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Thermodynamics of superfluid vortex tangles

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Abstract

We consider the main energetic and geometric aspects contributing to the entropy of the vortex tangles appearing in superfluid turbulence, and we propose a simplified model to evaluate them. We consider the tangle and the underlying superfluid system as two different systems, each one in internal quasi-equilibrium, but exchanging heat with each other. The resulting thermodynamic formalism is studied in two different significative situations.

Keywords: `Superfluid turbulence. Nonequilibrium thermodynamics.`

1. Introduction.

One topic of interest arising in thermodynamics and hydrodynamics of turbulent superfluids [1]– [3] is the form of their thermodynamic functions, in terms of L, the average vortex line density per unit volume, and of other relevant geometrical features of the tangle, related, for instance, to the orientation of the local unit tangent to the vortex line or to the local curvature of the vortex lines. Thus, the thermodynamics of turbulent superfluids is still open to many basic challenges, in contrast with the thermodynamics of their laminar flows. A deeper description of the tangle should also incorporate, for instance, the length distribution of the vortex loops composing the tangle, the number of their crossings, the spatial correlations between the tangent vectors or the curvature vectors at different positions, and so on [5]– [7]. Here, we will try to identify some of the most relevant and basic features and to provide a simple estimation for their contributions to the entropy.

To this purpose in the paper we shall consider first the entropic contribution to the entropy of the vortex-line density per unit volume (briefly

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called line density) $L = \mathcal{L}/V$, where $\mathcal{L}$ is the total vortex line length and V the volume. Line density L may be directly measured -for instance, by means of second-sound experiments- and is therefore the most used variable describing the tangle [1]– [3]. The line density L has two kinds of contributions to the entropy: on the one side, it carries out a well defined energy; on the other side, it is a source of microscopic disorder -related to the length distribution in vortex loops or, in simpler terms, to the average number of vortex loops N- and, therefore, contributes to the entropy also from this perspective.

In the paper, we will introduce several thermodynamic functions for the tangle, entropy S_{tangle}, energy U_{tangle}, chemical potential μ_{tangle}, pressure p_{tangle}, temperature T_{tangle}, ...; in the following, for the sake of notation simplicity we will neglect in all these functions the subindex $tangle$, writing simply S, U, μ, p, T, ...

Concerning the energy contribution U of the tangle we will simply assume that it is

$$U = NU_l, \tag{1}$$

where U_l is the energy of the loop of length l.

The energy (1) will be identified with the energy of the tangle, i.e., in contrast with other descriptions, where one directly considers the global energy of the two-fluid helium plus the tangle, here we give a separated consideration to the energies of both subsystems. This distinction between both energies will allow us to use a two-temperature model for the whole system, namely one for the liquid helium and another one for the tangle. Indeed, observing that the tangle is not a continuous line, but a superposition of many closed vortex loops, one might consider the tangle as an ideal gas of vortex loops and one may relate an effective temperature of the vortex tangle to the average energy of the vortex loops, defining $T_{Tangle} = T$, as

$$k_B T =< U_l > \tag{2}$$

where k_B is Boltzmann's constant.

Though this has not been done -to our knowledge- in this system, it is a usual assumption in many other systems [8]– [10]. For instance, one speaks of two temperature system in plasma physics, because ion and electron temperatures may be very different, and in the study of glassy systems where two different temperatures are defined: vibrational temperature (linked to the fast degrees of freedom, kinetic and vibrational ones, which is equal to the environment temperature) and configurational temperature (linked to the slow degrees of freedom, configurational ones) which turns out to

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be very high, but that it is not felt because of the extremely slow rate of exchange, which may be of the order of years, or even of centuries.

The system as a whole (the helium background and the vortex tangle) can be considered as a two-temperature system, where the difference in temperatures is sustained by the external forcing. This definition of temperature for the vortex tangle does not rely on any particular distribution for U_l, as well as the kinetic definition of temperature is used in kinetic theory even when the velocity distribution function is very different from that of Maxwell-Boltzmann. In the following sections, we will present two statistical distribution functions for l, corresponding to two different experimental situations.

