

Optimal Trajectories of Anti-satellite Missiles

Mauro Pontani¹, Bruce A. Conway²

¹*Scuola di Ingegneria Aerospaziale
University of Rome “La Sapienza”
mauro.pontani@uniroma1.it*

²*Department of Aerospace Engineering
University of Illinois at Urbana–Champaign, IL
bconway@uiuc.edu*

Abstract

In the last decades a number of countries have shown interest towards anti-satellite systems for strategic military purposes. In general, anti-satellite weapons are employed to intercept and destroy adversary or malfunctioning satellites. This study is concerned with the determination of the optimal, i.e. minimum-time trajectories leading to the interception of low Earth satellites by a ground-launched interceptor, which is represented by a two-stage rocket. The target satellite is assumed to be placed in a repeating ground track orbit, so the performance of the anti-satellite interceptor can be evaluated over the period of repetition of the repeating orbit. This paper addresses the additional optimization problem of determining the geographical regions where the launch site can be located so that interception can occur in a specified time. The numerical solutions are derived through the use of a direct optimization method, termed “direct collocation with nonlinear programming”.

Keywords: Trajectory optimization, anti-satellite systems.

1. Introduction.

Anti-satellite (ASAT) systems represent strategic weapons tailored to intercepting and destroying adversary satellites. The U.S. and the USSR have shown a significant interest towards such systems since the 1950’s, with the development of several projects. The U.S. air force started developing the *Weapon System WS-199A project*, which led to several tests with the use of the *Bold Orion* and *High Virgo* air-launched missiles. The USSR developed the *UR-200* rocket as launch vehicle for the *Istrebitel Sputnikov*, which was a co-orbital system. In the twentieth century the last successful test was performed by the U.S. through the use of the *ASM-135 ASAT missile*, carried by a modified F15 aircraft, and led to intercepting and

destroying the *Solwind P78-1* satellite. Most recently, in 2007 China carried out its first anti-satellite test, by intercepting the *FY-1C* weather satellite, orbiting Earth at an altitude of 865 km. Finally, in 2008 the U.S. employed the *Raytheon SM-3* rocket to intercept and destroy the *USA-193* spysat.

A consistent number of studies concern anti-satellite systems. Of course, a relevant number of them is classified for strategic (military) reasons. In the scientific literature, some researchers [1] focused on the analysis of complex battle scenarios, whereas other studies were mostly concerned with existent ASAT systems [2,3]. Kelley et al. [4] addressed the complementary problem of determining the evasive maneuvers that spacecraft can perform to avoid interception. A similar problem involving missiles (instead of orbiting spacecraft) was treated by Pontani and Conway [5].

The objective of the present research is to formulate and solve numerically the problem of interception of a low Earth satellite by a high-performance, two-stage rocket. More precisely, four distinct optimal control problems are being considered, in order to identify the minimum-time trajectories leading to interception and the geographical regions where the launch site can be located so that interception can occur in a specified time.

2. Problem definition.

This research addresses the problem of the optimal interception of a low Earth satellite by a two-stage, ground-launched interceptor rocket. Both the missile and the satellite are modelled as point masses in the context of a three-degree-of-freedom problem.

2.1. Repeating orbit of the target satellite

The target satellite is assumed to be placed in a circular, repeating, low Earth orbit at an altitude of 200 km. For such orbits, Earth oblateness (J_2 zonal harmonic) is the only perturbation that produces significant effects on orbital elements. In particular, J_2 perturbation affects the right ascension of the ascending node (RAAN) Ω , the argument of perigee ω , and the mean anomaly M :

$$(1) \quad \dot{\Omega}(a, i) = -\frac{3}{2} J_2 \frac{R_E^2}{a^{7/2}} \sqrt{\mu_E} \cos i$$

$$(2) \quad \dot{\omega}(a, i) = \frac{3}{2} J_2 \frac{R_E^2}{a^{7/2}} \sqrt{\mu_E} \left(2 - \frac{5}{2} \sin^2 i \right)$$

$$(3) \quad \dot{M}(a, i) = \frac{3}{2} J_2 \frac{R_E^2}{a^{7/2}} \sqrt{\mu_E} \left(1 - \frac{3}{2} \sin^2 i \right) + \sqrt{\frac{\mu_E}{a^3}}$$

