Communications to SIMAI Congress,
ISSN 1827-9015, Vol. 3 (2009) 290 (12pp)

DOI: 10.1685/CSC09290

Vortex line density in plane Couette flow in superfluid helium

Michele Sciacca, Maria Stella Mongiovì

Dipartimento di Metodi e Modelli Matematici Università di Palermo, c/o Facoltà di Ingegneria, Viale delle Scienze, 90128 Palermo, Italy
msciacca@unipa.it
mongiovi@unipa.it

David Jou

Departament de Física, Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia, Spain
david.jou@uab.es

Abstract

Recent interest in cryogenic applications has enhanced studies in coflow superfluid turbulence, which is characterized by two fluids, normal and superfluid, moving in the same direction. In our studies we are interested to deal with superfluid turbulence in plane Couette flow, as an illustration of this interesting phenomena. An equation previously proposed to describe the evolution of vortex line density in rotating counterflow turbulent tangles in ^{4}He is generalized to incorporate nonvanishing barycentric velocity and velocity gradients. Incorporating our generalized equation into a thermodynamical model previously proposed, we evaluate the vortex density in plane Couette flow.

Keywords: `quantum vortices, coflow and counterflow.`

1. Introduction.

The most well known hydrodynamical model of superfluid helium is the two-fluid model of Tisza [1] and Landau [2] which thinks helium II as a mixture of two fluid components, the normal fluid and the superfluid, having densities ρ_n and ρ_s respectively and velocities $\mathbf{v}_n$ and $\mathbf{v}_s$ respectively, with total mass density ρ and velocity $\mathbf{v}$ defined by $\rho = \rho_s + \rho_n$ and $\rho\mathbf{v} = \rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n$. The first component consists of thermally excited states that form a viscous fluid which carries the entire entropy content of the liquid. The second component is related to the quantum ground state and is an ideal fluid, which does not experience dissipation neither carries entropy.

The behaviour of helium II is very different from that of classical fluids, as confirmed by many experiments [3,4]. One example of non-classical

Received 13/02/2009, in final form 19/06/2009
Published 31/07/2009

behavior is heat transfer in counterflow experiments (absence of mass flux $\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s = 0$), characterized by an extremely high thermal conductivity (three million times larger than that of helium I). The two-fluid model explains the experimental counterflow situation. In fact, the heat is carried toward the bath by the normal fluid only, and the heat flux $\mathbf{q} = \rho s T \mathbf{v}_n$ where s is the entropy per unit mass and T the temperature. On the other side, being the net mass flux zero there is superfluid motion — in opposite direction with respect to normal component — toward the heater ($\mathbf{v}_s = -\rho_n \mathbf{v}_n / \rho_s$), hence there is a net internal counterflow $\mathbf{V}_{ns} = \mathbf{v}_n - \mathbf{v}_s = \mathbf{q}/(\rho_s s T)$ which is proportional to the applied heat flux $\mathbf{q}$.

Quantum turbulence is described as a chaotic tangle of quantized vortices characterized by a quantum of vorticity $\kappa = h/m_4$ (h being the Planck constant and m_4 the mass of ^{4}He atom) and measured by introducing a scalar quantity L, the average vortex line length per unit volume, briefly called *vortex line density*. An evolution equation which describes the dynamics of L in counterflow superfluid turbulence has been formulated by Vinen, who, neglecting the influence of the walls, proposed [5]:

$$\frac{dL}{dt} = \alpha_v V_{ns} L^{3/2} - \beta_v \kappa L^2, \tag{1}$$

with V_{ns} the averaged magnitude of the counterflow velocity $\mathbf{V}_{ns}$ and α_v and β_v dimensionless parameters. Equation (1) assumes homogeneous turbulence, i.e. that the value of L is the same everywhere in the system. This equation was extended in [6] to the combined situation of counterflow and rotation.

