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Some results on the reconstruction of a convolution kernel in an integrodifferential beam equation

Fabrizio Colombo[1], Davide Guidetti[2]

[1] *Dipartimento di Matematica*
Politecnico di Milano, Italy
fabrizio.colombo@polimi.it

[2] *Dipartimento di matematica,*
Università degli Studi di Bologna,Italy
guidetti@dm.unibo.it

Abstract

We illustrate some results of existence and uniqueness of local and global solutions to an inverse problem of reconstruction of a convolution kernel in a semilinear integrodifferential beam equation. These results are particular cases of corresponding results concerning certain abstract equations of the second order.

Keywords: `Beam equation with memory, nonhomogeneous boundary conditions, global in time existence and uniqueness result, nonlinear second order in time integrodifferential inverse problem.`

1. An abstract problem.

In this short note we shall illustrate the study of a certain inverse problem of reconstruction of a convolution kernel in an abstract second order differential equation. As we shall see, this admits an application to a semilinear integrodifferential beam equation. The proofs will be, at most, sketched. A complete version is going to appear in [2].

We shall consider a system of the form

$$(1.1) \qquad \begin{cases} u''(t) + Au(t) = h * (Bu + r)(t) + f(t, u(t)), \\ t \in (0, \tau), \\ u(0) = u_0, \\ u'(0) = u_1, \\ \Phi(u(t)) = g(t), t \in (0, \tau), \end{cases}$$

with u and h unknown. h is a scalar valued convolution kernel. The fact that h is unknown should be compensated by the knowledge of $\Phi(u(t))$ (for

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every t), with Φ suitable functional.

Our assumptions are the following:

(H1) $D(A)$, V, H *are real Banach spaces, such that*

$$D(A) \hookrightarrow V \hookrightarrow H,$$

with $D(A)$ *dense in* V *and in* H.

Let A *belong to* $\mathcal{L}(D(A), H)$*; we set*

$$\begin{cases} \mathcal{A} : D(A) \times V \to V \times H, \\ \\ \mathcal{A}(u, v) = (v, -Au), \end{cases} \tag{1.2}$$

and assume that $\mathcal{A}$ *is the infinitesimal generator of a strongly continuous semigroup* $(e^{t\mathcal{A}})_{t\geq 0}$ *in* $V \times H$.

(H2) $u_0 \in D(A)$.

(H3) $u_1 \in V$.

(H4) $B \in \mathcal{L}(V; H)$.

(H5) We assume the following regularity conditions on f*:* $f \in C([0, T] \times D(A); H)$, $f(., u) \in W^{1,1}(0, T; H)$, $\forall u \in D(A)$, $f(t, .)$ *is Frechét differentiable from* $D(A)$ *to* H, $\forall (t, u) \in [0, T] \times D(A)$. *The partial derivative* $(t, u) \to d_u f(t, u)$ *is extensible to an element of* $C([0, T] \times D(A); \mathcal{L}(V, H))$, *which we continue to indicate with* $d_u f(t, u)$*; finally, there exist* $L(t, R)$, *with domain* $0, T] \times \mathbb{R}^+$, *integrable in* t *for every* R, *and nondecreasing in* R, *and* $L(R)$, *nonnegative and nondecreasing, such that, for almost every* $t \in [0, T]$, *and* $\forall u_1, u_2 \in D(A)$, *with*

$$\max\{\|u_1\|_{D(A)}, \|u_2\|_{D(A)}\} \leq R,$$

one has

$$\begin{cases} \|\frac{\partial f}{\partial t}(t, u_1) - \frac{\partial f}{\partial t}(t, u_2)\|_H \leq L(t, R)\|u_1 - u_2\|_{D(A)}, \\ \\ \|d_u f(t, u_1) - d_u f(t, u_2)\|_{\mathcal{L}(V,H)} \leq L(R)\|u_1 - u_2\|_{D(A)}. \end{cases} \tag{1.3}$$

(H6) $r \in W^{1,1}(0, T; H)$.

(H7) $\Phi \in H'$, $\Phi \circ A$, *with domain* $D(A)$, *can be extended to an element* Ψ *of* V'.

(H8) $\Phi(Bu_0 + r(0)) \neq 0$.

