Communications to SIMAI Congress, ISSN 1827-9015, Vol. 3 (2009) 296 (9pp)

DOI: 10.1685/CSC09296

Planar small oscillations of a container acted by an elastic force partially filled by an heavy viscous liquid

Pierre Capodanno[1], Doretta Vivona[2]

[1] *Professor Emeritus, Facoltè de Sciences*
Universitè de Besançon, France
pierre.capodanno@neuf.fr

[2] *MEMOMAT, Facoltà di Ingegneria*
"Sapienza" Università di Roma, Italy
vivona@dmmm.uniroma1.it

Abstract

From the equations of the system container-liquid, one deduces the variational equation of the problem, and then an operatorial equation in a suitable Hilbert space. The study of the normal oscillations is reduced to the study of an operator bundle, the kind of which is well known. One obtains an infinity of aperiodic damped motions and, for a sufficiently small viscosity, a finite number of oscillatory damped motions.

Keywords: `Viscous liquid, small oscillations, operator bundle.`

1. Introduction.

The problem of the small oscillations of a container, partially filled by an inviscid heavy liquid, and acted by a given system of forces, has been studied by many authors: Sretenskii, Narimanov, Moiseyen and others [7].

Moreover, Kopachevskii, Krein NgoZuy Can in [6] have studied, in details, the problem of the small oscillations of an heavy viscous liquid which partially fills a fixed container; by using a little long method, the authors reduced this problem to a operatorial equation of a known class.

We can obtain this equation more easily by leaving from the variational equation or, in an equivalent way, from the Principle of the virtual works. This last method was used by Capodanno in [2], in an article on the oscillations of two viscous liquids placed one upon another.

We apply this method to the planar problem of the small oscillations of a container partially filled by an heavy viscous liquid which is acted by two horizontal forces: the first is an elastic force, proportional to its

Received 14/02/2009, in final form 30/06/2009
Published 31/07/2009

displacement, due, for example to two springs, the second is a dissipative force proportional to the velocity, due to the friction.

This problem is a part of the study of the general problem of the oscillations of a viscous liquid in a container, acted by a given system of forces, near a stable equilibrium position. In a previous paper [4] we have considered the problem of a pendulum partially filled by a viscous liquid.

2. Position of the problem

We restrict ourselves to a two dimensional problem.

We consider a liquid in a domain Ω, limited, in equilibrium, by a free horizontal line Γ and a regular curve S. We denote with $RC(0_1, X_1X_2)$ a system of orthogonal inertial planar axes, (0_1X_1 coincides with Γ), and with $Rc(0, x_1x_2)$ a system of orthogonal inertial planar axes, associated with the container, which coincides with RC in the equilibrium position.

The motion of the container is defined by $\overrightarrow{0_10} = X(t)\overrightarrow{e_1}$, where $\overrightarrow{e_1}$ is the unit vector of the axis 0_1X_1 or $0x_1$ and $X(t)$ is a unknown function of the time t , belonging to C^2.

The container moves in the horizontal direction, acted by the elastic force $-k^2X(t)\overrightarrow{e_1}$ and by the dissipative force $-\phi\dot{X}(t)\overrightarrow{e_1}$, where k and ϕ are positive constants. Between the wall of the container and the free surface there is a gas with constant pressure p_0.

3. Equations of the motion of the system liquid-container

We denote with ρ and μ the density and the coefficient of viscosity of the liquid (they are constant) and with $\overrightarrow{u}(t, x_1, x_2)$ the displacement of a particle of the liquid with respect to the axes of Rc from its equilibrium position. We suppose that the displacement is small and we study classically the problem in linear theory.

We write the **linearized Navier-Stokes equations of the motion of the liquid** in the following form:

$$(1) \quad \begin{cases} \rho\ddot{u}_i = \dfrac{\partial\sigma_{ij}(P,\dot{\overrightarrow{u}})}{\partial x_j} + \dfrac{\partial}{\partial x_i}(-\rho g x_2) - \rho\ddot{X}\delta_{i1} & (\mathrm{i} = 1, 2) \\ div\,\dot{\overrightarrow{u}} = 0 \qquad \text{in } \Omega \quad , \end{cases}$$

where $\sigma_{ij}(P, \dot{\overrightarrow{u}})$ are the components of the stress tensor and δ_{ij} indicates the Kronecker's symbol.(Here we have taken the sum with respect to i).