2. Polarized tangles: contributions of L.

In some regimes of superfluid turbulence produced by classical means, as that generated by a towing or vibrating grid, or in simultaneous presence of counterflow and rotation, the vortex tangle results polarized, and the reconnection of vortices is forbidden [11]. In this case we can suppose that the number N of vortex loops is constant and that the energy of a vortex loop is proportional to its length $U_l = \epsilon_V l$, with ϵ_V the energy per unit length of vortex line

$$\epsilon_V = \rho_s \kappa \tilde{\beta} \qquad \tilde{\beta} = \frac{\kappa}{4\pi} \ln\left(\frac{\delta}{a_0}\right), \tag{3}$$

where $\tilde{\beta}$ is the vortex tension parameter and $\delta \simeq L^{-1/2}$ is the average separation between vortices. Then, we assume [12]

$$k_B T = \epsilon_V < l >, \tag{4}$$

while, for the distribution function of vortex loops with respect to their length, we can choose [12]:

$$f(l) \propto \exp\left[-\frac{\epsilon_V l}{k_B T}\right]. \tag{5}$$

As we will see later, in counterflow, at extremely low temperatures, the vortex lines form a very entangled structure and possess fractal properties, so that U_l becomes proportional to some power of l.

We will concentrate our attention on the entropy of the vortex tangle, which will be assumed to be a function of the total length $\mathcal{L}$ of vortex lines in the tangle, of the average number N of vortices and of a tensor $\mathbf{\Pi}$, describing the orientations of the tangent vector $\mathbf{s}'$, which was introduced

in a previous work [13] and will be specified in Section 3. Then, we will search for

$$S_{tangle} = S(\mathcal{L}, N, \mathbf{\Pi}). \tag{6}$$

Now, we will consider the purely entropic contributions of L, i.e., the contributions related to the disorder. In this Section we will not consider the orientational disorder, which will be analyzed in the next Section, but another effect, related to the fact that the tangle is not a single continuous line, but a superposition of closed vortex loops. Therefore, there is a number of possibilities of distributing the total length in several closed loops of length l_j, each loop having the corresponding energy $U_j = l_j \epsilon_V$.

A simple way to estimate the entropy would be to take into account that $S = k_B \ln W$, with W the number of available microstates, which is proportional to the volume of phase space of the vortices. This phase space is related to the position and the length of the vortex loop. Therefore, the volume of the phase space available to vortices will be proportional to $V\mathcal{L}$. Since the different vortices are assumed to be independent, and since the entropy $S(\mathcal{L}, V, N)$ is a homogeneous function of the first order, in such a way that $S(\mathcal{L}, V, N) = NS[(\mathcal{L}/N), (V/N)]$ this leads to an expression for the entropy of the form

$$S = S(\mathcal{L}, N, V) = S_0 + k_B N \ln \left[\frac{\mathcal{L}}{N} \frac{V}{N} \right]. \tag{7}$$

One could also write this expression in terms of the internal energy U rather than $\mathcal{L}$, to stress the analogy with ideal gases, namely

$$S = S(U, V, N) = S_0 + k_B N \ln \left[\frac{U}{N} \frac{V}{N} \right], \tag{8}$$

In these equations S_0 is a reference constant. From (8) one has

$$\left[\frac{\partial S}{\partial U} \right]_{V,N} = \frac{k_B N}{U} \equiv \frac{1}{T} = \frac{k_B}{< l > \epsilon_V}. \tag{9}$$

This gives an entropic meaning to the effective temperature introduced in (2). In the thermodynamic analyses of vortex tangles one is also especially interested in the quantity $\partial S/\partial N$, which defines the chemical potential conjugate to variations in N. This is obtained from (8) as

$$\left[\frac{\partial S}{\partial N} \right]_{U,V} = -k_B \left(\ln \frac{N^2}{UV} + 2 \right) \equiv -\frac{\mu}{T}, \tag{10}$$