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(H9) $g \in W^{3,1}(0,T)$, $\Phi(u_0) = g(0)$, $\Phi(u_1) = g'(0)$, $\Phi(v_0) = g''(0)$, *with*

$$v_0 := -Au_0 + f(0, u_0). \tag{1.4}$$

(H10) We assume the following growth conditions on f*: there exist* $L_0 \in L^1(0,T)$, $L_1 \in \mathbb{R}^+$, *such that,* $\forall u \in D(A)$,

$$\begin{cases} \|\frac{\partial f}{\partial t}(t,u)\|_H \le L_0(t) & a.e. \text{ in } (0,T), \\ \\ \|d_u f(t,u)\|_{\mathcal{L}(V,H)} \le L_1 \ \forall t \in [0,T]. \end{cases}$$

A problem similar to (1.1) was considered in [4], where the more general case $B \in \mathcal{L}(D(A), H)$ is treated. However, it does not contain any result of global existence of solutions. This is due to the fact that the method adopted in this paper requires very regular coefficients (with respect to ours), so that it is difficult to get a priori estimates of the solution.

Concerning the assumptions, we observe that, owing to [3], (H1) is equivalent to the assumption that A is the infinitesimal generator of a strongly continuous cosine operator in H (if this happens it can be determined a space V such that (H1) holds).

Our main results are the following:

Theorem 1.1. *(Local in time existence). Let the assumptions* $(H1)$*–*$(H9)$ *hold. Then there exists* $\tau \in (0,T]$*, depending on the data, such that the inverse problem (1.1) has a solution* (u,h)*, such that*

$$u \in C^2([0,\tau];H) \cap C^1([0,\tau];V) \cap C([0,\tau];D(A)),$$
$$h \in L^1(0,\tau).$$

Theorem 1.2. *(Global in time uniqueness). Let the assumptions* $(H1)$*–*$(H9)$ *hold. Then, if* $\tau \in (0,T]$*, and the inverse problem (1.1) has two solutions* (u_j, h_j) $(j \in \{1,2\})$*, such that*

$$u_1, u_2 \in C^2([0,\tau];H) \cap C^1([0,\tau];V) \cap C([0,\tau];D(A)),$$
$$h_1, h_2 \in L^1(0,\tau).$$

then $u_1 = u_2$ *and* $h_1 = h_2$.

Theorem 1.3. *(Global in time existence and uniqueness). Let the assumptions* $(H1)$*–*$(H10)$ *hold. Let* $T > 0$*. Then the inverse problem (1.1) has a unique solution* (u,h)*, such that*

$$u \in C^2([0,\tau];H) \cap C^1([0,\tau];V) \cap C([0,\tau];D(A)),$$
$$h \in L^1(0,\tau).$$

Remark 1.1. We observe that, even in the case that f is linear, the system (1.1) is not linear, because there appears the convolution product $h * Bu$ with both h and u unknown. The result of global existence with the further assumption (H10) can be obtained employing a method developed (in the parabolic case) in [1], which is based on the observation that, in some sense, convolution becomes linear after a time interval of positive length.

2. The beam equation

Now we show an application of our abstract results to the problem which motivated our assumptions, namely the following system for the beam equation with memory, with u and h unknown:

$$
\begin{cases}
U_{tt}(t,x) + \Delta^2 U(t,x) - \Delta U(t,x) = \int_0^t h(t-s)\Delta U(s,x)ds \\
+F(t,x,U(t,x),D_x U(t,x),D_x^2 U(t,x)), \\
(t,x) \in (0,\tau) \times \Omega, \\
U(t,x) = g_0(t,x), \qquad (t,x) \in [0,T] \times \partial\Omega, \\
D_\nu U(t,x) = g_1(t,x), \qquad (t,x) \in [0,T] \times \partial\Omega, \\
U(0,x) = U_0(x), \qquad x \in \Omega, \\
U_t(0,x) = U_1(x), \qquad x \in \Omega, \\
\int_\Omega \phi(x) U(t,x)\, dx = G(t), \qquad t \in [0,T].
\end{cases}
\tag{2.5}
$$

We study problem (2.5) under the following assumptions:

(K1) *Ω is an open bounded subset of $\mathbb{R}^n$, with $n \leq 3$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class C^5 .*

(K2) *We indicate with (t,x,u,p,q) the generic element of $[0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}$. We assume that $F \in C^1([0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2})$; moreover, the first order derivatives are Lipschitz continuous with respect to u, p and q, uniformly in bounded subsets of $[0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}$.*

(K3) *$U_0 \in H^{4+\varepsilon}(\Omega)$, $U_1 \in H^{2+\varepsilon}(\Omega)$, for some $\varepsilon \in \mathbb{R}^+$.*