where R_E , μ_E , and J_2 are the Earth radius, gravitational parameter, and oblateness coefficient, whereas a and i are the orbit semi-major axis (SMA) and inclination, respectively. For circular orbits the perigee is not defined, so only the argument of latitude, $\theta (= M + \omega)$, is meaningful. Let Ω_{00} , ω_{00} , M_{00} , and θ_{00} denote the values of Ω , ω , M , and θ at the reference time t_{00} , which is set to 0 and is assumed to correspond to the ascending node crossing. It follows that

$$(4) \quad \Omega(t) = \Omega_{00} + \dot{\Omega}(a, i)t \quad \text{and} \quad \theta(t) = \theta_{00} + \dot{\theta}(a, i)t$$

where $\dot{\theta} = \dot{M} + \dot{\omega}$ and $\theta_{00} = M_{00} + \omega_{00}$. The time t_{00} corresponds to the ascending node crossing and this circumstance implies that $\theta_{00} = 0$.

An orbit is termed ‘‘repeating’’ when phased with Earth rotation, i.e. when the trajectory ground track is periodically repeated. This occurs if the satellite completes N_t orbits in m nodal days:

$$(5) \quad mD_n(a, i) = N_t T_n(a, i)$$

In Eq. (5) $T_n = 2\pi/\dot{\theta}$ denotes the nodal orbital period (i.e. the time interval between two consecutive ascending node crossings), whereas $D_n = 2\pi/(\omega_E - \dot{\Omega})$ represents the nodal day (i.e. the time required for the Earth to complete a rotation with respect to the orbital plane); ω_E is the Earth rotation rate. Once the SMA (or the altitude for circular orbits), N_t and m are specified, the relationship (5) determines the inclination i . In this work N_t is set to 16 and m is set to 1, and the inclination i for a repeating orbit is $i = 52.9$ deg. The related ground track is illustrated in Fig. 1, under the assumption that the Greenwich sidereal time and the RAAN at t_{00} are zero, i.e. $\theta_{g00} = 0$ and $\Omega_{00} = 0$.

2.2. Missile characteristics and equations of motion

The anti-satellite (ASAT) rocket is composed of two stages with specified characteristics. The propulsive performance of the interceptor missile is defined by simply specifying the following three parameters:

- the specific impulse of the propulsive system of each stage, ISP_i ($i = 1, 2$)
- the thrust-to-mass ratio at ignition of each stage, n_{0i} ($i = 1, 2$)
- the ratio $u_0^{(i)} = m_f^{(i)}/m_0^{(i)}$, where $m_0^{(i)}$ and $m_f^{(i)}$ denote the initial and final mass of each stage, respectively ($i = 1, 2$)

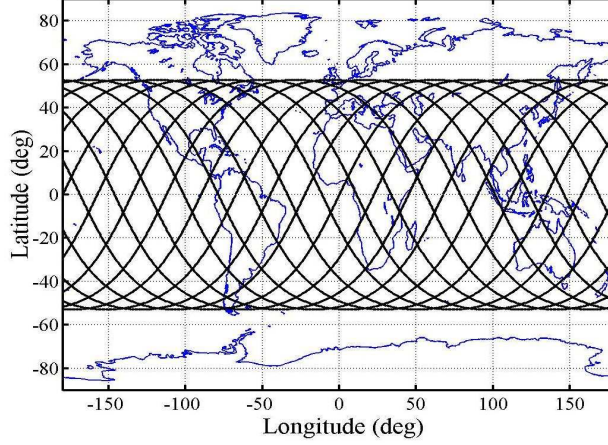


Fig. 1. Repeating ground track of the target satellite.