The experiments of thermal counterflow and rotating sample were extensively studied over the years, and now special emphasis is addressed to the situations in which the barycentric velocity $\mathbf{v}$ is not zero, such as Couette flow. Here, we are interested to extend previous results for rotating superfluid turbulence to include also situations where the barycentric velocity is not zero, which have practical interest, for instance, in cryogenic applications. To this aim we generalize a previous equation proposed for rotating counterflow superfluid turbulence [6] by emphasizing more explicitly the dynamical role of the rotational of the superfluid velocity $\mathbf{v}_s$, related to quantized vortices. This allows us to write a proposal for the evolution equations of vortices in plane Couette and Poiseuille flows. Furthermore, we will define a quantum Reynolds number for the superfluid turbulence as well as the classical Reynolds number for the turbulence in viscous fluids.

2. Hydrodynamical model.

The image of helium II as composed by a mixture of two fluid requires two different evolution equations for normal and superfluid velocities. A set of equations frequently used are the Hall-Vinen-Bekarevich- Khalatnikov (HVBK) equations, which in an inertial frame are written as

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} + \rho_n(\mathbf{v}_n \cdot \nabla)\mathbf{v}_n = -\frac{\rho_n}{\rho}\nabla p_n - \rho_s S \nabla T + \mathbf{F}_{ns} + \eta \nabla^2 \mathbf{v}_n, \tag{2}$$

$$\rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \rho_s(\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = -\frac{\rho_s}{\rho}\nabla p_s + \rho_s S \nabla T - \mathbf{F}_{ns} + \rho_s \mathbf{T}, \tag{3}$$

where $\nabla p_n = \nabla p + (\rho_s/2)\nabla V^2$, $\nabla p_s = \nabla p - (\rho_n/2)\nabla V^2$, p is the total pressure, S is the entropy, η is the dynamic viscosity of the normal component and $\rho_s \mathbf{T}$ is the vortex tension force, which vanishes for rectilinear vortices and for isotropic vortex tangles, but which may be relevant in other situations. Here we will assume that $\mathbf{T} = \mathbf{0}$. $\mathbf{F}_{ns}$ is the mutual friction force between normal and superfluid components, which depends on the presence of vortices (it is null in the laminar regime) as

$$\mathbf{F}_{ns} = \alpha \rho_s \kappa L \left[\hat{\mathbf{p}} \times [\mathbf{p} \times (\mathbf{V} - \mathbf{v_i})] + \frac{\alpha'}{\alpha} \hat{\mathbf{p}} \times (\mathbf{V} - \mathbf{v_i}) \right], \tag{4}$$

with α and α' being friction coefficients depending on temperature, and the "self-induced velocity" $\mathbf{v_i}$ approximated by $\mathbf{v_i} = \tilde{\beta} \nabla \times \hat{\mathbf{p}}$. The polarity vector $\mathbf{p}$ was first defined by Lipniacki in [7] and then used by us in [8], and it is linked to the rotational of the averaged superfluid velocity by

$$\mathbf{p} = < \mathbf{s}' > = \frac{\int \mathbf{s}' d\xi}{\int d\xi} = \frac{\nabla \times \mathbf{v}_s}{\kappa L}, \tag{5}$$

averaged in a mesoscopic volume Λ, where $\mathbf{s}'$ is the first derivative of the curve $\mathbf{s}(\xi)$ describing a vortex line with respect to the arc-length ξ. Note that in the transient interval when the turbulence has not a homogeneous distribution in the whole system, the polarity vector $\mathbf{p}$ depends on the spatial position of Λ in the system. On the other side, when the homogeneous situation is reached, any volume Λ in the system can be assumed to have the same polarity $\mathbf{p}$ (see Fig. 1 of Ref. [8]) [9]. From (5) one can note also that $|\mathbf{p}| \in [0, 1]$ measures the directional anisotropy of the tangent to the vortex lines: in particular $|\mathbf{p}| = 1$ for a system of parallel vortices and $|\mathbf{p}| = 0$ for isotropic tangles.