(K4) *For some* $\varepsilon \in \mathbb{R}^+$:

$$g_0 \in W^{1+\varepsilon,1}(0,T; H^{\frac{7}{2}}(\partial\Omega)) \cap W^{\frac{11}{4}+\varepsilon,1}(0,T; L^2(\partial\Omega)),$$

$$g_1 \in W^{1+\varepsilon,1}(0,T; H^{\frac{5}{2}}(\partial\Omega)) \cap W^{\frac{9}{4}+\varepsilon,1}(0,T; L^2(\partial\Omega)).$$

(K5) *Compatibility conditions on* g_0, g_1, U_0 *and* U_1:

$$\gamma U_0 = g_0(0,.), \quad D_\nu U_0 = g_1(0,.), \quad \text{in} \quad \partial\Omega,$$

$$\gamma U_1 = D_t g_0(0,.), \quad D_\nu U_1 = D_t g_1(0,.) \quad \text{in} \quad \partial\Omega,$$

with γ *trace operator on* $\partial\Omega$.

(K6) $\phi \in H_0^2(\Omega)$.

(K7) $\int_\Omega \phi(x)\Delta U_0(x)dx \neq 0$.

(K8) $G \in W^{3,1}(0,T)$.

(K9) $\int_\Omega \phi(x)U_0(x)dx = G(0)$, $\int_\Omega \phi(x)U_1(x)dx = G'(0)$, $\int_\Omega \phi(x)V_0(x)dx = G''(0)$, *with*

$$V_0 := -\Delta^2 U_0 + \Delta U_0 + F(0, U_0, D_x U_0, D_x^2 U_0).$$

(K10) *The first order derivatives of* F *are uniformly bounded in* $[0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}$.

The following results hold:

Theorem 2.1. *(Local in time existence). Let the assumptions* $(K1)$*–*$(K9)$ *hold. Then there exists* $\tau \in (0,T]$*, depending on the data, such that problem (2.5) has a solution* (U,h)*, such that*

$$U \in C^2([0,\tau]; L^2(\Omega)) \cap C^1([0,\tau]; H^2(\Omega)) \cap C([0,\tau]; H^4(\Omega)),$$

$$h \in L^1(0,\tau).$$

Theorem 2.2. *(Global in time uniqueness). Let the assumptions* $(K1)$*–*$(K9)$ *hold. Then, if* $\tau \in (0,T]$*, and problem (2.5) has two solutions* (U_j, h_j) *(*$j \in \{1,2\}$*), such that*

$$U_1, U_2 \in C^2([0,\tau]; L^2(\Omega)) \cap C^1([0,\tau]; H^2(\Omega))$$

$$\cap C([0,\tau]; H^4(\Omega)),$$

$$h_1, h_2 \in L^1(0,\tau),$$

then $U_1 = U_2$ *and* $h_1 = h_2$.

Theorem 2.3. *(Global in time existence and uniqueness). Let the assumptions* $(K1)$–$(K10)$ *hold. Then problem (2.5) has a unique solution* (U, h)*, such that*

$$U \in C^2([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^2(\Omega))$$
$$\cap C([0,T]; H^4(\Omega)),$$
$$h \in L^1(0,T).$$

The first step, in order to study problem (2.5), is to reduce it to vanishing boundary conditions. One can show that, if the assumptions (K3)-(K5) hold, there exists $\tilde{U}$ such that

$$\tilde{U} \in W^{3,1}(0,T; L^2(\Omega)) \cap W^{2,1}(0,T; H^2(\Omega))$$
$$\cap W^{1,1}(0,T; H^4(\Omega)),$$

such that

$$\tilde{U}(0,x) = U_0(x), \quad D_t\tilde{U}(0,x) = U_1(x),$$
$$\gamma\tilde{U}(t,x) = g_0(t,x), D_\nu\tilde{U}(t,x) = g_1(t,x).$$