Under the assumption of constant, maximum thrust for each of the two stages, the thrust-to-mass ratio (T/m) of the interceptor rocket is given by:

$$(6) \quad \frac{T}{m} = \begin{cases} \frac{n_{01}g_0}{1-q_1(t-t_L)} & \text{if } t_L \leq t \leq t_L + \Delta t_{B1} \\ \frac{n_{02}g_0}{1-q_2(t-t_L-\Delta t_{B1})} & \text{if } t_L + \Delta t_{B1} \leq t \leq t_L + \Delta t_{B1} + \Delta t_{B2} \end{cases}$$

where $\Delta t_{Bi} = (1 - u_0^{(i)})/q_i$ represents the burn-out time of each stage, t_L denotes the unspecified launch time, $g_0 = 9.798 \text{ m/sec}^2$, and $q_i \doteq n_{0i}/I_{SPi}$. The following values are assumed for the above parameters:

$$(7) \quad \begin{array}{l} \text{Stage 1: } n_{01} = 3, \quad u_0^{(1)} = 0.4, \quad I_{SP1} = 350 \text{ sec} \\ \text{Stage 2: } n_{02} = 2, \quad u_0^{(2)} = 0.2, \quad I_{SP2} = 350 \text{ sec} \end{array}$$

Using the data (7), Eq. (6) yields reasonable values for the acceleration provided by the propulsive system of each stage: for the first stage the thrust-to-mass ratio (T/m) varies from $3g_0$ to $7.5g_0$, whereas for the second stage the same ratio goes from $2g_0$ to $10g_0$. In addition, the burn-out times turn out to be $\Delta t_{B1} = 70 \text{ sec}$ and $\Delta t_{B2} = 140 \text{ sec}$.

The trajectory of the ASAT missile can be described through a convenient set of variables, related to the orbital frames $(\hat{r}, \hat{\theta}, \hat{h})$ and $(\hat{n}, \hat{v}, \hat{h})$, where \hat{r} and \hat{v} are the unit vectors aligned with the position and the velocity vectors, \mathbf{r} and \mathbf{v} , whereas \hat{h} is associated to the specific angular momentum $\mathbf{h} (= \mathbf{r} \times \mathbf{v})$. This set of variables defines the state of the ASAT

rocket and includes: the instantaneous radius r , the absolute longitude ξ , the latitude ϕ , the flight path angle γ , the velocity v , and the coazimuth angle ζ , which is portrayed in Fig. 2. These variables form the state vector $\mathbf{x} = [r \ \xi \ \phi \ \gamma \ v \ \zeta]^T$. The control is performed through the thrust direction, described by the two angles α and β defined in Fig. 2. Hence, the control vector \mathbf{u} is given by $\mathbf{u} = [\alpha \ \beta]^T$. If one neglects the aerodynamic forces, the three-dimensional equations of motion are written as follows:

$$(8) \quad \begin{cases} \dot{r} = v \sin \gamma \\ \dot{\xi} = \frac{v \cos \gamma \cos \zeta}{r \cos \phi} \\ \dot{\phi} = \frac{v \cos \gamma \sin \zeta}{r} \end{cases} \quad \begin{cases} \dot{\gamma} = \frac{v \cos \gamma}{r} + \frac{T \sin \alpha \cos \beta}{m v} - \frac{\mu_E \cos \gamma}{r^2 v} \\ \dot{v} = \frac{T}{m} \cos \alpha \cos \beta - \frac{\mu_E \sin \gamma}{r^2} \\ \dot{\zeta} = \frac{T \sin \beta}{m v \cos \gamma} - \frac{v \cos \gamma \sin \phi \cos \zeta}{r \cos \phi} \end{cases}$$

The equations (8) are also referred to as the “state equations” and can be written in compact form as $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$.

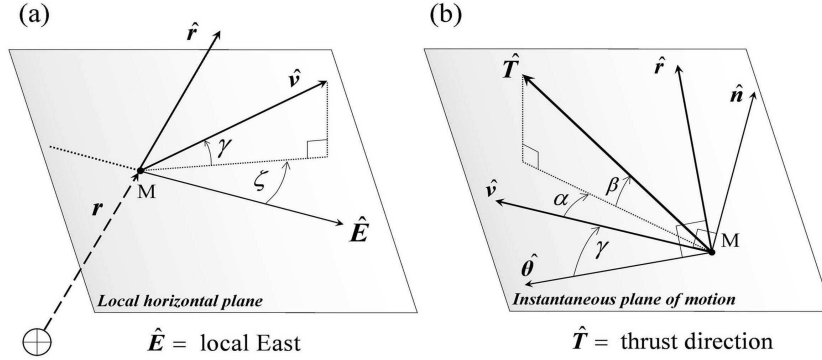


Fig. 2. Axes and angles related to the local horizontal plane (a) and to the instantaneous plane of motion (b)

