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Integral Closure of Monomial Ideals

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Abstract

Let R be a polynomial ring over a field K. If J is an ideal of R generated by square-free monomials, then J is integrally closed. We consider an ideal I of R not generated by square-free monomials and we compute the integral closure of I, $\overline{I}$. The integral closure $\overline{I}$ is again a monomial ideal. Therefore, the integral closure is a new combinatoric object associated to the ideal. Since monomial ideals are associated to graphs, interactions will occur in various field: networks, transports, computer science, etc.
We want to highlight these issues.

Keywords: `Monomial Ideal, Integral Closure, Graph Theory.`

1. Introduction.

Let $R = K[X_1, X_2, \ldots, X_n]$ be a polynomial ring over a field K. The class of monomial ideals of R has been intensively studied and many problems arise when we would study good properties of monomial ideals, such that the integral closure and normality. The aim of this paper is to study the integral closure of monomial ideals generated by monomials of degree two in the variables X_i, $i = 1, \ldots, n$, not necessary square-free. This type of ideals deal with graph theory. In particular if we consider a graph G with loops, we can associate to G a polynomial ring with one variable X_i for each vertex x_i, and we can consider the edge ideal associated to G. The edge ideal is an ideal of R, generated by monomials of degree two, $x_i x_j$ such that $\{x_i x_j\}$ is an edge of G. We want to study the integral closure of the edge ideal associated to some classes of graphs. We will show the integral closure of an edge ideal can be considered an edge ideal of a new graph $\overline{G}$. In 2 we study the integral closure of edge ideal associated to some graphs and we compute the integral closure of a particular class of monomial ideals called ideals of mixed products. In 3 we give an application of the new edges arising from the computation of the integral closure. The author is grateful to Professor Gaetana Restuccia for useful discussions about the results of this paper.

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2. Integral closure of monomial ideals and graphs.

Definition 2.1. Let $R = K[X_1, X_2, \dots, X_n]$ be a polynomial ring over a field K, and $J \in R$ a monomial ideal. The integral closure of J is the set of all elements of R which are integral over J.

Remark 2.1. The integral closure of a monomial ideal is again a monomial ideal.

In ([9]) it is given the following description for the integral closure of J:

$$\overline{J} = \{f | f \text{ is a monomial in } R \text{ and } f^i \in J^i, \text{ for some } i \geqslant 1\ \}$$

Definition 2.2. If $J = \overline{J}$, J is said to be integrally closed. If $J^k = \overline{J^k}$ for all k then J is said to be normal.

Definition 2.3. A graph G consists of a finite set $V = \{x_1, \dots, x_n\}$ of vertices and a collection $E(G)$ of subsets of V, called edge, such that every edge of G is a pairs $\{x_i, x_j\}$ for some x_i, $x_j \in V$.

Definition 2.4. A graph G is complete if each pair $\{x_i, x_j\}$ is an edge of G for all x_i, $x_j \in V$.

Definition 2.5. A graph G is bipartite if the vertex set V can be partitioned into two disjoint subsets V_1 and V_2, and any edge joins a vertex of V_1 with a vertex of V_2.

Let G be a graph with vertex set $V = \{x_1, \dots, x_n\}$ and let $R = K[X_1, X_2, \dots, X_n]$ be a polynomial ring over a field K, with one variable X_i for each vertex x_i.

Definition 2.6. The edge ideal $I(G)$ associated to a graph G is the ideal of R generated by monomials of degree two, x_ix_j, on the $X_1, \dots, X_n$ variables, such that $\{x_i, x_j\} \in E(G)$ for $1 \leq i < j \leq n$:

$$I(G) = (\{X_iX_j | \{x_ix_j\} \in E(G)\}\,.$$

Definition 2.7. Let G be a graph with vertex set $V = \{x_1, \dots, x_n\}$, if for all edge $\{x_i, x_j\} \in E(G)$, $x_i \neq x_j$, G is called simple graph.

Proposition 2.1. *Let $I(G)$ be the edge ideal associated to a simple graph then $I(G)$ is integrally closed ([9]).*

Definition 2.8. Let G be a graph with vertex set V, if G is a bipartite graph and $E(G)$ contains every edge joining V_1 and V_2, then G is called complete bipartite graph.

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Proposition 2.2. *Let $I(G)$ be the edge ideal associated to a complete bipartite graph then $I(G)$ is integrally closed ([9], Cor 7.5.10).*

Proof. Since The edge ideal $I(G)$ associated to a simple graph G is generated by square-free monomials, the result follows from proposition 2.1 □

Definition 2.9. Let G be a graph with vertex set $V = \{x_1, \ldots, x_n\}$, G has loops if it not requiring $x_i \neq x_j$ for all edge $\{x_i, x_j\} \in E(G)$.