Setting

$$u(t,x) := U(t,x) - \tilde{U}(t,x), \tag{2.6}$$

we are reduced to look for a solution (u, h) such that

$$u \in [C^2([0,\tau]; L^2(\Omega)) \cap C^1([0,\tau]; H^2(\Omega)) \cap C([0,\tau]; H^4(\Omega)),$$
$$h \in L^1(0,\tau),$$

of the problem

(2.7)
$$\begin{cases} u_{tt}(t,x) + \Delta^2 u(t,x) - \Delta u(t,x) = \int_0^t h(t-s)(\Delta u(s,x) + \Delta\tilde{U}(s,x))ds \\ +F(t,x,\tilde{U}(t,x) + u(t,x), D_x\tilde{U}(t,x) + D_x u(t,x), D_x^2\tilde{U}(t,x) + D_x^2 u(t,x)) \\ -(\tilde{U}_{tt}(t,x) + \Delta^2\tilde{U}(t,x) - \Delta\tilde{U}(t,x)), \quad (t,x) \in (0,\tau) \times \Omega, \\ u(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ D_\nu u(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \\ u(0,x) = 0, \quad x \in \Omega, \\ u_t(0,x) = 0, \quad x \in \Omega, \\ \int_\Omega \phi(x)u(t,x)\,dx = G(t) - \int_\Omega \phi(x)\tilde{U}(t,x)\,dx, \quad t \in [0,T]. \end{cases}$$

We set

$$H := L^2(\Omega), \quad V := H_0^2(\Omega), \quad D(A) = H^4(\Omega) \cap H_0^2(\Omega). \tag{2.8}$$

Lemma 2.1. *Assume that (K1) is satisfied. We adopt notation (2.8) and introduce the operators*

$$\begin{cases} \mathcal{A} : D(A) \times V \to V \times H, \\ \\ \mathcal{A}(u, v) := (v, \Delta^2 u - \Delta u). \end{cases} \tag{2.9}$$

Then $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup in $V \times H$.

We set

$$\begin{cases} B : V \to H, \\ \\ Bv := \Delta v, \end{cases} \tag{2.10}$$

$$r(t, x) := \Delta \tilde{U}(t, x), \tag{2.11}$$

(2.12)

$$\begin{cases} f : [0, T] \times D(A) \to H, \\ \\ f(t, u)(x) := -(\tilde{U}_{tt}(t, x) + \Delta^2 \tilde{U}(t, x) - \Delta \tilde{U}(t, x)) \\ \\ +F(t, x, \tilde{U}(t, x) + u(x), D_x \tilde{U}(t, x) + D_x u(x), D_x^2 \tilde{U}(t, x) + D_x^2 u(x)), \end{cases}$$

As $n \leq 3$, Sobolev embedding theorem implies that $D(A) \hookrightarrow C^2(\overline{\Omega})$, so that, if $|\alpha| \leq 2$, $D_x^\alpha \tilde{U} \in W^{1,1}(0, T; C(\overline{\Omega}))$. This implies that f, as defined in (2.12), satisfies assumption (H5). We observe only that, if $u, z \in D(A)$

$$\begin{gathered} d_u f(t, u)(z)(x) \\ = \tfrac{\partial F}{\partial u}(t, x, \tilde{U}(t, x) + u(x), D_x \tilde{U}(t, x) + D_x u(x), \\ D_x^2 \tilde{U}(t, x) + D_x^2 u(x)) z(x) + D_p F(t, x, \tilde{U}(t, x) + u(x), D_x \tilde{U}(t, x) \\ +D_x u(x), D_x^2 \tilde{U}(t, x) + D_x^2 u(x)) \cdot D_x z(x) \\ +D_q F(t, x, \tilde{U}(t, x) + u(x), D_x \tilde{U}(t, x) + D_x u(x), \\ D_x^2 \tilde{U}(t, x) + D_x^2 u(x)) \cdot D_x^2 z(x) \end{gathered} \tag{2.13}$$

and $d_u f(t,u)$ is continuously extensible to an element of $\mathcal{L}(V,H)$ depending continuously on (t,u).

Finally, we set

$$\begin{cases} u_0, u_1 : \Omega \to \mathbb{R}, \\ u_0(x) = u_1(x) = 0, \quad x \in \Omega, \end{cases} \tag{2.14}$$

$$\begin{cases} g : [0,T] \to \mathbb{R}, \\ g(t) = G(t) - \int_\Omega \phi(x)\tilde{U}(t,x)\, dx, \end{cases} \tag{2.15}$$

$$\Phi(f) := \int_\Omega \phi(x) f(x) dx, \quad \text{for} \quad f \in H. \tag{2.16}$$

We can show that the assumptions (H1)-(H9) are consequences of (K1)-(K9), while (H10) follows from (K10). For example, we observe that, if $u \in D(A)$, using (K6), we have

$$\Phi(Au) = \int_\Omega (\Delta\phi(x)\Delta u(t,x) - \phi(x)\Delta u(t,x)) dx,$$

so that $\Phi \circ A$ is continuously extensible to V.

