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A defense strategy by edge ideals

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Abstract

We study a defense strategy for a region from enemy attacks by using tools from the graph theory together with commutative algebra methods.

Keywords: `Graphs, unmixed graphs`

1. Introduction.

In this paper we analyze the following problem:

Let R be a region to be defended by enemy attacks. We suppose that the region can be reached by n independent foreign sites $\{y_1, \dots, y_n\}$ and that n are the inner sites of the region R to be defended, $\{x_1, \dots, x_n\}$.

We also assume that there exist some passes from y_i to x_j, with $i, j \in \{1, \dots, n\}$, and moreover that some "strategical" inner ways between the $x_i's$ could exist.

In order to watch the security of the region R a sentry or a spy can station himself in x_i or in y_i to control the passes and the strategical inner ways.

We are interested to study the shape of G when exactly n persons are necessary and sufficient to control the passes and the inner ways. The solution of this problem is given by some classes of unmixed graphs G with $2n$ vertices and whose edge ideals $I(G)$ have height equals to n. Such classes of graphs were studied in [3]. Recall that by the edge ideal $I(G)$ of a graph G with $2n$ vertices we mean a squarefree monomial ideal of a polynomial ring in $2n$ variables over a field, equigenerated in degree 2.

In section 1, we recall some notions that we will use in the article.

In section 2, we discuss the classes of graphs that allow us to reach a solution of our problem.

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In section 3, we analyze a property that is satisfied by the classes of graphs solution of our problem.

2. Preliminaries and notations.

In this section we recall some concepts and notations on graphs that we will use in the article.

Let G be a simple graph with vertex set $V(G) = \{x_1 \dots, x_n\}$ and edge set $E(G)$. Let K be a field and let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables, the edge ideal $I(G)$, associated to G, is the ideal of S generated by the set of all squarefree monomials $x_i x_j$ so that x_i is adjacent to x_j.

A subset $C \subset V(G)$ is a minimal vertex cover of G if:

(1) every edge of G is incident with one vertex in C, and
(2) there is no proper subset of C with the first property.

Observe that C is a minimal vertex cover of G if and only if $V(G) \setminus C$ is a maximal independent set that is no two vertices of $V(G) \setminus C$ are adjacent.

For every graph G, define $\alpha_0(G)$ as the smallest number of vertices in any minimal vertex cover and $\beta_1(G)$ as the maximum number of independent edges in G. A set of edges in a graph G is independent if no two of them have a vertex in common.

It is easy to observe that height of $I(G)$ is equal to $\alpha_0(G)$.

A graph G is called *unmixed* if all the minimal vertex covers of G have the same number of elements. In this case the vertex covering number $\alpha_0(G)$ is equal to the height of the ideal $I(G)$.

For more details on unmixed graphs see [14].

3. Two examples.

In this section we discuss the classes of graphs that can give a solution to our problem. The idea of examining problems related to the security of a region by using algebraic tools is also contained in ([10], [12]).

The situation described in our problem can be represented by a graph G with vertex set $V(G) = \{x_1, \dots, x_n, y_1, \dots, y_n\}$ and a solution can be obtained under the following hypotheses:

1. G is unmixed;
2. $X = \{x_1, \dots, x_n\}$ is a minimal vertex cover of G.

In this case it is:

$$2\text{height} I(G) = |V(G)|.$$

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There are two nice classes of graphs that satisfy our hypotheses:

(a) the unmixed bipartite graphs;
(b) the grafted graphs.

Now we briefly discuss such important classes of graphs.

First of all recall that a graph G is bipartite if its vertex set $V(G)$ can be partitioned into disjoint subsets V_1 and V_2 such that every edge of G joints V_1 with V_2.

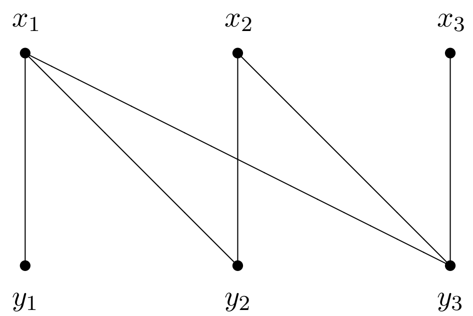
The class of unmixed bipartite graphs has been analyzed by J. Herzog, T. Hibi and R. Villarreal ([9], [15]), that have recently given the following combinatorial characterization:

Theorem 3.1. *Let G be a bipartite graph without isolated vertices. Then G is unmixed if there is a bipartition $V_1 = \{x_1, \dots, x_n\}$, $V_2 = \{y_1, \dots, y_n\}$ of G such that:*

(i) *$x_i y_i \in E(G)$ for all i, and*
(ii) *if $x_i y_j, x_j y_k \in E(G)$ and i, j, k are distinct, then $x_i y_k \in E(G)$.*

For a bipartition (V_1, V_2) of a graph G we mean that $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and every edge of G joins V_1 with V_2.

