sides of rhomb in the same way, while HSF fiters (on the right) obtains best results on the directions in which they are symmetric, where the edge is more sharp and the thickness of the edge is more fine.

5.2. *Industrial application*

A milling machine is a machine tool used for the complex shaping of metal and other solid materials. Milling machines may be operated manually or under computer numerical control (CNC). A CNC machine receives the picture of the object to reproduce, which is used by software packages to indicate the cutting operations to apply. The computer center of the machine can store a restricted number of information about the object to shape, so it is necessary to provide the machine with the minimal fundamental information for a better reproduction.

Edge detection is suitable to provide the details of a picture, that is the primary features, while the symmetry property of the filters used in edge detection allows to get more accurate edges.

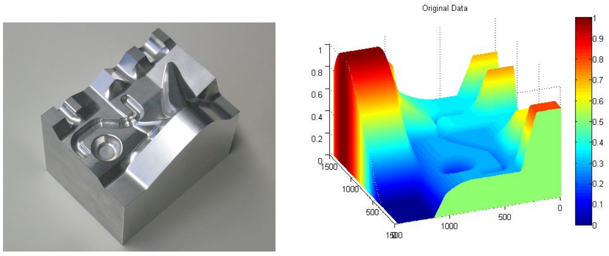

Fig. 4. Real solid object on the left and its digital representation on the right.

We applied edge detection to some $3-D$ data, provided by Siemens Company, representing a real solid test object (Fig. 4). Fig. 5 shows on the left the Siemens data showed as image, a square matrix of 1472×1472 pixels. We apply HSF filters compared with Biorthogonal 2.2 filters, showing the absolute values of the edges for 5 levels of decomposition. As one can observe from Fig. 5, edge detection with our filters (on the right) is visually better than what obtained with the Biorthogonal 2.2 filters (in the center). HSF filters detect some details that Biorthogonal 2.2 filters do not show. Furthermore our filters minimize phase distortion and produce, in particular across the boundaries, less visual artifacts than the other filters.

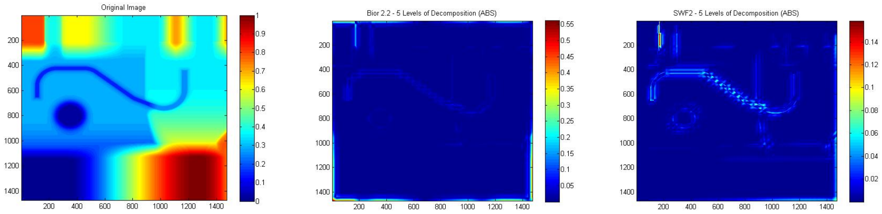

Fig. 5. Absolute values of the edge using 5 levels of decomposition. Original image on the left, Biorthogonal 2.2 filters in the center and HSF filters on the right.

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