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The chemical potential of vortex tangle is seen to be

$$\mu(U,V,N) = k_B T \left(\ln \frac{N^2}{UV} + 2 \right) = \mu^{(0)} + \epsilon_V < l > \ln \frac{\mathcal{L}}{\mathcal{L}^*}, \tag{11}$$

with $\mu^{(0)} = -\epsilon_V < l > [\ln(\epsilon_V < l > V) - 2]$ a term which depends on temperature and volume but not on N. One could write the chemical potential of vortex tangle in terms of the temperature T rather than the internal energy U, namely

$$\mu(T,V,N) = k_B T \left(\ln \frac{N}{k_B T V} + 2 \right) = \mu^{(0)}(T,V) + k_B T \ln \frac{\mathcal{L}}{\mathcal{L}^*}, \tag{12}$$

with $\mu^{(0)} = \mu^{(0)}(T,V) = -k_B T [\ln(k_B T V) - 2]$. Here we will take for the reference length $\mathcal{L}^*$ the value $\mathcal{L}^* =< l >$. In this way, the logarithm in (12) will vanish when the tangle is composed of a single closed loop, because in that case the average length $< l >$ will coincide with the total vortex length $\mathcal{L}$ and there will be only a single microstate for the vortex loop distribution.

The variation of the volume at constant total length allows us to introduce the concept of tangle pressure, given in thermodynamic terms as

$$\frac{p}{T} = \left(\frac{\partial S}{\partial V} \right)_{U,N} = \frac{k_B N}{V}. \tag{13}$$

In view of our interpretation (2) of T we would have

$$p = \frac{k_B N T}{V} = \frac{\epsilon_V \mathcal{L}}{V} = \epsilon_V L. \tag{14}$$

The total pressure of the system will be the pressure of the two-component helium fluid plus the contribution of the vortex tangle given by (14). This result has been also obtained in Ref. [14], in a macroscopic thermodynamic framework and in Ref. [15], in a different simplified microscopic description.

In this Section, we have neglected the configurational entropy, which will be considered in the next Section.

3. Geometric contributions: orientation of vortex lines.

The geometric characteristics of the tangle, beside its length, also contribute to the entropy. An essential geometric characteristic is the vector field $\mathbf{s}'(\xi)$, which gives the unit tangent vector along the vortex lines. The first and the second moments of the orientational distribution of $\mathbf{s}'$ (namely polarization and anisotropy of the tangle, respectively) can be determined

through measurements of the second sound attenuation, when the wave is propagating in some given direction, as for instance along the rotation vector or the counterflow velocity.

To recall the physical relevance of these vectors, we must have in mind that the friction force per unit length exerted by the normal fluid flowing with velocity $\mathbf{v}_n$ on the superfluid is [1]:

$$\mathbf{f} = -\alpha\rho_s\kappa\mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_{sl})] - \alpha'\rho_s\kappa\mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_{sl}), \tag{15}$$

where $\mathbf{v}_{sl}$ is any externally applied superfluid velocity $\mathbf{v}_s$ plus the self-induced velocity of the vortex line, $\mathbf{v_i}$. In the usual local-induction approximation, $\mathbf{v_i}$ is given by $\mathbf{v_i} \simeq \tilde{\beta}\mathbf{s}' \times \mathbf{s}''$ [1]. It is evident that $\mathbf{s}'$ and $\mathbf{s}''$ (and also their vectorial product $\mathbf{s}' \times \mathbf{s}''$), play a relevant mechanical role in the dynamics of the vortex lines. Here, we will discuss the role of $\mathbf{s}'$ in thermodynamics.