Example 3.2. The following graph is an unmixed bipartite graph.



Its minimal vertex covers are $\{x_1, x_2, x_3\}$, $\{x_1, x_2, y_3\}$, $\{x_1, y_2, y_3\}$, $\{y_1, y_2, y_3\}$.

The class of grafted graphs was studied by S.Faridi in ([5]).

In order to introduce the notion of grafted graph, we need the notion of *grafting* of a graph ([5]) or that is the same the notion of *suspension* of a graph in the sense of ([14]).

More precisely, let G be a graph on the vertex set $V(G) = \{x_1, \ldots, x_n\}$ and choose a new set of vertices $y_1, \ldots, y_n$, we define the grafting of G to be the graph obtained by attaching to each vertex x_i the new vertex y_i and the edge $x_i y_i$.

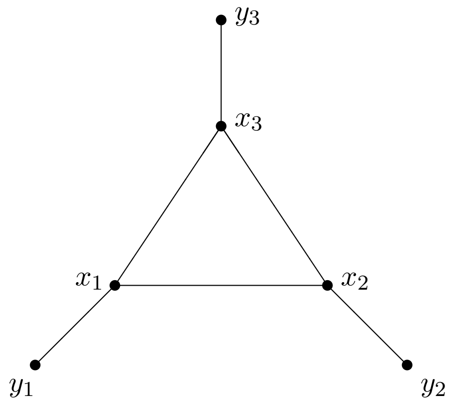
The new graph G' is called *grafted*. Observe that

$$V(G') = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}, \;\; E(G') = E(G) \cup \{x_1y_1, \ldots, x_ny_n\}$$

and

$$E(G) \cap \{x_1y_1, \ldots, x_ny_n\} = \emptyset, \quad \{x_i, y_i\} \cap \{x_j, y_j\} = \emptyset, \; \forall i \neq j.$$

Example 3.3. The following graph is a grafted graph.



It is an unmixed graph since its minimal vertex covers are $\{x_1, x_2, x_3\}$, $\{x_1, x_2, y_3\}$, $\{x_1, x_3, y_2\}$, $\{x_2, x_3, y_1\}$.

The next result holds ([5], Theorem 7.6):

Theorem 3.4. *Every grafted graph is unmixed.*

4. Perfect matching.

The unmixed bipartite graphs and the grafted graphs described in Section 3 have a property in common i.e. they have *a perfect matching.*

A pairing off of all vertices of a graph G is called a *perfect matching.*

Then a graph G has a perfect matching if and only if G has an even number of vertices and there is a set of independent lines "covering" (containing) all the vertices.

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A very nice theorem that characterizes perfect matching due to P. Hall ([8]) is the following:

Theorem 4.1. *If G is a bipartite graph with vertex set V, then the following are equivalent:*

(a) *G has a perfect matching;*
(b) *$|A| \leq |N(A)| = |\{v \in V(G) \,:\, v$ is adjacent to some vertex in $A\}|$ for all $A \subset V$ independent set of vertices.*

The next result assures the existence of a perfect matching for the classes of graphs solution of our problem. It could be obtained from ([7], Remark 2.2), but we can give a direct and alternative proof of it.

Proposition 4.2. *Let G be an unmixed graph with $2n$ not isolated vertices and let $\alpha_0(G) = n$.*

Then G has a perfect matching.

Proof. Since height $I(G) = n$, we label the vertices of G in the following way:
set $V(G) = X \cup Y$, $X \cap Y = \emptyset$ where $X = \{x_1, \ldots, x_n\}$ is a minimal vertex cover of G and $Y = \{y_1, \ldots, y_n\}$ is a maximal independent set of $V(G)$.

Suppose G does not have a perfect matching. First of all observe that for every $x_i \in X$ there exists a vertex $y_j \in Y$ such that $x_i y_j \in E(G)$. In fact, if we have $x_i y_j \notin E(G)$ for every $y_j \in Y$, then $X \setminus \{x_i\}$ is a vertex cover of G. A contradiction. We define the bipartite graph $B \subset G$ by

$$B = (X \cup Y, \{xy \in E(G) : x \in X,\ y \in Y\}).$$

Then B has no isolated vertex. Since G has no perfect matching, B has no perfect matching, either. Therefore by Theorem 4.1, it follows that there exists $A \subset Y$ such that $|A| > |N(A)|$ where

$$N(A) = \{v \in V(B) \,:\, v \text{ is adjacent to some vertex in } A\}.$$

Set $A = \{y_{i_1}, \ldots, y_{i_r}\}$ and $N(A) = \{x_{j_1}, \ldots, x_{j_s}\}$ with $r > s$. Put $y := y_{i_1}$. We may assume that A is minimal with respect to the inclusion among all the subsets $\tilde{A}$ of Y with $|\tilde{A}| > |N(\tilde{A})|$. Therefore for every $A' \subsetneq A$, we have $|A'| \leq |N(A')|$. In this case we have $r = s + 1$. In fact, we have $r > s$ and $r - 1 = |(A \setminus \{y\})| \leq |N(A \setminus \{y\})| \leq s$.