We add the laws between the stresses and the velocities of deformation:

DOI: 10.1685/CSC09296

(2) $$\sigma_{ij}(P,\overset{\cdot}{\overrightarrow{u}}) = -P\delta_{ij} + 2\mu\varepsilon_{ij}(\overset{\cdot}{\overrightarrow{u}}) \qquad ,$$

where P is the pressure and $\varepsilon_{ij}(\overset{\cdot}{\overrightarrow{u}}) = \frac{1}{2}(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i})$ are the components of the tensor of the velocities of deformation.

In linear theory, we can integrate the equation $div\overset{\cdot}{\overrightarrow{u}} = 0$ from the date of the equilibrium position to the instant t, and we obtain

(1') $div\ \vec{u} = 0 \quad$ in Ω.

Now we consider the **boundary conditions:**

- the no-slip condition at the rigid wall S is:

$$\overset{\cdot}{\overrightarrow{u}}\ |_S = 0 \quad ;$$

in the same way, we can replace it by

(3) $$\overrightarrow{u}\ |_S = 0 \quad ;$$

- if $x_2 = \zeta(t, x_1)$ indicates the equation of the free line, we have in the same way

(4) $$\zeta = u_2\ |_\Gamma$$

(this condition gives only ζ, i.e. the free line, if $\vec{u}$ is knonw);

and the dynamic conditions :

(5) $$\sigma_{12} = 0, \qquad \sigma_{22} = -p_0 \quad , \quad \text{on } \Gamma,$$

i.e. (5') $$\frac{\partial \dot{u}_1}{\partial x_2} + \frac{\partial \dot{u}_2}{\partial x_1} = 0 \quad , \quad -P + 2\mu\frac{\partial \dot{u}_2}{\partial x_2} = -p_0 \quad \text{on } \Gamma \quad .$$

The system container-liquid is acted by the elastic force and the dissipative force, by the weight and the normal components of the reactions of the framework, which are vertical (we know that the forces due to the constant pression of the gas form classically a null system).

Then, the projection on the axis $0x_1$ of the theorem of the quantity of motion gives the following equation, where M is the mass of the container:

(6) $$M\ddot{X} + \int_\Omega \rho(\ddot{u}_1 + \ddot{X})d\Omega = -k^2 X - \phi\dot{X} \qquad ,$$

or, if $M_0 = M + \rho\ meas(\Omega)$ is the mass of the system:

(6') $$M_0\ddot{X} + \rho\int_\Omega \ddot{u}_1 d\Omega = -k^2 X - \phi\dot{X}.$$

From these equations we want to deduce the variational equation of the problem.

For a formal calculation, we introduce the **space of the admissible displacements**:

$W = \{\overrightarrow{w}/\ div\overrightarrow{w} = 0 \text{ on } \Omega, \overrightarrow{w}\ |_S = 0\}$,

with $\vec{w}$ sufficiently smooth. This space will be precised later.

First, as $div\overrightarrow{w} = 0$, we have $\int_\Omega div\overrightarrow{w}\ d\Omega = 0$, and therefore by using Green formula and (3),$\int_\Gamma -p_0\ w_2\ |_\Gamma\ d\Gamma = 0$. By multipling the i-th Navier-Stokes' equation for $\overline{w}_i$, by adding with respect to i, and by integrating on Ω, we obtain

$$\int_\Omega \rho\ddot{u}_i\overline{w}_i\ d\Omega = \int_\Omega \frac{\partial\sigma_{ij}(P,\overset{\cdot}{\overrightarrow{u}})}{\partial x_j}\overline{w}_i d\Omega + \int_\Omega \frac{\partial}{\partial x_i}(-\rho g x_2)\overline{w}_i d\Omega - \rho\ddot{X}(t)\int_\Omega \overline{w}_1 d\Omega.$$