In a previous analysis of the structure of the vortex tangle [13], we have proposed to use for the description of the effects of $\mathbf{s}'$ the tensorial variables

$$\mathbf{\Pi}^s \equiv \frac{3}{2} < \mathbf{U} - \mathbf{s}'\mathbf{s}' >, \qquad \mathbf{\Pi}^a \equiv \frac{3}{2}\frac{\alpha'}{\alpha} < \mathbf{W} \cdot \mathbf{s}' >, \tag{16}$$

where $\mathbf{U}$ is the unit tensor, $\mathbf{s}'\mathbf{s}' = \mathbf{s}' \otimes \mathbf{s}'$ the diadic product, and the angular brackets stand for average over the total line length, whereas $\mathbf{W}$ is the Ricci tensor, a completely antisymmetric third-order tensor; α' and α are dimensionless friction coefficients appearing in equation (15). The antisymmetric tensor $\mathbf{\Pi}^a$ is related to the polarization. In a purely isotropic tangle, $\mathbf{\Pi}^a$ vanishes and $\mathbf{\Pi}^s$ is equal to $\mathbf{U}$. The latter definition deserves some explanation. Indeed, since the average value of $\mathbf{s}'$ is zero along any closed curve, the polarization related to the average value of $\mathbf{s}'$ will be zero when the tangle is composed entirely of vortex loops. The use of the term "polarization" is analogous to that used in magnetic of dielectric dipole systems, where the polarization is related to the net value of the magnetic or electric moment, rather than with the fact of the dipoles being oriented along an axis without taking into account their sign. In this setting, polarized tangles would require the existence of open vortex lines, which are also possible provided they being and end on the walls. This is the case, for instance, of the vortex lines produced in rotating superfluids, which form an array of straight vortex lines parallel to the rotation axis in absence of counterflow. A detailed analysis of the dynamical role of (16) may be found in Ref. [13]. Here we will focus our attention on purely thermodynamic aspects of $\mathbf{s}'$.

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To deal with the geometric effects related to the orientation of $\mathbf{s}'$, we make a parallelism with an analogous problem in polymer physics or in liquid crystals, where the geometrical configurational characteristics are described by the so-called configuration tensor $\mathbf{c}$ [16]– [18], defined as $\mathbf{c} =< \mathbf{RR} >$, where $\mathbf{R}$ is the end to end vector (in polymers) or the orientational vector (in nematic liquid crystals or in rigid dumbbell molecular models). The corresponding expression for the configurational entropy used in polymer physics is [16]– [18]:

$$s(\mathbf{c}) = k_B \left[\frac{1}{2} Tr(\mathbf{U} - \mathbf{c}) + \ln det(\mathbf{c})\right]. \tag{17}$$

In the situation considered in this paper, we would have, in view of the definition (16) of $\mathbf{\Pi}^s$:

$$s_{anisotr}(\mathbf{\Pi}'^s) = k_B \left[\frac{1}{2} Tr\mathbf{\Pi}'^s + \ln det(\mathbf{U} - \mathbf{\Pi}'^s)\right], \tag{18}$$

where we use $\mathbf{\Pi}'^s = (2/3)\mathbf{\Pi}^s =< \mathbf{U} - \mathbf{s}'\mathbf{s}' >$. Indeed, $< \mathbf{s}'\mathbf{s}' >$ plays an analogous role to that of $\mathbf{c}$ in polymer physics. Introducing here the form of $\mathbf{\Pi}'^s$ we would obtain an explicit expression of the configurational geometrical entropy $s_{anisotr}(\mathbf{\Pi}^s)$.

To give an explicit example we may consider the vortex tangles under the simultaneous presence of counterflow $\mathbf{V}_{ns}$ and rotation $\mathbf{\Omega}$, assumed parallel to each other [19]. In this case, in Ref. [13] for the vortex orientational distribution was proposed the Langevin expression:

$$Pr(\mathbf{s}') \sim \exp\left[-x\widehat{\mathbf{\Omega}} \cdot \mathbf{s}'\right] = \exp\left[-a\cos\theta\right], \tag{19}$$

where θ is the angle between $\mathbf{s}'$ and $\mathbf{\Omega}$. The distribution (19) reflect the competition between the orienting effects of $\mathbf{\Omega}$ and the randomizing effects of $\mathbf{V}_{ns}$, respectively analogous to the orienting effects of a magnetic field H on magnetic dipoles μ and the randomizing effects of temperature, in such a way that in a magnetic systems x would be $x = \frac{\mu H}{kT}$. The value of x was found to be $x = 11\frac{L_R}{L_H}$, where L_R would be the line density if we only had rotation and L_H would be the line density if we only had counterflow.