Definition 2.10. The edge $\{x_i, x_i\}$ is said a loop of G.

Definition 2.11. Let G be a graph with vertex set $V = \{x_1, \ldots, x_n\}$, if for all $x_i, x_j \in V, \{x_i, x_j\} \in E(G)$ and $\{x_i, x_i\}$ is a loop, then G is said a complete graph with loops.

Proposition 2.3. *Let G be a graph complete with loops, then $I(G)$ is integrally closed.*

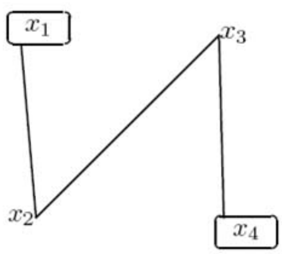
Proof. Let $V = \{x_1, \ldots, x_n\}$ be the vertex set of G, then the edge ideal associated to G is:

$$I(G) = (X_1^2, \ldots, X_n^2, X_1X_2, \ldots, X_1X_n, \ldots, X_{n-1}X_n) = (X_1, \ldots, X_n)^2 = (I_1)^2$$

where I_1 is the monomial ideal of $R = K[X_1, X_2, \ldots, X_n]$ generated by all the monomials of degree one, then $I(G)$ is integrally closed ([9], Prop 7.4.5). □

Remark 2.2. In general the ideal $I(G)$ associated to a graph with loops is not integrally closed.

Example 2.1. Let G the following graph with loops:



Then

$$I(G) = (X_1^2, X_1X_3, X_2X_3, X_2X_4, X_4^2)$$

and

$$\overline{I(G)} = (X_1^2, X_1X_3, X_2X_3, X_2X_4, X_4^2, X_1X_4)$$

2.1. *Adding edges.*

Now we consider graphs whose edge ideals are not integrally closed and we compute the integral closure.

Proposition 2.4. *Let $R = K[X_1, X_2, \dots, X_n]$ be a polynomial ring over the field K and $I(G)$ be the edge ideal of a graph with loops G, then the integral closure of $I(G)$ is generated by binomial of degree 2.*

Proof. Let $\underline{X}^{a_1}, \underline{X}^{a_2}, \dots, \underline{X}^{a_r}$ be the generators of $I(G)$ where $\underline{X}^{a_1}, \underline{X}^{a_2}, \dots, \underline{X}^{a_r}$ are monomials of degree 2 of type X_k^2 or X_lX_m. Using the geometric description of the integral closure ([9], 7.3.4) we have:

$$\overline{I(G)} = \left(\left\{\underline{X}^{\lceil\alpha\rceil} | \alpha \in conv(a_1, a_2, \dots, a_n)\right\}\right)$$

with

$$conv(a_1, a_2, \dots, a_r) = \left\{\sum_{i=1}^{r} \lambda_i a_i | \sum_{i=1}^{r} \lambda_i = 1, \lambda_i \in \mathbb{Q}_+\right\}$$

Let f be a generator of $\overline{I(G)}$, $f = \underline{X}^{\lceil\alpha\rceil}$, then

$$\alpha = \left(\sum_{i=1}^{r} \lambda_i a_{i_1}, \dots, \sum_{i=1}^{r} \lambda_i a_{i_n}\right) \in \mathbb{Q}_+^n$$

with

$$a_i = (a_{i_1}, a_{i_2}, \dots, a_{i_n}) \text{ and } a_{i_j} \in \{0, 1, 2\}$$

The generic element of α, α_j, $1 \le j \le n$, is

$$\sum_{i=1}^{r} \lambda_i a_{i_j} = \lambda_1 a_{1_j} + \lambda_2 a_{2_j} + \dots + \lambda_k \cdot 2 + \dots + \lambda_r a_{r_j} = 2\lambda_k + (1 - \lambda_k) = \lambda_k + 1 \le 2$$

□

Corollary 2.1. *Let X_k^2 and X_j^2 be two generators of the edge ideal of the graph G, then X_kX_j is a generator of the integral closure of $I(G)$.*

Proof. We consider the obtained convex hull placing $\lambda_k = \frac{1}{2}$ and $\lambda_j = \frac{1}{2}$ and $\lambda_i = 0\,\forall i \ne k, j$. Then

$$\alpha = \left(0, \dots, \underbrace{1}_{\alpha_k}, \dots, \underbrace{1}_{\alpha_j}, \dots, 0\right) \in conv(a_1, \dots, a_n)$$

□

Example 2.2. Let G the following graph:

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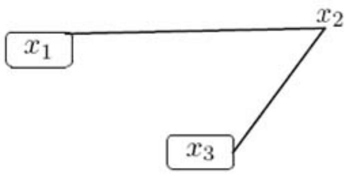