But
$\int_\Omega \frac{\partial \sigma_{ij}(P,\overset{\cdot}{\overrightarrow{u}})}{\partial x_j}\overline{w}_i d\Omega = \int_\Omega [\frac{\partial}{\partial x_j}(\sigma_{ij}(P,\overset{\cdot}{\overrightarrow{u}})\overline{w}_i) - \sigma_{ij}(P,\overset{\cdot}{\overrightarrow{u}})\frac{\partial \overline{w}_i}{\partial x_j}]d\Omega,$
and the Green's formula gives
$\int_\Omega [\frac{\partial}{\partial x_j}(\sigma_{ij}(P,\overset{\cdot}{\overrightarrow{u}})\overline{w}_i)]d\Omega = \int_{\Gamma+S} \sigma_{ij}(P,\overset{\cdot}{\overrightarrow{u}})n_j \overline{w}_i dS,$
where n_j are the components of the external normal unit vector $\overrightarrow{n}$ to the boundary of Ω.

Taking into account that $\overrightarrow{w}\mid_S = 0$, and (5), we get easily
$\int_\Omega \frac{\partial}{\partial x_j}(\sigma_{ij}(P,\overset{\cdot}{\overrightarrow{u}})\overline{w}_i)d\Omega = \int_\Gamma -p_0\ w_2\mid_\Gamma d\Gamma = 0;$
by using (2) , we have
$-\int_\Omega \sigma_{ij}(P,\overset{\cdot}{\overrightarrow{u}})\frac{\partial \overline{w}_i}{\partial x_j}d\Omega = \int_\Omega [P\delta_{ij} - 2\mu\varepsilon_{ij}(\overset{\cdot}{\overrightarrow{u}})]\frac{d\overline{w}_i}{dx_j}d\Omega =$
$\int_\Omega P\frac{d\overline{w}_i}{dx_i}d\Omega - 2\mu\int_\Omega \varepsilon_{ij}(\overset{\cdot}{\overrightarrow{u}})\frac{d\overline{w}_i}{dx_j}d\Omega.$
As $\frac{d\overline{w}_i}{dx_i} = div\overrightarrow{w} = 0$ and ε_{ij} are symmetric,
$-\int_\Omega \sigma_{ij}(P,\overset{\cdot}{\overrightarrow{u}})\frac{\partial \overline{w}_i}{\partial x_j}d\Omega = -2\mu\int_\Omega \varepsilon_{ij}(\overset{\cdot}{\overrightarrow{u}})\varepsilon_{ij}(\overline{\overrightarrow{w}})d\Omega$
and finally
$\int_\Omega \frac{\partial \sigma_{ij}(P,\overset{\cdot}{\overrightarrow{u}})}{\partial x_j}\overline{w}_i d\Omega = -2\mu\int_\Omega \varepsilon_{ij}(\overset{\cdot}{\overrightarrow{u}})\varepsilon_{ij}(\overline{\overrightarrow{w}})d\Omega.$
On the other hand,
$\int_\Omega \frac{\partial}{\partial x_i}(-\rho g x_2)\overline{w}_i d\Omega = \int_\Omega \overrightarrow{grad}(-\rho g x_2)\cdot\overline{\overrightarrow{w}}d\Omega = \int_\Omega [div(-\rho g x_2)\overline{\overrightarrow{w}} + \rho g x_2 div\overline{\overrightarrow{w}}]d\Omega;$
by applying the Green formula, it is
$\int_\Omega \frac{\partial}{\partial x_i}(-\rho g x_2)\overline{w}_i d\Omega = -\int_\Gamma \rho g\ u_2\mid_\Gamma \overline{w_2}\mid_\Gamma d\Gamma.$
Therefore, we get
$\int_\Omega \rho\overset{\cdot\cdot}{\overrightarrow{u}}\cdot\overline{\overrightarrow{w}}d\Omega + \rho\int_\Omega \ddot{X}\cdot\overline{w_1}d\Omega + 2\mu\int_\Omega \varepsilon_{ij}(\overset{\cdot}{\overrightarrow{u}})\varepsilon_{ij}(\overline{\overrightarrow{w}})d\Omega$
$+\int_\Gamma \rho g u_2\mid_\Gamma \overline{w_2}\mid_\Gamma d\Gamma = 0.$