The hydrodynamic model (2) and (3) has to be completed by an evolution equation for the vortex line density L which has field properties: it depends on the coordinates, it has a drift velocity $\mathbf{v}_L$, and it has associated a diffusion flux $\mathbf{J}^L$. This evolution equation in inhomogeneous situation takes the form

$$\frac{\partial L}{\partial t} + \nabla \cdot \mathbf{J}^L = \sigma^L, \tag{6}$$

where σ_L stands for the production term, which in the counterflow experiments is the right hand side of Vinen equation (1). The form of $\mathbf{J}^L$ contains a convective contribution, $L\mathbf{v_L}$ with $\mathbf{v_L}$ the velocity of vortex lines with respect to the laboratory frame, and a diffusive contribution. Here, neglecting the relaxation time of $\mathbf{J}^L$ and considering isothermal situations, we take for $\mathbf{J}^L$ the following simple law, where the diffusive contribution is analogous to Fick's diffusion law

$$\mathbf{J}^L = -\tilde{D}\nabla L + L\mathbf{v_L}. \tag{7}$$

The coefficient $\tilde{D}$ (of the order of κ [10], [11]) is the diffusion coefficient of vortex lines. When turbulence reaches an homogeneous distribution in the vessel then the flux $\mathbf{J}^L$ is almost zero, and hence can be neglected in equation (6) becoming the Vinen equation (1).

After some transient time when the homogeneous situation is reached, an evolution equation for the dynamics of quantum vortices in rotating helium under counterflow was proposed in [6], describing the influence of the heat flow and of angular velocity on the vortex line density. In particular, the vortex-line density L was assumed to obey the following equation

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\beta\kappa L^2 + \left[\alpha_1 V + \beta_2\sqrt{\kappa\Omega}\right] L^{3/2} - \left[\beta_1\Omega + \beta_4 V\sqrt{\frac{\Omega}{\kappa}}\right] L, \tag{8}$$

where β, α_1, β_2, β_1, and β_4 are dimensionless coefficients and $\Omega = |\mathbf{\Omega}|$ is the angular velocity of the container. The coefficients are seen to satisfy the relations $\beta_4 = \sqrt{2}\alpha_1$ and $\beta_1 = \sqrt{2}\beta_2 - 2\beta$, which are required on relatively general arguments about the form of solutions. Their particular values were obtained in Ref [6] by comparison with experimental data of [12], and were confirmed by a different calculation carried out in [13]. When $\Omega = 0$, equation (8) reduces to the Vinen's equation (1), with parameters α_1 and β being respectively related to the production and destruction of vortices per unit volume and time.

Equation (8) lacks an important source of vorticity, namely a barycentric velocity gradient, which is known to produce turbulence as for instance in Couette flow. In [8] this equation was generalized by incorporating barycentric velocity gradients, simply interpreting equation (8) in some deeper terms, which will be useful for a consistent incorporation of the velocity gradient. The main topic was to note that, in the particular case of pure rotation, Ω is related to rot $\mathbf{v}_s$ as $2\Omega = |\text{rot } \mathbf{v}_s|$, $\mathbf{v}_s$ being the macroscopic superfluid velocity. So, writing an equation such as (8) in terms of rot $\mathbf{v}_s$ and V rather than in terms of Ω and V would be more general, because it would reduce to (8) for rotation, and it could be applied to other flows as plane Couette flow, where $|\text{rot } \mathbf{v}_s| = dv_{sx}(z)/dz$, x being the direction of the fluid motion, z the direction orthogonal to the parallel plates, and $v_{sx}(z)$ the macroscopical superfluid velocity, depending only on z.

But, the direct replacement of the quantity $2\Omega = |\text{rot } \mathbf{v}_s|$ in the equation (8) is not completely correct because whereas Ω is taken as an externally fixed parameter in (8), rot $\mathbf{v}_s$ is a dynamical quantity, which must be described by a suitable evolution equation. It will be correct when one assumes equation (6) with the production term given by the right hand side of equation (8). Under this assumption the production term σ^L becomes