In this case, by assuming that the vortex tangle is isotropic in the plane orthogonal to the rotation first axis, we have [13]:

$$\mathbf{\Pi}^s = \begin{pmatrix} 1-b & 0 & 0 \\ 0 & 1+\frac{b}{2} & 0 \\ 0 & 0 & 1+\frac{b}{2} \end{pmatrix}, \qquad \mathbf{\Pi}^a = \frac{3}{2}\frac{\alpha'}{\alpha} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix} \tag{20}$$

where coefficients b and c are linked to moments of the tangent vector $\mathbf{s}'$ by:

$$< s'_x >= c = \coth x - \frac{1}{x}, \tag{21}$$

$$< s'^{\,2}_x >= \frac{1+2b}{3} = 1 + \frac{2}{x}\left[\frac{1}{x} - \coth x\right], \tag{22}$$

With respect to the form of the contribution of the polarity to the entropy it may be obtained implicitly. Note in fact that equation (21) relates x with the average polarity $< \cos\theta >$. The entropy related to (19) turns out to be [13], in terms of x,

$$S_{polarity}(\Omega, V_{ns}) = k_B \left[1 + \ln 4\pi - x\coth x + \ln\left(\frac{1}{x}\sinh x\right)\right]. \tag{23}$$

Combination of the two above equations would allow one to express the polarity contribution to the entropy in terms of the polarity itself.

4. The fractal model.

In pure counterflow, an interesting case to study is when the temperature T_0 of the helium background is so small ($T_0 < 1$K) that the normal fluid fraction of helium II is negligible. Hence viscous dissipation and mutual friction are absent. In this low temperature range, superfluid turbulence takes its purest form: a tangle of reconnecting vortex filaments which move under the velocity field induced by the presence of the other vortices. In this case the tangle can be supposed isotropic, so that $\mathbf{\Pi}^s = \mathbf{U}$ and $\mathbf{\Pi}^a = 0$.

The vortex breaking and reconnections are the prevalent phenomena; they randomize the vortex tangle and cause the formation of helical Kelvin waves [20]– [21]. A process of generation of smaller and smaller scales proceeds, without loss of kinetic energy, to the smallest scales in which the kinetic energy of the highly curved and cusped fragments of the vortices is radiated away as sound. If the above processes act in a self-similar way on several orders of spatial lengths, one expects the vortex tangle to exhibit fractal features. In this case, we have led to a probability distribution function of the length of loops $n(l)$ which obeys a power law of the form

$$n(l) = c\frac{l^{-a}}{(l_{min})^{4-a}}, \tag{24}$$

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where c is a positive dimensionless constant, a an exponent related to the fractal dimension of the vortex tangle, and l_{min} the minimum length of the vortex loops. The values of N and L may be immediately obtained as

$$N = \int_{l_{min}}^{\infty} c \frac{l^{-a}}{(l_{min})^{4-a}} dl = \frac{c}{a-1} (l_{min})^{-3}, \tag{25}$$

$$L = \int_{l_{min}}^{\infty} \frac{l^{-a+1}}{(l_{min})^{4-a}} dl = \frac{c}{a-2} (l_{min})^{-2}. \tag{26}$$

Putting $\lambda = \frac{c^{1/2}(a-1)}{(a-2)^{3/2}}$, one has also:

$$< l >= \frac{L}{N} = \frac{a-1}{a-2} l_{min} = \lambda L^{-1/2}, \qquad N = \frac{1}{\lambda} L^{3/2}. \tag{27}$$

Result $< l >= \lambda L^{-1/2} \simeq \lambda \delta$ is in accord with some microscopic analyses of vortex tangles which strongly suggest that the average length of the vortex loops is of the order of the intervortex spacing.