By multiplying the (6') for $\overline{\xi}$, where ξ is any complex number and adding, we obtain the *variational equation*:

(7) $\int_\Omega \rho\overset{\cdot\cdot}{\overrightarrow{u}}\cdot\overline{\overrightarrow{w}}d\Omega + M_0\ddot{X}\overline{\xi} + \rho\int_\Omega(\ddot{X}\overline{w}_1 + \ddot{u}_1\overline{\xi})d\Omega +$
$+2\mu\int_\Omega \varepsilon_{ij}(\overset{\cdot}{\overrightarrow{u}})\varepsilon_{ij}(\overline{\overrightarrow{w}})d\Omega + \phi\dot{X}\overline{\xi} + \int_\Gamma \rho g u_{2|\Gamma}\overline{w_2}\mid_\Gamma d\Gamma + k^2 X\overline{\xi} = 0,$
$\forall\overrightarrow{w}\in W,\ \ \xi\in\mathcal{C},$
where Ω and Γ are now the fluid domain and the free line in the equilibrium position.

Directly we could have obtained this equation, by using the Principle of virtual works: in fact, the first line represents the opposite of the virtual work of the inertia forces of the systems, the second one is, classically, the virtual work of the viscosity forces, of the elastic force, of the dissipative force and of the weight, respectively.

DOI: 10.1685/CSC09296

4. Variational formulation of the problem

In this section we are giving the exact variational formulation of the problem. For this reason we introduce the space (of the **admissible displacements of the liquid**):

$V = \{\overrightarrow{u} \in \mathcal{H}^1(\Omega) = [H^1(\Omega)]^2 / \; div\overrightarrow{u} = 0; \; \overrightarrow{u} \mid_S = 0\}$,

equipped with the scalar product

$$(\overrightarrow{u}, \overrightarrow{w})_V = 2\mu \int_\Omega \varepsilon_{ij}(\overrightarrow{u})\varepsilon_{ij}(\overline{\overrightarrow{w}})d\Omega.$$

Its associated norm is equivalent to the classical norm $\| \cdot \|_V$ of $\mathcal{H}^1(\Omega)$ by virtue of Korn inequality. We denote with $\mathcal{V}$ the product space $\mathcal{C}\times V$ of the pairs $U = \{X, \overrightarrow{u}\}$, $\widetilde{U} = \{\xi, \overrightarrow{w}\}$ with the scalar product

$\left(U, \widetilde{U}\right)_\mathcal{V} = \phi\,(X, \xi)_\mathcal{C} + (\overrightarrow{u}, \overrightarrow{w})_V = \phi X\overline{\xi} + (\overrightarrow{u}, \overrightarrow{w})_V$.

Now we put,

$\left(U, \widetilde{U}\right)_\mathcal{H} = \int_\Omega \rho\overrightarrow{u} \cdot \overline{\overrightarrow{w}}d\Omega + M_0\ddot{X}\overline{\xi} + \rho\int_\Omega(\ddot{X}\overline{w}_1 + u_1\overline{\xi})d\Omega;$

also in this case, it is easy to see that it is a scalar product and its associate norm $\| \cdot \|_\mathcal{H}$ is equivalent to the norm of the space of $\mathcal{C} \times \mathcal{L}^2(\Omega) = \mathcal{C} \times [L^2(\Omega)]^2$,defined by

$\| U \|^2 = \mid X \mid^2 + \int_\Omega \mid \overrightarrow{u} \mid^2 d\Omega.$

We denote with H the functional completion of V for the norm of $\mathcal{L}^2(\Omega)$. It can be shown [3] that

$H = \{\overrightarrow{u} \in \mathcal{L}^2(\Omega) / \; div\overrightarrow{u} = 0; \; u_n = 0 \text{ on } [H_{00}^{1/2}(S)]'\}.$