(9)
$$\sigma_L = \left[\alpha_1 V + \frac{\beta_2}{\sqrt{2}}\sqrt{\kappa|\text{rot } \mathbf{v}_s|}\right] L^{\frac{3}{2}} - \left[\frac{\beta_1}{2}|\text{rot } \mathbf{v}_s| + \frac{\beta_4 V}{\sqrt{2}}\sqrt{\frac{|\text{rot } \mathbf{v}_s|}{\kappa}}\right] L - \beta\kappa L^2,$$

which reduces to the right-hand side of (8) for pure rotation. Note that in (8) it is assumed that $|\text{rot } \mathbf{v}_s|$ is equal to 2Ω, even if it will take some time for $\mathbf{v}_s$ to get these values, so that expressions (6) and (9) generalize (8) also on dynamical grounds. Then, the form (8) will be useful after some transient interval, whereas (6) and (9) is expected to be valid also for fast changes in $\mathbf{v}_s$. Thus, equation (6) with σ_L expressed by (9) is the central point of this paper, as it generalizes (8) both to a wider set of external conditions and to a wider domain of dynamical variations.

Now, using the polarity vector $\mathbf{p}$ introduced above and mimicking in some way the form of the original Vinen's equation, equation (6) and (9) becomes

(10)
$$\frac{\partial L}{\partial t} + \nabla \cdot \mathbf{J}^L = \sigma^L = \alpha_1 V L^{3/2}\left[1 - A\sqrt{|\mathbf{p}|}\right] - \beta\kappa L^2\left[1 - \sqrt{|\mathbf{p}|}\right]\left[1 - B\sqrt{|\mathbf{p}|}\right],$$

where $B = \frac{\beta_1}{2\beta}$ and $A = \frac{\beta_4}{\sqrt{2}\alpha_1}$, which in [8] was assumed equal to 1 motivated by the comparison with the experimental data performed in [6]. The polarization comes from pinned vortex lines, which begin and end on the walls of the container. In rotating containers, a part of the vortices go from one end to the other of the system, more or less parallel to the angular velocity vector. Near the walls, the polarization is a little bit higher than in the bulk, because the proportion of pinned vortices is higher, with respect to closed loops, and equation (10) predicts a reduction in the rate of formation and destruction of vortex lines, as compared with the bulk.

Another argument which we will focus our attention on is the characterization of the transition from laminar flows to turbulent flows. In classical fluids, this transition is often characterized in terms of the dimensionless Reynolds number. It would be interesting to have a similar characterization of the transition to superfluid turbulence in an analogous way, by defining a suitable dimensionless number. This is also confirmed by the fact that when the counterflow velocity (related to the heat flux) is high enough, vortices appear in the superfluid, and that when the barycentric velocity of the superfluid is high enough, vortices appear. But, in superfluids there is not a typical viscosity, as the value of the viscosity of the superfluid component is zero. However, the quantum of vorticity has the same dimensions as kinematic viscosity, and therefore, one may define a quantum Reynolds number as

$$Rey_q = \frac{v_s D}{\kappa} \tag{11}$$

where D is the typical size of the the duct or of the object and v_s is the modulus of the superfluid velocity. Of course, number (11) holds true also for a different choice of the velocity, such as the counterflow velocity — which in counterflow situation is linked to superfluid velocity — or the barycentric velocity. However, the transition to superfluid turbulence is due to the fact that the superfluid component becomes turbulent.

The choice of κ in place of the viscosity is not accidental, but it is supported by the fact that the turbulence in superfluids is characterized by the presence of a tangle of quantized vortices which are formed if the circulation of the relative velocity between normal and superfluid component exceeds the quantum of vorticity κ. In spite of its physical appel, this number is not as widely used as it could be expected, but it is useful to characterize quantum turbulence.

3. Vortex-line density in rotating counterflow and plane Couette flow.

In this Section, we investigate the proposed equation (10) for a rotating superfluid helium inside a cylindric container in the presence of counterflow when the homogeneous situation is reached, and for a barycentric motion as plane Couette flow (without external heat flux) between two parallel plates.