Then

$$I(G) = (X_1^2, X_1X_2, X_2X_3, X_3^2) \subset K[X_1, X_2, X_3]$$

$$\overline{I(G)} = \left\{ (X_1X_2X_3)^{\lceil \alpha \rceil} | \alpha \in conv(a_1, a_2, a_3, a_4) \right\}$$

with

$$a_1 = (2,0,0),\ a_2 = (1,1,0),\ a_3 = (0,1,1),\ a_4 = (0,0,2)$$

$$conv(a_1, a_2, a_3, a_4) = \left\{ \sum_{i=1}^{4} \lambda_i a_i | \sum_{i=1}^{4} \lambda_i = 1, \lambda_i \in \mathbb{Q}_+ \right\}$$

and

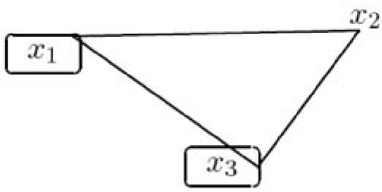
$$\alpha = \left(\sum_{i=1}^{4} \lambda_i a_{i_1}, \sum_{i=1}^{4} a_{i_2}, \sum_{i=1}^{4} \lambda_i a_{i_3} \right) \in \mathbb{Q}_+^3$$

Then

$$\overline{I(G)} = (X_1^2, X_1X_2, X_2X_3, X_3^2, X_1X_3)$$

Remark 2.3. Since the integral closure of an edge ideal $I(G)$ is again a monomial ideal of degree 2, we can associate $\overline{I(G)}$ to a graph, denoted by $\overline{G}$.

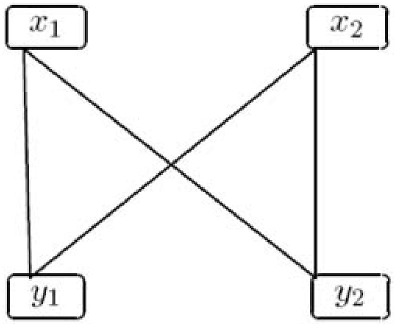
Example 2.3. In the example 2.2 the graph $\overline{G}$ associated to $\overline{I(G)}$ is:



Definition 2.12. Let G be a graph with vertex set $V = \{x_1, \ldots, x_n\}$, if G is a complete bipartite graph and for each vertex of V there is a loop, then G is said a strong quasi-bipartite graph ([3]).

Remark 2.4. Let $I(G)$ be the edge ideal associated to a strong quasi-bipartite graph, in general $I(G)$ is not integrally closed.

Example 2.4. Let G be the following strong quasi-bipartite graph:



Then

$$I(G) = (X_1^2, X_1Y_1, X_1Y_2, X_2^2, X_2Y_1, X_2Y_2, Y_1^2, Y_2^2)$$
$$\overline{I(G)} = (X_1^2, X_1X_2, X_1Y_1, X_1Y_2, X_2^2, X_2Y_1, X_2Y_2, Y_1^2, Y_1Y_2, Y_2^2)$$

Proposition 2.5. *Let G be a strong quasi-bipartite graph, let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ be the polynomial ring over a field K associated to G, then*
$$\overline{I(G)} = I_1^2 + I_1J_1 + J_1^2$$
where $I_1 = (X_1, \ldots, X_n)$, $J_1 = (Y_1, \ldots, Y_m)$ ([3]).

2.2. *Adding lattices points.*

A particular class of monomials ideals are the ideals generated by mixed products introduced by Restuccia and Villarreal in ([7]).

Definition 2.13. Let $R = K[X_1, X_2, \ldots, X_n; Y_1, \ldots, Y_m]$ be a polynomial ring in two disjoint sets of variables over a field K. Given non negative integers k, r, s, t such that $k + r = s + t$, the ideals of mixed products L are:

$$L = I_kJ_r + I_sJ_t$$

where I_k is the ideal of R generated by the square-free monomials of degree k in the variables X_i and J_r is the ideal of R generated by the square-free monomials of degree r in the variables Y_j.

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Theorem 2.1. *Let $R = K[X_1, X_2, \ldots, X_n; Y_1, \ldots, Y_m]$ be a polynomial ring in two disjoint sets of variables over a field K. The only normal mixed products ideals are ([7]):*

(i) $L = I_k J_r + I_{k+1} J_{r-1}$, $k \geq 0$, $r \geq 1$
(ii) $L = I_k J_t$, $k \geq 1$, $t \geq 1$
(iii) $L = J_r + I_m J_t$, *and* $L = I_k + J_m I_t$

We are interesting to study the integral closure of the mixed products ideals that are not normal.