We shall call $\mathcal{H} = \mathcal{C}\times H$ the functional completion of $\mathcal{V}$ for the norm $\| \cdot \|_\mathcal{H}$. Setting:

$b\left(U, \widetilde{U}\right) = \left(U, \widetilde{U}\right)_\mathcal{V} =: 2\mu\int_\Omega \varepsilon_{ij}(\overrightarrow{u})\varepsilon_{ij}(\overline{\overrightarrow{w}})d\Omega + \phi X\overline{\xi},$

$a\left(U, \widetilde{U}\right) = \int_\Gamma \rho g u_{2|\Gamma}\overline{w_2} \mid_\Gamma d\Gamma + k^2 X\overline{\xi}\,.$

Then, we can deduce from (7) the following:

Theorem 4.1. *The exact variational formulation of the problem is: to find* $U(t) = \{X(t), \overrightarrow{u}(t)\} \in \mathcal{V}$ *such that*

$$\left(\ddot{U}, \widetilde{U}\right)_\mathcal{H} + b\left(\dot{U}, \widetilde{U}\right) + a\left(U, \widetilde{U}\right) = 0 \qquad \forall\widetilde{U} \in \mathcal{V}. \tag{8}$$

5. Operatorial equation

In this paragraph we want to deduce an operatorial equation from (8), by studying the forms $b\left(U, \widetilde{U}\right)$ and $a\left(U, \widetilde{U}\right)$.

First, the embedding of $\mathcal{V}$ into $\mathcal{H}$ is, obviously, dense and continuous. It is, also, compact: in fact, let $\{U_n\} = \{X_n, \overrightarrow{u_n}\}$ be a sequence weakly converging in $\mathcal{V}$. By virtue of Rellich Theorem, $\{\overrightarrow{u_n}\}$ strongly converges in $\mathcal{L}^2(\Omega)$, so $\{U_n\}$ strongly converges in $\mathcal{C} \times \mathcal{L}^2(\Omega)$ and then in $\mathcal{H}$.

Second, the sesquilinear form $b\left(U, \widetilde{U}\right)$ is Hermitian, continuous and coercive in $\mathcal{V} \times \mathcal{V}$. To this form, classically, we can associate an *unbounded* operator $\mathcal{B}$ of $\mathcal{H}$, whose the domain of definition $D(\mathcal{B}\ )$ is the set of the $U \in \mathcal{V}$, such that $b\left(U, \widetilde{U}\right)$ is continuous in $\mathcal{V}$ with respect to the topology of $\mathcal{H}$, i.e. such that $\mid \mathcal{B}\left(U, \widetilde{U}\right) \mid \leq K_U \parallel \widetilde{U} \parallel_{\mathcal{H}}, \forall\ \widetilde{U} \in \mathcal{V}$, where K_U depends on U. We have

$$b\left(U, \widetilde{U}\right) = \left(\mathcal{B}U, \widetilde{U}\right)_{\mathcal{H}} \qquad \forall U \in D(\mathcal{B}\ ), \forall \widetilde{U} \in \mathcal{V}\ .$$

The properties of the operators $\mathcal{B}^{\ -1}, \mathcal{B}^{\ 1/2}, \mathcal{B}^{\ -1/2}$ are well known [8] and they are recalled in the paragraph 5.

Finally, we study the sesquilinear form $a\left(U, \widetilde{U}\right)$. For every $u_2 \in L^2(\Gamma)$, (not only for the normal trace on Γ of any element $\overrightarrow{u}$ of V), $X \in\ \mathcal{C}$ and $\widetilde{U} = \{\xi, \overrightarrow{w}\} \in \mathcal{V}$, we have (also in this case we denote

$$a\left(U, \widetilde{U}\right) = \int_\Gamma \rho g u_2 w_2 \mid_\Gamma d\Gamma + k^2 X \overline{\xi}\ )$$

$$\mid a\left(U, \widetilde{U}\right) \mid \leq \rho g \parallel u_2 \parallel_{L^2(\Gamma)} \parallel w_2 \mid_\Gamma \parallel_{L^2(\Gamma)} + k^2 \mid X \mid \mid \xi\ \ \mid .$$