3.1. *Application to rotating counterflow.*

In this case we consider the experimental situation of a rotating container filled of helium II with an external counterflow $\mathbf{V}$ parallel to the angular velocity $\mathbf{\Omega}$ of the container by Swanson *et al.* [12]. For high angular velocities, after some transient time, when the steady state is reached, these authors observed two critical counterflow velocities V_{c1} and V_c such that for $0 \leq V \leq V_c$ the line density L is approximately independent on V, undergoing only a small step (about 0.4%) at the first critical velocity V_{c1} whereas for $V \geq V_c$ the line density L grows with V^2. Here, we will neglect the small step of the vortex density at the first critical velocity V_{c1} because it is far from the scopes of this paper (see Ref. [8] for further investigations).

In this situation equation (10), together with the HVBK equations, should be valid also in transient situation, even if numerical simulation and experiments are needed. Therefore, we restrict our interest to the homogeneous situation in such a way that the vortex flux $\mathbf{J}^L$ can be neglected, and any variation in L is linked to the production term σ^L as well as the polarity vector $\mathbf{p}$ can be approximately assumed independent on the spatial coordinates because the small volume Λ, used to define it, has the property that it does not depend on the position vector $\mathbf{x}$ (see Fig. 1 of Ref. [8]). Therefore, the only equation needs to describe the homogeneous situation is the new evolution equation (10) for L (with $\mathbf{J}^L = 0$). The polarity vector $\mathbf{p}$ is parallel to the direction of rotation and external counterflow, and its modulus depends on the counterflow velocity. In fact, from the definition of the vector $\mathbf{p}$, one notes that $|\mathbf{p}| = \mathbf{1}$ for $V < V_c$, because $L \approx 2\Omega/\kappa \approx |\text{rot } \mathbf{v}_s|$ in this situation, whereas $|\mathbf{p}| < \mathbf{1}$ for $V > V_c$ because $|\text{rot } \mathbf{v}_s| = 2\Omega$ and the vortex line density is higher than $2\Omega/\kappa$ (see Fig. 2 of Ref. [8]). In the homogeneous situation equation (10) can be written

$$\frac{\mathrm{d}L}{\mathrm{d}t} = L^{3/2}\left(1 - \sqrt{|\mathbf{p}|}\right)\left[\alpha_1 V - \beta\kappa L^{1/2}\left(1 - B\sqrt{|\mathbf{p}|}\right)\right], \tag{11}$$

whose stationary solutions are

$$|\mathbf{p}| = 1 \qquad \text{and} \qquad L^{1/2} = \frac{\alpha_1}{\beta\kappa}V + B\sqrt{\frac{|\nabla \times \mathbf{v}_s|}{\kappa}}. \tag{12}$$

The stability of the solution $|\mathbf{p}| = 1$ can be studied assuming that the perturbation δ does not modify the vorticity $\vec{\omega} = \text{rot } \mathbf{v}_s$ in such a way the relation $\delta|\mathbf{p}| = -(|\mathbf{p}|/L)\delta L$ is obtained. Therefore, linearizing equation (11) the following evolution equation for the perturbation δL is obtained

$$\left(\frac{\partial \delta L}{\partial t}\right)_{|\mathbf{p}|=1} = \left[\frac{\alpha_1 V}{2L^{1/2}} - \frac{1}{2}\beta\kappa(1 - B)L\right]\delta L, \tag{13}$$

from which it follows that the solution $|\mathbf{p}| = 1$ is stable for V less than

$$V_c = \frac{\beta}{\alpha_1}(1-B)\sqrt{|\nabla \times \mathbf{v}_s|\kappa}, \tag{14}$$

which corresponds to the critical velocity V_c in the experiments of Swanson et al. [12]. The value $B = 1$ vanishes the critical counterflow velocity (14) for which the straight vortex lines parallel to the rotation axis become unstable. But, this is not the case because in [6] from a comparison with experimental data the value $B = 0.89 < 1$ was found.

For counterflow velocity higher than the critical velocity (14), the solution $|\mathbf{p}| = 1$ becomes unstable, and the line density L assumes the value (12b) which depends on V and $|\mathrm{rot}\ \mathbf{v}_s|$. There, in the second term in the right hand side, namely $B\sqrt{\frac{|\nabla\times\mathbf{v}_s|}{\kappa}}$, for low values of the counterflow velocity, the vorticity is essentially due to the rotation, and therefore we put $|\nabla \times \mathbf{v}_s| = 2\Omega$, recovering the results obtained in [6].