The respective initial conditions depend on the launch time t_L and on the launch site (identified by the geographical longitude, μ_{ls} , and by the latitude, ϕ_{ls}) and are given by:

$$(9) \quad \begin{aligned} r(t_L) &= R_E & \xi(t_L) &= \theta_{g00} + \omega_E t_L + \mu_{ls} & \phi(t_L) &= \phi_{ls} \\ \gamma(t_L) &= 0 & v(t_L) &= \omega_E R_E \cos \phi_{ls} & \zeta(t_L) &= 0 \end{aligned}$$

The launch time is unspecified, i.e. chosen as an optimization variable, and this implies that also the absolute longitude at t_L is unspecified, unlike the remaining initial values of the state components. Interception terminates the problem and occurs when the missile reaches the instantaneous position of the target satellite.

3. Formulations of the optimal control problem.

In this research, the rocket is assumed to perform a ten-second vertical ascending arc after launch, then the trajectory is optimized in order to minimize the objective function of interest. The problem of interception can be formulated as an optimal control problem defined over the time interval $[t_0, t_f]$, where both $t_0 (= t_L + 10 \text{ sec})$ and t_f are unspecified. The time interval $[t_0, t_f]$ is partitioned into two subarcs (each corresponding to the propulsion of a single stage): $[t_0, t_S]$ and $[t_S, t_f]$, where $t_S = t_L + \Delta t_{B1}$. More specifically, this paper considers four distinct optimal control problems. If φ denotes the objective function, the following optimal control problems are defined:

A. Determine

- the optimal initial time $t_0^* (= t_L^* + 10 \text{ sec})$
- the optimal control law \mathbf{u}^*
- the optimal initial absolute longitude of the launch site, $\xi^*(t_0)$

such that the time for interception, $t_f - t_L$, is minimized ($\varphi = t_f - t_0$)

B. Given the geographical longitude of the launch site, μ_{ls} , determine

- the optimal initial time $t_0^* (= t_L^* + 10 \text{ sec})$
- the optimal control law \mathbf{u}^*

such that the time for interception, $t_f - t_L$, is minimized ($\varphi = t_f - t_0$)

C/D. Given the time of flight, $t_f - t_L$, which is set to 3 min, determine

- the optimal initial time $t_0^* (= t_L^* + 10 \text{ sec})$
- the optimal control law \mathbf{u}^*
- the optimal initial absolute longitude of the launch site, $\xi^*(t_0)$

such that

- C** the geographical longitude of the launch site, μ_{ls} , is minimized
- D** the geographical longitude of the launch site, μ_{ls} , is maximized

For all the above cases, the latitude of the launch site is assumed specified, whereas the geographical longitude, μ_{ls} , is given by $\mu_{ls} = \xi(t_0) - \theta_g(t_0)$, where $\theta_g(t_0) = \omega_E t_0$. Hence, for case **C**: $\varphi = \mu_{ls} = \xi(t_0) - \theta_g(t_0)$, whereas for case **D**: $\varphi = -\mu_{ls} = \theta_g(t_0) - \xi(t_0)$.

After introducing the Hamiltonian $H = \boldsymbol{\lambda}^T \mathbf{f}$ and the function of terminal conditions Φ (depending on φ and on the boundary conditions of

the problem), the optimal control problem can be translated into a two-point boundary-value problem (TPBVP) [6]. Omitting details for the sake of brevity, through the Pontryagin minimum principle the optimal control can be expressed as a function of the state vector \mathbf{x} , of the adjoint variable $\boldsymbol{\lambda}$ (conjugate to the state equations), and of the time t : $\mathbf{u} = \mathbf{u}(\mathbf{x}, \boldsymbol{\lambda}, t)$. In addition, for the problem at hand the Weierstrass–Erdmann corner conditions [6] state that the adjoint variable must be continuous across the time t_S (when the first stage separates from the rest of the rocket). This circumstance implies that the control must be continuous at $t = t_S$, since also the state \mathbf{x} is continuous at t_S . This property is a consequence of the necessary conditions for optimality and must be fulfilled by the numerical solution of the optimal control problem in each specific case.