Later, C_1, C_2, C_3 will be positive constants. As the trace application is continuous, we get

$$\mid a\left(U, \widetilde{U}\right) \mid \leq C_1 \left[\ \parallel u_2 \parallel_{L^2(\Gamma)} \parallel \overrightarrow{w} \parallel_V + \mid X \mid \mid \xi\ \ \mid \right],$$

from which we obtain

$$\mid a\left(U, \widetilde{U}\right) \mid \leq C_1(\mid X \mid^2 + \parallel u_2 \parallel^2_{L^2(\Gamma)})^{1\backslash 2} \cdot (\mid \xi\ \ \mid^2 + \parallel \overrightarrow{w} \parallel^2_V)^{1\backslash 2},$$

but we can write

$$\xi^{\ 2} + \parallel \overrightarrow{w} \parallel^2_V \leq C_2 \parallel \widetilde{U} \parallel^2_{\mathcal{V}}\ .$$

On the other hand, we introduce the space $\widehat{\mathcal{V}} = \mathcal{C} \times L^2(\Gamma)$; if $U_2 = \{X, u_2\}$ and $\widetilde{U}_2 = \{\xi, w_2\}$ belong to this space, we can define a scalar product in it as $\left(U_2, \widetilde{U}_2\right)_{\widehat{\mathcal{V}}} = X\overline{\xi} + \int_\Gamma u_2 \overline{w_2} d\Gamma$.

Then, we have

$$(9) \qquad \mid a\left(U, \widetilde{U}\right) \mid \leq C_3 \parallel U_2 \parallel^2_{\widehat{\mathcal{V}}} \parallel \widetilde{U} \parallel_{\mathcal{V}}, \forall U_2 \in \widehat{\mathcal{V}}, \forall \widetilde{U} \in \mathcal{V}.$$

When U_2 is fixed in $\widehat{\mathcal{V}}$, $a\left(U, \widetilde{U}\right)$ is an antilinear continuous form in $\mathcal{V}$, by virtue of Riesz Theorem, the equation

$$b\left(W, \widetilde{U}\right) = a\left(U, \widetilde{U}\right) \qquad \forall\ \widetilde{U} \in \mathcal{V}$$

DOI: 10.1685/CSC09296

has a unique solution in $\mathcal{V}$, which shall denote $W = \mathcal{T}\ U_2$, where $\mathcal{T}$ is a linear operator of the whole space $\widehat{\mathcal{V}}$ in $\mathcal{V}$. We prove that $\mathcal{T}$ is bounded. In the relation $\left(\mathcal{T}\ U_2, \widetilde{U}\right)_{\mathcal{V}} = a\left(U, \widetilde{U}\right)$, we take $\widetilde{U} = \mathcal{T}\ U_2$, from (9) we obtain

$$\| \mathcal{T}\ U_2 \|_{\mathcal{V}} \leq C_3 \| U_2 \|_{\widehat{\mathcal{V}}} \qquad \forall U_2 \in \widehat{\mathcal{V}}.$$

Therefore we can write the *variational equation* of the problem:
to find $U(t) \in V$ *such that*

$$(10) \qquad \left(\ddot{U}, \widetilde{U}\right)_{\mathcal{H}} + b\left(\dot{U}, \widetilde{U}\right) + b\left(\mathcal{T}\ U_2, \widetilde{U}\right) = 0 \qquad \forall \widetilde{U} \in \mathcal{V},$$

where $U_2 = \{X, u_{2|\Gamma}\}$, and $u_{2|\Gamma}$ is now the normal trace on Γ of an element $\overrightarrow{u}$ of $\mathcal{V}$. By virtue of a classical result [3] this variational equation is equivalent to an operatprial equation. More precisely, we have the

Theorem 5.1. *The operatorial equation is:*
to find $U(t) \in \mathcal{V}$ *such that*

$$(11) \qquad \ddot{U} + \mathcal{B}\dot{U} + \mathcal{B}(\mathcal{T}\ U_2) = 0.$$