As already pointed out, in [7] a hydrodynamical model of superfluid turbulence was proposed by Lipniacki, which is valid for plane Couette flow and rotating counterflow but only for $V > V_c$, as also remarked by the author himself. His proposal is

$$\frac{dL}{dt} = \tilde{\alpha} I_0 c_{10} V L^{3/2}\left[1-|\mathbf{p}|^2\right] - \tilde{\beta}\alpha c_{20}^2 L^2 \left[1-|\mathbf{p}|^2\right]^2, \tag{15}$$

where $I_0 = \mathbf{I}\cdot\hat{\mathbf{V}}$, and the subscript 0 stands for independence of I_0 on V and L. The author chooses for I_0 the same values found in pure counterflow, in such a way to not consider the anisotropy of the vortex tangle, which is present owing of the high values of rotation considered in the experiments by Swanson *et al.*. $\mathbf{I}$ is the binormal vector already defined by Schwarz [3]

$$\mathbf{I} = \frac{<\mathbf{s}'\times\mathbf{s}''>}{<|\mathbf{s}''|>}. \tag{16}$$

The stationary solutions of the equation (15) are $|\mathbf{p}| = 1$ (which however is unstable) and

$$L = \frac{L_H}{\left(1-(L_\omega/L)^2\right)^2}, \tag{17}$$

where

$$L_H = V^2\left(\frac{c_{10}I_0}{\beta c_{20}^2}\right)^2 \quad \text{and} \quad L_\omega = \frac{|\mathrm{rot}\mathbf{v}_s|}{\kappa} = \frac{2\Omega}{\kappa} \tag{18}$$

DOI: 10.1685/CSC09290

are the steady state vortex-line density in pure counterflow and in pure rotation, respectively.

In Fig. 2 of Ref. [8], we compare the results of equations (10) and (15) with the experimental data of Swanson's experiments. It follows that our model yields the horizontal branch of the experimental data for $V < V_c$, which are not described by equation (15), and also equation (10) (black line) describes better the experimental data (solid circle) than (15) (dashed line) for $V > V_c$.

A reason for the difference between proposals (10) and (15) could be related to a different microscopical interpretation of some terms in the evolution equation for L. It is known that in rotating superfluid helium the vortices grow near the walls due to the rotation, and drift towards the center of the system, where they find a repulsion due to other vortices. It could then be that the vanishing of the terms in (10) as $1 - \sqrt{|\mathbf{p}|}$ had a different physical origin than the vanishing proposal by Lipniacki from a different model. These open questions stress the need of the inclusion of rotational effects in a more general version of Schwarz's derivation of Vinen's equation.

3.2. *Application to plane Couette flow.*

Equation (10) may be also used to describe situations with barycentric motion as plane Couette and Poiseuille flows, without external heat flux, between two parallel plates [8]. Here, we consider two plane surfaces at $z = 0$ and $z = D$ such that the second one moves parallely to the first one at the velocity $\mathbf{V}_0$, sufficiently small to maintain laminar the normal component and sufficiently high to neglect the "effects of the walls". When the upper plate starts suddenly moving with respect to the lower plate, the normal component starts moving under the action of the viscous force and non-slip condition, whereas the superfluid component will remain initially insensitive to the motion of the plate. From the HVBK equations one finds a Newtonian profile $\mathbf{v}_n = z\mathbf{V}_0/D$ for the normal fluid and $\mathbf{v}_s = \mathbf{0}$ for the superfluid because the lack of vortices (see Fig. 1). Then, a relative velocity (the counterflow velocity) $\mathbf{V}(z) = \mathbf{v}_n - \mathbf{v}_s$ will arise between the two components: $\mathbf{V}$ is maximum for $z = D$ (upper plate) and zero for $z = 0$ (lower plate). When the counterflow velocity reaches a critical value near the moving plane, the remnant vortices, always present in He II, begin to lengthen and reconnect to form other vortices, which diffuse towards the lower plate (at rest) because of vortex line flux $\mathbf{J}^L$ and the second term of the mutual friction force (4), forming, in the stationary situation, an array of vortices along the y-direction. The transition to the superfluid turbulence