4. Method of Solution.

This study employs the direct collocation with nonlinear programming (DCNLP) algorithm to achieve the numerical solution to the ASAT optimal interception problem. The DCNLP algorithm is a direct method of optimization [7], which has been successfully employed to solve a wide variety of aerospace optimal control problems. Its features are the following:

- the continuous problem is discretized in time, so only the values of the state and of the control at discrete times are employed by the algorithm
- the state equations are translated into nonlinear algebraic equations by means of high-order quadrature rules (the highly accurate Gauss–Lobatto fifth-order quadrature rules [8] in this research)
- the resulting nonlinear programming problem is solved by a numerical optimizer, for instance NPSOL [9], which does not utilize explicitly the adjoint variables

For the transcription of the optimal control problem into a discrete problem, each subarc i is partitioned into $N^{(i)}$ subintervals. The values $N^{(1)} = N^{(2)} = 10$ have been used for the problem at hand. The method of solution requires the discretization of the continuous variables, which are to be represented by a set of parameters. Their number corresponds to the degree of the polynomial that approximates each state variable in each subinterval. With the fifth-order Gauss–Lobatto quadrature rules, each state is represented by the values at the initial, at the central, and at the terminal point of each subinterval. Similarly, each control component is represented by the respective values at the initial, at the central, and at the terminal point of each subinterval, and also by two additional values corresponding

to two interior collocation points (cf. [8] for further details). Once the DCNLP algorithm has produced the numerical result, the state components are interpolated through fifth-degree polynomials, which represent the continuous approximations of their optimal time histories.

5. Numerical results.

The DCNLP algorithm has been successfully employed to solve the four optimal control problems defined in Section 3 with reference to different latitudes of the launch site.

The optimal control problem **A** is related to the determination of the optimal location of the launch site (i.e., the optimal geographical longitude $\mu_{l_s}^*$, as the latitude ϕ_{l_s} is specified). For all the cases that have been taken into account the optimal locations lie along the ground track of the target satellite. The optimal trajectories are represented (approximately) by vertical ascending trajectories (with the thrust direction \hat{T} always aligned with \hat{r}). The time for interception, $t_f - t_L$, is nearly independent of the launch latitude and is equal to 1.96 min. Figures 3–5 portray the optimal control laws, the ground tracks, and the optimal trajectories (represented in the local frame centered in the launch site) for three illustrative examples. All of them correspond to a launch latitude of 40 deg. With regard to problems **B** (Fig. 3), **C** (Fig. 4), and **D** (Fig. 5), in the local frame the optimal trajectories are associated to a constant azimuth angle and are nearly perpendicular to the target trajectory at interception. These two properties turn out to be common to all the numerical solutions, also for different launch latitudes. For all cases, the control variables are continuous at the first stage separation, as predicted by the analytical necessary conditions for optimality. The related time histories are strongly nonlinear, as it emerges from the inspection of Figs. 3(c), 4(c), and 5(c).

Figure 6 reports the launch sites (indicated with points) for the test cases that have been considered. In particular, the points corresponding to the Figs. 3, 4, and 5 are marked. For each latitude, the solution of problems **C** and **D** allows determining the longitudinal range where the launch site can be located so that interception can occur within a specified time (3 minutes). This leads to defining the stripe portrayed in Fig. 6, which represents the region where the launch site can be located so that interception is guaranteed in 3 minutes or less. Due to symmetry, it is relatively straightforward to determine the stripe with respect to the remaining portion of the target ground track. With reference to a single orbit of the target satellite, Fig. 7 portrays the entire “stripe of interception”, which includes the possible launch locations that guarantee interception in 3 minutes or less.