Proposition 2.6. *Let $R = K[X_1, X_2, \ldots, X_n; Y_1, \ldots, Y_m]$ be a polynomial ring over a field K and $L = I_s + J_s$. If $n = m = s + 1$, then for $i \geq s - 1$*

$$\overline{L^{s+i}} = \left(L^{s+i}, hL^{i-1}\right),$$

where $h = X_1^{s-1} \cdots X_{s+1}^{s-1} Y_1 \cdots Y_{s+1}$ ([4])

Remark 2.5. If we consider the mixed products ideals, $L = I_k J_r + I_s J_t$ with $s \geq k + 2$, $k \geq 1$, $t \geq 1$, there is a computation evidence for the expression of the powers of the integral closure $\overline{L}$ given by

$$\overline{L^i} = (L^i, fL^{i-3}),$$

where $f = X_1^2 X_2^2 X_3^2 Y_1^2 Y_2^2 Y_3^2$ and $i \geq 3$.

Remark 2.6. Computing the integral closure of a monomial ideals we have new lattice points. In the remark 2.5 the point $(2, 2, 2; 2, 2, 2)$ is a new lattice point of $\overline{L^3}$. Geometric realizations of the new lattices points arising from the computation of the integral closure are very important for the security problems.

3. Applications.

In this section we are going to explain some applications in the direction of the topic of security. The same direction has been followed in the paper ([1], [2]). For our aim, we are interesting to study the graphs whose edge ideals are not integrally closed that is to say graph with loops.
We suppose two entities, denoted by **A** and **B**, want to communicate in secret way. **A** send a message to **B**, **B** elaborates the message and he discovers the true meaning.
We suppose the message is a graph with loops, for example associated to a map of a city, a road-map, a rail network, etc., **B** computes the integral closure of $I(G)$, and then **B** obtains a new graph associated to $\overline{I(G)}$ which is the true meaning of the message.

Fig. 1. A map of a city

Example 3.1. We consider a map of a city.
We can identify some streets with edges. We turn of white the vertices with loop in white and in yellow the others. The loops represent the traffic circles.

Fig. 2. A graph

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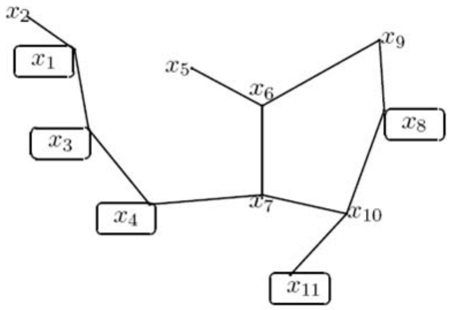
Now, we simplify our graph:

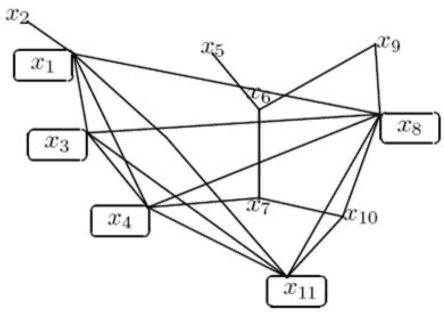


We compute the integral closure of $I(G)$, $\overline{I(G)}$, and we obtain:

$$I(G) = (X_1^2, X_1X_2, X_1X_3, X_3^2, X_3X_4, X_4^2, X_4X_7, X_5X_6, X_6X_7,$$
$$X_6X_9, X_7X_{10}, X_8^2, X_8X_9, X_8X_{10}, X_{10}X_{11}, X_{11}^2)$$

$\overline{I(G)} = (X_1^2, X_1X_2, X_1X_3, X_3^2, X_3X_4, X_4^2, X_4X_7, X_5X_6, X_6X_7, X_6X_9, X_7X_{10}, X_8^2,$

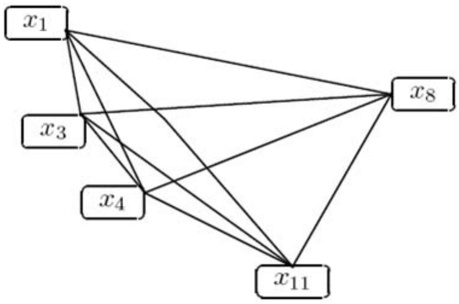
$X_8X_9, X_8X_{10}, X_{10}X_{11}, X_{11}^2, X_1X_4, X_3X_8, X_1X_8, X_4X_8, X_8X_{11}, X_3X_{11}, X_1X_{11}, X_4X_{11})$
The graph associated to $\overline{I(G)}$ is:



Remark 3.1. If x_i and x_j are vertices with loop then $X_iX_j \in \overline{I(G)}$

Remark 3.2. The subgraph G^* obtained from $\overline{G}$ erasing the vertices without loops is a complete graph.

In the example 3.1, G^* is:



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