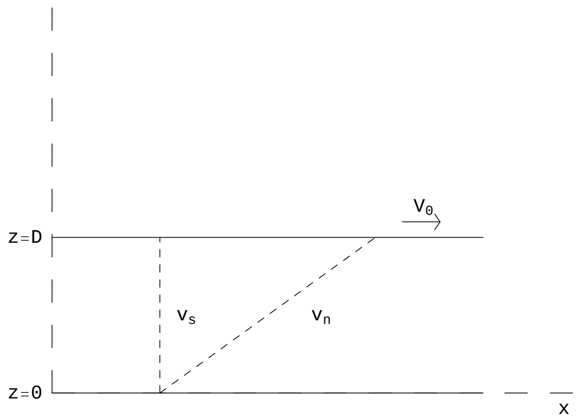

Fig. 1. Initial profile of the velocities of superfluid and normal components.

will be characterized by the quantum Reynolds number (11) with V in place of v_s.

Through the vortices, the normal and the superfluid components become coupled by the mutual friction force $\mathbf{F}_{ns}$, and the superfluid will tend to match its velocity with that of the normal fluid ($V \to 0$); this will introduce a rotational $|\text{rot } \mathbf{v}_s| = |\partial \mathbf{v}_s/\partial z| \neq 0$ in the superfluid.

After a sufficiently long time, it is expected that a steady shear flow will have formed, with $\mathbf{v}_n = \mathbf{v}_s$ depending only on z and having the x direction and such that $\partial \mathbf{v}_n/\partial z = \partial \mathbf{v}_s/\partial z = \text{rot } \mathbf{v}_s = \mathbf{V}_0/D$, corresponding to the physical Newtonian linear profile, which follows from the hydrodynamical model (2) and (3) with vanishing tension force $T = 0$. In terms of Rey_q, vortices will be created for Rey_q higher than a critical value, and the creation of them will be stopped when Rey_q decreases under this critical value.

Since we are interested to consider steady state and relatively slow variations with respect to steady states, then rot $\mathbf{v}_s$ with its own nontrivial dynamics should not be considered in the dynamical equation of L. Then, the dynamics of L in this case is described by

(19)
$$\frac{\mathrm{d}L}{\mathrm{d}t} = L\left(L^{1/2} - \sqrt{\frac{1}{\kappa}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right|}\right)\left[\alpha_1 V - \beta\kappa\left(L^{1/2} - B\sqrt{\frac{1}{\kappa}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right|}\right)\right] - \overbrace{\nabla \cdot \mathbf{J}^L}^{\simeq 0}$$

In the stationary situation $V \approx 0$ and, according to (19), there will be a completely polarized array of vortices, perpendicular to the velocity and to the velocity gradient, with uniform areal density given by

$$L = \frac{1}{\kappa}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right| = \frac{V_0}{\kappa D}, \tag{20}$$

which, according to the intrinsic feature of equation (19), is stable for V less than

$$V_c = \frac{\beta}{\alpha_1}\left[2\frac{\beta_4}{\alpha_1} - \frac{\beta_2}{\beta}\right]\sqrt{\frac{\kappa}{2}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right|} \cong c'\sqrt{\frac{\kappa}{2}\left|\frac{\partial \mathbf{v}_s}{\partial z}\right|}, \tag{21}$$

with $c' \approx 3.7$, according to the values of the coefficients used in (8) to describe the value of V_c in rotating counterflow velocity.

4. Conclusion.

In this paper, by writing the local average rotational of the superfluid component instead of the angular velocity, we have enlarged the set of applications of the theory in two main aspects: (a) our proposal (9), in contrast to the previous one (8), may be applied not only to rotation but also to shear flows, as illustrated in plane Couette flow; (b) since in our proposal (9) rot $\mathbf{v}_s$ appears, then it becomes deeply coupled to the HVBK equations (2) and (3) for $\mathbf{v}_n$ and $\mathbf{v}_s$.