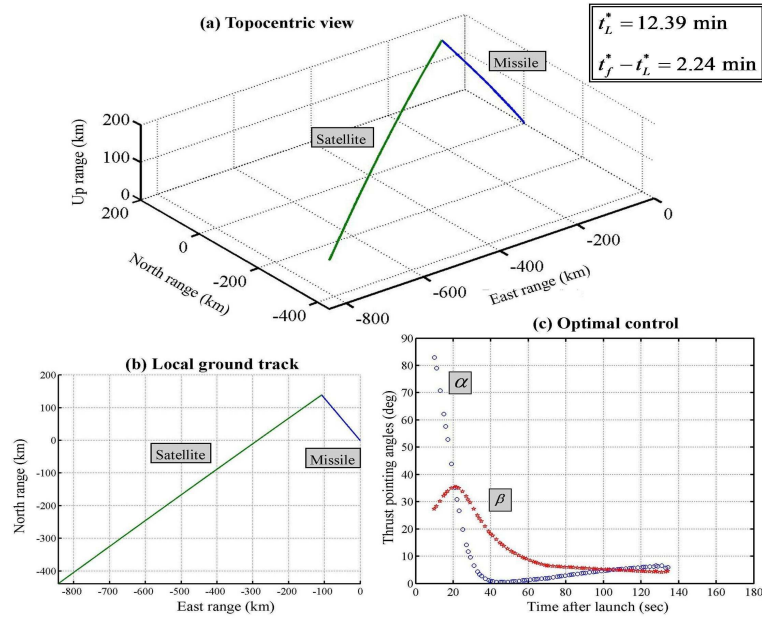


Fig. 3. Problem B: optimal trajectory (a), ground track (b), and optimal control time history (c)

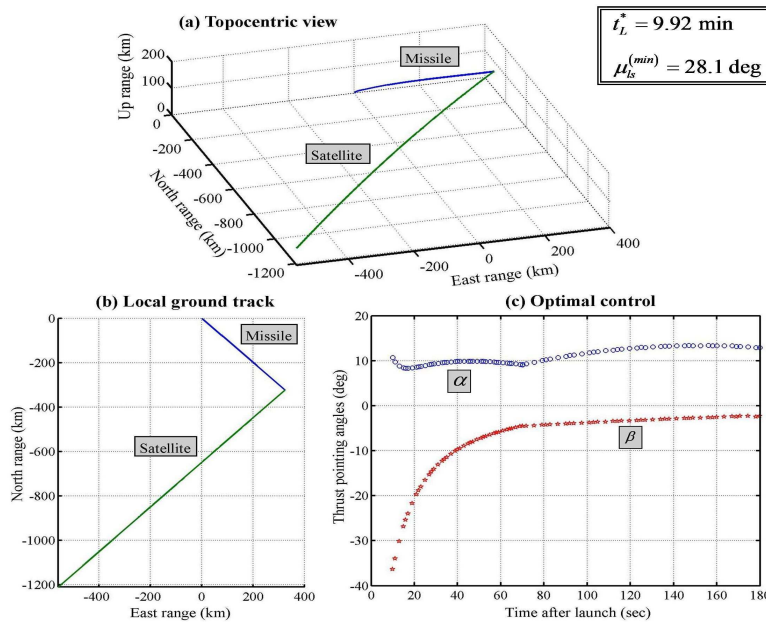


Fig. 4. Problem C: optimal trajectory (a), ground track (b), and optimal control time history (c)

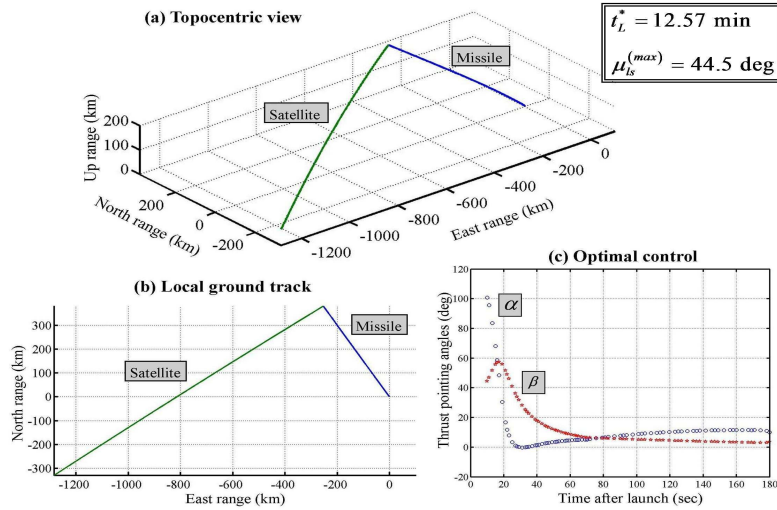


Fig. 5. Problem D: optimal trajectory (a), ground track (b), and optimal control time history (c)