The corresponding behavior of the entropy is

$$\frac{S}{k_B} = -\int_{l_{min}}^{\infty} n(l) \log n(l) dl = ac \left[1 - \frac{a-1}{a} \log(l_{min}^{-4} c) \right] \tag{28}$$

and in terms of L and λ

$$\frac{S}{k_B} = \frac{1}{\lambda} L^{3/2} \left[\frac{a}{a-1} + \log \frac{(a-2)\lambda^2 L^{-2}}{(a-1)^2} \right]. \tag{29}$$

As one sees, the leading term in this expression is in $L^{3/2}$.

The temperature of the vortex tangle is obtained as the average energy of the vortex loops:

$$k_B T =< U_l >= \frac{1}{N} \int_{l_{min}}^{\infty} n(l) U_l(l) dl \tag{30}$$

We assume that U_l is proportional to a power of l

$$U_l = d \left(\frac{l}{l_{min}} \right)^{\chi} \epsilon_V l, \tag{31}$$

with the exponent χ linked to the fractal dimension of the tangle, and d a dimensionless coefficient. It is, for $a > 2 + \chi$:

$$k_B T = \frac{(a-1)d}{a-(2+\chi)} \epsilon_V l_{min} = d \frac{(a-2)\lambda}{a-(2+\chi)} \epsilon_V L^{-1/2}. \tag{32}$$

The energy per unit volume is

$$U = \frac{cd}{a-(2+\chi)}\epsilon_V l_{min}^{-2} = \frac{(a-2)d\lambda}{a-(2+\chi)}\epsilon_V L. \tag{33}$$

A potential distribution function $n(l) \propto l^{-a}$ for the length of the loops was proposed by Nemirovskii [22], which found $a = 5/2$ as a stationary solution of a master equation describing the evolution of the length distribution function.

The main results of the previous analysis are $< l > \propto L^{-1/2}$, $N \propto L^{3/2}$, $T \propto L^{-1/2}$ and S with a leading term in $L^{3/2}$, or, in terms of the tangle temperature:

$$< l > \propto T \qquad N \propto T^{-3} \qquad U \propto T^{-2} \tag{34}$$

and S with a leading term in T^{-3}.

Note that, owing to the relation $U \propto T^{-2}$, the decay of the vortex tangle leads to a decrease in L and, therefore, an increase in $< l >$. Thus, when the system loses energy its temperature increases. Furthermore, this behavior implies that the heat capacity of the vortex tangle is negative: $\partial U/\partial T \propto -T^{-3} < 0$.

These features share some analogies with the thermodynamics of black holes [23], for which it is known that $U \propto T^{-1}$ and $\mathcal{S} \propto T^{-2}$. Since the entropy increases faster that energy, the heat capacity of the black holes is negative. Thus, fractal vortex tangles and black holes are thermodynamically unstable by themselves, and they must be accompanied by sufficient electromagnetic radiation (Hawking radiation) or by a superfluid in the presence of a heat flux.

5. Conclusions

We have proposed a thermodynamic analysis of superfluid vortex tangles by focusing our attention on the explicit form of several different contributions to the entropy. We have examined two kinds of contributions: on the one side, the energetic and entropic contributions of L, which has led us to introduce the vortex temperature, in terms of the average energy of the closed vortex loops constituting the tangle, and on the other side, the contributions arising from the geometrical details of the vortex lines which describe the entropy associated to the anisotropy and the polarization of the tangle. Our approach may be taken as a first exploration, with the aim of clarifying the origin of the different contributions to the entropy of the tangle. The model borrows ideas and methods of classical mechanics for gases, polymers and magnetic systems, related, respectively to the entropy

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of the length distribution of loops, of the orientation and curvature of loops, and of the orientation of vortex lines in the presence of external rotations. Future analyses will lead to more elaborate expressions for these contributions, based on a deeper understanding of the required nonequilibrium distribution functions.

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