The application of our model to plane Couette flow gives an ordered array of vortices, parallel to the plates and orthogonal to the velocity $\mathbf{V}_0$, which is stable until $V < V_c$. This means that, as V_0 grows, the regular array of rectilinear vortices is still present and the velocities $\mathbf{v}_n$, $\mathbf{v}_s$ and $\mathbf{V}$ have rectilinear profiles, but with slightly different slope. The value of V is maximum near the plane $z = D$. When the counterflow velocity V reaches the critical value V_c the regular Couette array of vortices becomes unstable and a disordered tangle of vortex lines appears in the zone between the two plates.

This new evolution equation (9) for the vortex line density combined with HVBK equations, or with other thermodynamical systems (already proposed in our paper), could be applied to Couette or Poiseuille flows to study the stability of the stationary solutions. The same models could be useful for numerical simulations. The main results of the present paper, as well as an application to plane Poiseuille flow, were published in Ref. [8].

Acknowledgements.

We acknowledge the support of the Acció Integrada España-Italia (Grant No. S2800082F HI20040316 of the Spanish Ministry of Science and

Technology and Grant No. IT2253 of the Italian MIUR). D.J. acknowledges the financial support from the Dirección General de Investigación of the Spanish Ministry of Education under Grant No. FIS 200612296-C0201 and of the Direcció General de Recerca of the Generalitat of Catalonia, under Grant No. 2005 SGR- 00087. M.S.M. and M.S. acknowledge the financial support from "Fondi 60" of the University of Palermo. M.S. is supported by an "Assegno di ricerca MIUR" of Italy.

REFERENCES

1. L. Tisza, Transport phenomena in He II, *Nature*, **141** (1938), pp. 913.
2. L.D. Landau, The theory of superfluidity of He II, *J. Phys.*, **5** (1941), pp. 71.
3. R. J. Donnelly, *Quantized vortices in Helium II*, Cambridge University Press, Cambridge, 1991.
4. C.F. Barenghi, R.J. Donnelly and W.F. Vinen, *Quantized Vortex Dynamics and Superfluid Turbulence*, Springer, Berlin, 2001.
5. W.F. Vinen, Mutual friction in a heat current in liquid helium II. III. Theory of the mutual friction, *Proc. R. Soc. London* **A 240** (1957), pp. 493–515.
6. D. Jou and M. S. Mongiovì, Phenomenological description of counterflow superfluid turbulence in rotating containers, *Phys. Rev. B*, **69** (2004), pp. 094513 (7 pages).
7. T. Lipniacki, Dynamics of superfluid ^{4}He: Two-scale approach, *Eur. J. Mech. B Fluids*, **25**, (2006), pp. 435-458.
8. D. Jou, M. Sciacca and M. S. Mongiovì, Vortex dynamics in rotating counterflow and plane Couette and Poiseuille turbulence in superfluid helium, *Phys. Rev. B*, **78** (2008), pp. 024524 (12 pages).
9. S. K. Nemirovskii, Gaussian model of vortex tangle in He II, *Phys. Rev. B*, **57** (1998), pp. 5972–5986.
10. M.S. Mongiovì, D. Jou, Thermodynamical derivation of a hydrodynamical model of inhomogeneous superfluid turbulence, *Phys. Rev. B*, **75** (2007), pp. 024507 (14 pages).
11. D. Jou, M. S. Mongiovì and M. Sciacca, Vortex density waves and high-frequency second sound in superfluid turbulence hydrodynamics, *Phys. Lett. A*, **368** (2007), pp. 7–12.
12. C. E. Swanson, C. F. Barenghi and R. J. Donnelly, Rotation of a tangle of quantized vortex lines in He II, *Phys. Rev. Lett.*, **50** (1983), pp. 190–193.
13. N. Anderson, T. Sidery and G. L. Comer, Superfluid neutron star turbulence, *Mon. Not. R. Astron. Soc.*, **381** (2007), pp. 747–756.