Now consider the restricted bipartite graph $B_0 = B|_{A \cup N(A)}$ with the partition A and $N(A)$. We prove that if C_0 is a vertex cover of B_0 with $y \in C_0$, then $|C_0| > s$. We observe that by the "minimality" of A, for each subset of A of cardinality s, there exists a perfect matching $\{x_{j_1} y_{k_1}, \ldots, x_{j_s} y_{k_s}\}$.

There exists a perfect matching $\{x_{j_1}y_{k_1}, \dots, x_{j_s}y_{k_s}\}$ in $B_0 \setminus \{y\}$ by the assumption and Theorem 4.1. Since C_0 is a vertex cover of B_0 and $\{x_{j_1}y_{k_1}, \dots, x_{j_s}y_{k_s}\} \subset E(B_0)$, either x_{j_ℓ} or y_{k_ℓ} is in C_0 for $\ell = 1, \dots, s$. Hence the assertion follows.

Now let C be a minimal vertex cover of the graph G with $y \in C$. Then $C_0 = C \cap V(B_0)$ is a vertex cover of B_0. Hence $|C_0| > s = |N(A)|$. It is easy to check that

$$C' = C \setminus C_0 \cup N(A)$$

is also a vertex cover of G. Since $|C'| < |C|$, the unmixed graph G is not unmixed, which is a contradiction. $\square$

Remark 4.3. The hypothesis that G is unmixed in Proposition 4.2 is essential. Consider the graph G with vertex set $V(G) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and edge set $E(G) = \{x_1y_1, x_1y_2, x_2y_3, x_3y_3, x_2x_3\}$. Then $I(G)$ is not unmixed ideal with height 3. In fact, $\{x_1, x_2, x_3\}$ and $\{x_3, y_1, y_2, y_3\}$ are minimal vertex covers of G. But G has no perfect matching.

From Proposition 4.2 an unmixed graph G with $2n$ vertices, which are not isolated, and with height $I(G) = n$, has a perfect matching. Hence we may assume that

(*) $V(G) = X \cup Y$, $X \cap Y = \emptyset$, where $X = \{x_1, \dots, x_n\}$ is a minimal vertex cover of G and $Y = \{y_1, \dots, y_n\}$ is a maximal independent set of G such that $\{x_1y_1, \dots, x_ny_n\} \subset E(G)$.

The characterization of the unmixed graph satisfying our hypotheses is the following ([3], Proposition 2.3):

Proposition 4.4. *Let G be a graph with $2n$ vertices, which are not isolated, and with height $I(G) = n$. We assume the condition (*). Then G is unmixed if and only if the following conditions hold:*

(i) If z_ix_j, $y_jx_k \in E(G)$ then $z_ix_k \in E(G)$ for distinct i, j and k and for $z_i \in \{x_i, y_i\}$.
(ii) If $x_iy_j \in E(G)$ then $x_ix_j \notin E(G)$.

An example of an unmixed graph of the type above is the *B-grafted graph* introduced in [3] that generalizes the notion of grafted graph.

Such graph is defined as follows:
let H_0 be a graph with the labeled vertices $\{1, \dots, p\}$.

For every $i = 1, \dots, p$, let B_i be a bipartite graph with labeled partition X_i and Y_i such that $|X_i| = |Y_i| = n_i$.

We construct a new graph

$$G = G(H_0; B_1, \dots, B_p)$$

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with vertex set

$$V(G) := X_1 \cup \ldots \cup X_p \cup Y_1 \cup \ldots \cup Y_p$$

and with edge set $E(G)$ defined as follows:

$$xy \in E(G) \text{ if and only if}$$

either

$$\text{there exist } i,\ j \text{ such that } x \in X_i,\ y \in X_j,\ \{i,j\} \in E(H_0)$$

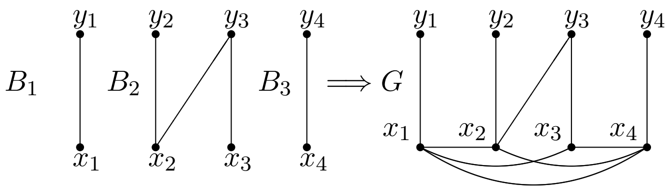
or

$$\text{there exists } i \text{ such that } x \in X_i,\ y \in Y_i,\ xy \in E(B_i).$$

We call such a graph G the *B-grafted graph.* It is $|V(G)| = 2(\sum_{i=1}^{p} n_i)$.

The B-grafted graph $G = G(H_0; B_1, \ldots, B_p)$ is unmixed if and only if every B_i is unmixed for $i = 1, \ldots, p$. ([3], Theorem 5.3)

For example if H_0 is a triangle, by the following bipartite graphs B_1, B_2, B_3, we obtain the B-grafted graph G:



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