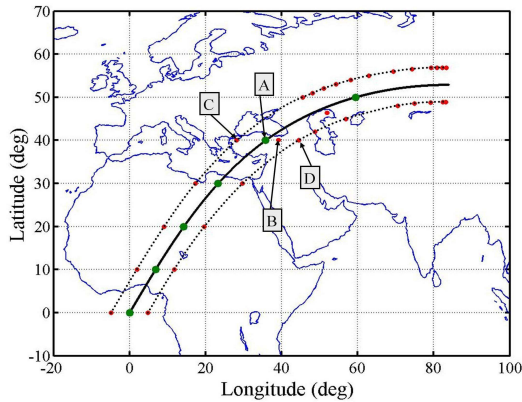


Fig. 6. Stripe of interception within 3 minutes with reference to a quarter orbit

Similar stripes can be traced with regard to the remaining orbits in the period of repetition of the target satellite and this process leads to identifying the “region of interception within 3 minutes” all over the world.

The method employed in this research is capable of determining the optimal ASAT trajectories also when the atmospheric drag is considered. Several test cases have been performed by employing a realistic atmospheric model. The optimal control time histories turn out to change as a function of the ballistic coefficients of the two stages, as well as the performance at-

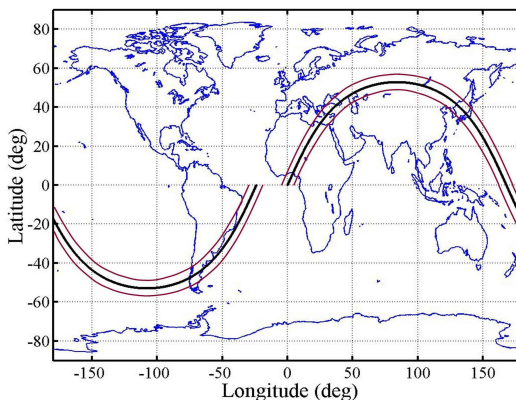


Fig. 7. Stripe of interception within 3 minutes with reference to a single orbit

tainable by the interceptor missile. For instance, with reference to a launch latitude of 40 deg, the longitudinal extension of the launch region that guarantees interception within 3 minutes is a function of the ballistic coefficient at ignition of the first stage, B_{01} . Under the assumption that atmospheric drag is negligible by the time of ignition of the second stage, the results reported in Table 1 have been obtained.

Table 1. Feasible region for the longitude as a function of B_{01} ($\phi_{ls} = 40$ deg).

| B_{01} (m^2/kg) | $\mu_{ls}^{(min)}$ (deg) | $\mu_{ls}^{(max)}$ (deg) |
|-------------------------------------|--------------------------|--------------------------|
| 0 | 28.1 | 44.5 |
| 1×10^{-4} | 28.2 | 44.4 |
| 5×10^{-4} | 28.4 | 44.1 |
| 1×10^{-3} | 28.6 | 43.9 |
| 5×10^{-3} | 29.8 | 42.4 |

As expected, the inclusion of atmospheric drag reduces the performance of the rocket and narrows the feasible region for the launch longitude.

6. Concluding remarks.

This research addresses the problem of optimal interception of a low Earth satellite by a high-performance, ground-launched, two-stage rocket. Four kinds of optimal control problems have been considered, to determine the minimum-time trajectories leading to interception and the geographical region where the launch site can be located so that interception can occur in a specified time. The absolute minimum-time trajectories correspond to

the optimal locations of the launch site, which must lie along the ground track of the target satellite.

The results reported in Section 5 neglect the effect of the atmosphere in order to determine the rocket performance regardless of its mass. However, depending on the ballistic coefficients of the two stages, the atmosphere can worsen considerably the performance attainable by the interceptor missile.

Real-time guidance is a central issue in the concrete realization of an anti-satellite system. Due to the short time of flight, its implementation is nontrivial. In addition, the performance attainable by the interceptor rocket can depend significantly on the possible bounds on the control variables (not considered in this study). However, the determination of the optimal trajectories leading to interception is an essential premise to the definition of the guidance strategy, and defines the best performance attainable by an anti-satellite rocket with specified propulsive characteristics, such as that considered in this research.

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