6. Operator bundle of the problem and esistence of the eigenvalues

We are finding solutions of the equation (11) under the form:

$$U(t, x_1, x_2) = e^{-\lambda t}\widehat{U}(x_1, x_2), \qquad \lambda \in \mathcal{C}\ ;$$

we obtain

$$(12) \qquad \lambda^2\widehat{U} - \lambda\mathcal{B}\widehat{U} + \mathcal{B}(\mathcal{T}\widehat{U}_2) = 0$$

In order to prove the existence of the eigenvalues, we want to substitute the equation (12) with an equation on the space $\mathcal{H}$. Setting $U^* = \mathcal{B}^{1/2}\ \widehat{U}$ and applying the operator $\mathcal{B}^{-1/2}$, we obtain

$$(13) \qquad U^* = \lambda\mathcal{B}^{-1}U^* + \tfrac{1}{\lambda}\mathcal{B}^{1/2}[\mathcal{T}(\mathcal{B}^{-1/2}U^*)_2],$$

where the unknown function $U^* \in \mathcal{H}$.

The equation (13) is of the kind considered by **Askerov, Krein, Laptev,** [1] i.e.

$$f = \lambda\mathcal{P}f + \tfrac{1}{\lambda}\mathcal{Q}f,$$

where f belongs to a Hilbert space $\mathcal{H}$, $\mathcal{P}$ is a self adjoint, compact, positive definited operator and $\mathcal{Q}$ ia a self adjoint, compact and non negative operator.

In this case, we have

1) $\mathcal{P} = \mathcal{B}^{-1/2}$. It is classically a compact operator of $\mathcal{H}$ in $\mathcal{H}$, self adjoint and positive definite;

2) Set $\mathcal{Q} = \mathcal{B}^{1/2}[\mathcal{T}(\mathcal{B}^{-1/2}\ \cdot)_2]$. The application

$U^* \in \mathcal{H} \longrightarrow (\mathcal{B}^{1/2}U^*)_2 \in \widetilde{\mathcal{V}}.$

is compact. Indeed, we can decompone it in

$U^* \in \mathcal{H} \longrightarrow \hat{U} = (\mathcal{B}^{1/2}U^*) \in \mathcal{V} \longrightarrow \hat{U}_2^* = (\mathcal{B}^{1/2}U^*)_2 \in \hat{\mathcal{V}}.$

$\mathcal{B}^{1/2}$ is classically continuous from $\mathcal{H}$ in $\mathcal{V}$ [8]. The application $\hat{U} \longrightarrow \hat{U}_2$ is compact, because the application normal trace from V into $L^2(\Gamma)$ is compact and the dimension of $\mathbb{C}$ is finite.

On the other hand, $\mathcal{T}$ is continuous from $\hat{\mathcal{V}}$ into $\mathcal{V}$ and $\mathcal{B}^{1/2}$ is continuous from $\mathcal{V}$ into $\mathcal{H}$, consequently, $\mathcal{Q}$ is compact from $\mathcal{H}$ into $\mathcal{H}$. Moreover for every $U^* \in \mathcal{H}$,$\widetilde{U^*} \in \mathcal{H}$, we have

$\left(\mathcal{Q}U^*, \widetilde{U^*}\right)_{\mathcal{H}} = \left(\mathcal{B}^{1/2}(\mathcal{T}[\mathcal{B}^{-1/2}U^*]_2, \widetilde{U^*}\right)_{\mathcal{H}};$

setting $\widetilde{U^*} = \mathcal{B}^{1/2}\hat{\widetilde{U}}$, we get

$\left(\mathcal{Q}U^*, \widetilde{U^*}\right)_{\mathcal{H}} = \left(\mathcal{B}^{1/2}(\mathcal{T}\widehat{U_2}), \mathcal{B}^{1/2}\hat{\widetilde{U}}\right)_{\mathcal{H}} = \left(\mathcal{T}U_2, \hat{\widetilde{U}}\right)_{\mathcal{V}} = a(U, \hat{\widetilde{U}}),$

that means that $\mathcal{Q}$ is selfadjoint as $a(\cdot,\cdot)$ is Hermitian; $\mathcal{Q}$ is also non negative because $(\mathcal{Q}U^*, U^*)_{\mathcal{H}} = a(U,U) \geq 0$.