We conclude that, if assumptions (K1)-(K9) hold, Theorems 1.1 and 1.2 are applicable, and we obtain Theorems 2.1 and 2.2. (H10) follows from (K10), taking into account (2.13). So, if the assumptions (K1)-(K10) are satisfied, we obtain Theorem 2.3.

3. Some ideas concerning the proofs of Theorems 1.1-1.3

Here we try to give some ideas concerning the proofs of Theorems 1.1-1.3.

If we think of h as the unknown in the first equation in (1.1), we observe that we have a Volterra integral equation of the first kind. In order to reduce it to a Volterra integral equation of the second kind, we differentiate with respect to t. If $v := \partial_t u$, we get

$$\begin{cases} v_{tt} + Av = h(t)(Bu_0 + r(0)) + h * (Bv + r') \\ + \frac{\partial f}{\partial t}(t, u_0 + 1 * v(t)) + d_u f(t, u_0 + 1 * v(t))v(t), \\ v(0) = u_1, \\ v'(0) = -Au_0 + f(0, u_0). \end{cases} \tag{3.17}$$

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Applying Φ and the condition $\Phi(u(t)) = g(t)$, we obtain

$$(3.18)\qquad \begin{aligned} &g''' + \Phi(Av) = h(t)\Phi(Bu_0 + r(0)) + h * \Phi(Bv + r') \\ &+\Phi[\tfrac{\partial f}{\partial t}(t, u_0 + 1 * v(t)) + d_u f(t, u_0 + 1 * v(t))v(t)], \end{aligned}$$

which is a Volterra equation of the second type in the unknown h.

Now we indicate with $C(t)$ the cosine operator function generated by A, and with $S(t)$ the corresponding sine operator function. Then, from (3.17), we obtain

(3.19)

$$\begin{cases} v(t) = C(t)u_1 + S(t)[-Au_0 + f(0, u_0)] + \int_0^t S(t-s)\{h(s)(Bu_0 + r(0)) \\ +h * (Bv + r')(s) + \mathcal{F}(v)(s)\}ds, \quad a.e. \;\; in \;\; (0, \tau), \\ h(t) = \chi\{g'''(t) + \Psi(v(t)) - \Phi[\mathcal{F}(v)(t) + h * (Bv + r')(t)]\}, \\ a.e. \;\; in \;\; (0, \tau), \end{cases}$$

where we have set

$$(3.20)\qquad \chi := \Phi(Bu_0 + r(0))^{-1},$$

and

$$(3.21)\qquad \mathcal{F}v(t) := \frac{\partial f}{\partial t}(t, u_0 + 1 * v(t)) + d_u f(t, u_0 + 1 * v(t))v(t).$$

Using a fixed point argument, we are able to show that, if τ is sufficiently small, (3.19) has a unique solution (v, h), such that $v \in C([0, \tau]; V)$, $1 * v \in C([0, \tau]; D(A))$ and $h \in L^1(0, \tau)$. Setting now

$$(3.22)\qquad u(t) := u_0 + 1 * v,$$

it is not difficult to verify that u has the desired regularity and (u, h) solves (1.1) in $[0, \tau]$.

Concerning the existence of a global solution, we observe that assumption (H10) implies a sort of "sublinear" growth of the nonlinear function f. So the main difficulty lies in trying to control the convolution $h * Bu$, which, at least formally, has quadratic growth in (u, h). In order to get this aim, we can observe that, in some sense, after a positive time the convolution "becomes linear".

In fact, assume that h and v are known in $[0, \tau]$ and unknown in $[\tau, 2\tau]$.

If

$$\tilde{h} := h_{[0,\tau]}, \tilde{v} := v_{[0,\tau]}$$

(known)

$$h_\tau(t) := h(\tau + t), v_\tau(t) := v(\tau + t)$$

(unknown)

for t in $[0, \tau]$ we have

$$\begin{gathered}(h * v)(\tau + t) \\ = (\tilde{h} * v_\tau)(t) + (h_\tau * \tilde{v})(t) + \int_t^\tau \tilde{h}(\tau + t - s)\tilde{v}(s)ds.\end{gathered}$$

So, we can estimate u and h in $[\tau, 2\tau]$ in a linear way in terms of $u_{[0,\tau]}$ and $h_{[0,\tau]}$. This allows to obtain an a priori estimate, which implies the conclusion.

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