Now we are ready to give the main theorem deduced from the Askerov, Krein, Laptev theorem [1]. First we recall that the operator $\mathcal{B}$ depends on coefficient of viscosity μ and on coefficient of friction ϕ

$\left(\mathcal{B}U, \widetilde{U}\right)_{\mathcal{H}} = b\left(U, \widetilde{U}\right) = \left(U, \widetilde{U}\right)_{\mathcal{V}} =: 2\mu \int_{\Omega} \varepsilon_{ij}(\overrightarrow{u})\varepsilon_{ij}(\overrightarrow{\overline{w}})d\Omega + \phi X\overline{\xi}).$

Theorem 6.1.

The problem has a countably eigenvalues λ with real part positive admitting $\lambda = 0$ e $\lambda = \infty$ for points of accumulation.

There are countably aperiodic motions arbitrary strongly damped (λ real $\rightarrow \infty$) and countably aperiodic motions arbitrary weakly damped (λ real$\rightarrow$ 0).

*1) If $4 \parallel \mathcal{B}^{-1} \parallel \parallel \mathcal{Q} \parallel < 1$, all eigenvalues are real and there are **no oscillatory motions** (it is the case in which μ and/or ϕ are sufficiently great).*

*2) If $4 \parallel \mathcal{B}^{-1} \parallel \parallel \mathcal{Q} \parallel > 1$, there is at most a finite number of complex eigenvalues, in the circular ring $\frac{1}{2\|\mathcal{B}^{-1}\|} \leq \mid \lambda \mid \leq 2 \parallel \mathcal{Q} \parallel$; **oscillatory dampened motions** correspond to these eigenvalues..*

Remark 6.1. The problem is studied in the framework of the linear theory. It seems difficult to take into acount of non linear terms in the Navier-Stokes equations . Indeed, in the chapter 3 of the classical book [9], the author presents existence and uniqueness theorems and stability results for non linear problems involving the Navier-Stokes equations. But, besause the conditions on the free surface, it seems tricky to extend these results to our problem.

DOI: 10.1685/CSC09296

7. Conclusion

The problem is reduced to the study of a classical Askerov, Krein , Laptev pencil. The small motions of the system dapend on the dissipative coefficient. There are always damped motions, but damped oscillatory motions can appear only for weak dissipative coefficient.

REFERENCES

1. N.G.Askerov, S.G.Krein and G.I.Laptev, A class of non selfadjoint boundary value problems, *Dokl.Akad.Nauk.SSSR.*, **155,3** (1964),pp. 499-502 .
2. P.Capodanno, Etude mathématique des petites oscillations de deux liquides pesante non miscibles dans un container, *Mechanics Research Communications*, **23,1**, (1996), pp.75-80.
3. P.Capodanno, Piccole oscillazioni di una massa liquida in presenza ed in assenza di gravitá, Conferenze all'Universitá di Roma, Facoltá di Ingegneria, Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, (2001).
4. P.Capodanno and D.Vivona, Small oscillations of a pendulum containig viscous liquid, *Mechanics Research Communications*, **30** (2003), pp.3-8.
5. R.Dautray and J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, **2**, Springer-Verlag, Berlin, 2000.
6. N.D.Kopachevski, S.G.Krein and Ngo Zuy Can, *Operator approach in linear problems of hydrodynamics, self-adjoint problem*, **1**, Birkha''user-Verlag, 2001.
7. N.N.Moiseyen and V.V.Rumiantsev *Dynamic stability of bodies containig fluid*, Springer-Verlag, Berlin, 1968.
8. J.Hubert Sanchez and E.Palencia Sanchez *Vibration and coupling of continuous systems. Asymptotic methods*, Springer- Verlag, Berlin, 1989.
9. R.Temam *Navier-Stokes equations*, North Holland, Amsterdam, 1979.