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Second Gradient Model of Consolidation: a One-Dimensional Variational Approach

Angela Madeo[1]

[1] *Dipartimento Me.Mo.Mat, Università degli Studi di Roma "La Sapienza", Italy.*
angela.madeo@uniroma1.it

Abstract

Governing equations for a one-dimensional, second gradient porous system with a solid-material discontinuity are deduced using an extended Hamilton-Rayleigh variational principle. Darcy-Birnkman dissipation is included in this model, while inertial effects are neglected. The linearized problem is used to study the problem of the consolidation of a soil layer. First and second gradient numerical solutions are compared and discussed.

Keywords: `Terzaghi Consolidation, Porous Media, Second Gradient Theory, Hamilton-Rayleigh Variational Principle`

1. Introduction.

In this work we study the mechanics of one-dimensional fluid-filled porous media in the framework of second gradient theories by using a variational approach. In order to do so, we first introduce a kinematical scheme suitable for describing the motion of a one-dimensional, fluid-filled porous medium with a solid-material discontinuity (see also [2], [3]). As a second step, we introduce a Hamilton-Rayleigh variational principle, extended to continuous systems, in which we take into account Darcy-Brinkman dissipation due to viscous flow of the fluid inside the pores of the solid matrix, but we neglect inertial effects. We also assume that the deformation energy of the porous system depends not only on the basic kinematical fields, but also on their space derivatives (second gradient theory). As a result, we obtain governing equations for the porous medium and jump conditions on the considered discontinuity. We particularize these equations and jump conditions to the limit case of a porous medium surrounded by a pure fluid.

We finally apply the obtained differential system to the problem of the consolidation of a soil layer originally considered by Terzaghi. We find some numerical solutions for this problem and compare first and second gradient solutions underlying that second gradient ones are appropriate to describe boundary layer effects in the vicinity of the boundaries of the layer.

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2. Kinematics

In this section we set up a kinematical framework suitable for describing the one-dimensional motion of a porous medium with a solid-material surface discontinuity. Let B_s and B_f be two open subsets of $\mathbb{R}$ (usually referred to as the Lagrangian configurations of the two constituents), $(0,T)$ be a time interval and let

$$\chi_s : B_s \times (0,T) \mapsto \mathbb{R}, \qquad \chi_f : B_f \times (0,T) \mapsto \mathbb{R}$$

be the maps which represent the placement of the solid and fluid constituents respectively. Moreover, let $\phi_s : B_s \times (0,T) \mapsto B_f$ be the map which locates, for any instant $t \in (0,T)$, the fluid material particle X_f which is in contact with the solid material particle X_s. The three introduced maps are related by $\phi_s = \chi_f^{-1} \circ \chi_s$. We also assume that χ_s and ϕ_s are continuous in B_s, while their space derivatives χ_s' and $\phi_s'$are continuous a.e. in B_s except in a point S_s which is assumed to be fixed in the solid-Lagrangian configuration[a]. This means that, correspondingly to the point S_s, the fields χ_s' and ϕ_s' suffer a jump. From now on we also assume that $\chi_s(B_s,t) = \chi_f(B_f,t)$ and we denote $B_e(t)$ this time-varying sub-domain of $\mathbb{R}$ usually referred to as Eulerian configuration of the porous system. This means that we are adopting a macroscopic kinematical model since each point x of the physical space is assumed to be simultaneously occupied both by a solid and a fluid material particle.

As far as the solid constituent is concerned, we introduce on B_s the displacement field u and the Green-Lagrange deformation field ε, which in the considered one-dimensional case are given by

$$u(X_s,t) := \chi_s(X_s,t) - X_s, \quad \text{and} \quad \varepsilon := \frac{1}{2}\left((\chi_s')^2 - 1\right). \tag{2}$$

In the sequel, owing to Eq. (2), we replace χ_s' with $1 + u'$ since we want to deal only with physical quantities.

Notation 1. Given two fields $a(x,t)$ and $b(X_f,t)$ defined on $B_e(t)$ and B_f respectively, we denote by $a^{Ⓢ}(X_s,t)$ and $b^{Ⓢ}(X_s,t)$ the corresponding fields transported on B_s as: $a^{Ⓢ} = a \circ \chi_s$ and $b^{Ⓢ} = b \circ \phi_s$.

Let now η_s and η_f be the solid and fluid Lagrangian apparent densities, defined on B_s and B_f respectively. We introduce the solid-Lagrangian fluid density as $m_f := \eta_f^{Ⓢ}\, \phi_s'$ and in the sequel we assume that η_f is constant

[a] From now on the apex $'$ always indicates the derivative with respect to the variable X_s.

in space. We also assume that no creation (or dissolution) of mass occurs during the motion of the porous system so that $\partial \eta_s / \partial t = 0$ and $\partial \eta_f / \partial t = 0$. Following classical poromechanics (see e.g. [1], [2], [3] for details) the fluid conservation equation is pulled-back on the solid-Lagrangian configuration B_s and in the one-dimensional case reads

$$\dot{m}_f - \left(\eta_f^{\circledS} \, \dot{\phi}_s \right)' = 0, \tag{3}$$

where $\dot{m}_f = \partial m_f / \partial t$ and $\dot{\phi}_s = \partial \phi_s / \partial t$. In the sequel a superposed dot always indicates the time derivative of a solid-Lagrangian field.

3. Extended Hamilton-Rayleigh Variational Principle

We now assume that the volume deformation energy density $\Psi(\varepsilon, m_f, \varepsilon', m_f')$ depends not only on the deformation of the solid matrix and on the density of the fluid (classical theory), but also on the space derivatives of these fields (second gradient theory). Assuming that inertial effects are negligible, we can define the action functional $\mathcal{A}$ for the one-dimensional porous system as

$$\mathcal{A} := - \int_{B_s \times (0,T)} \Psi \left(\varepsilon, m_f, \varepsilon', m_f' \right). \tag{4}$$

Let $\kappa := K \eta_f^{\circledS} / m_f (1 + u') \dot{\phi}_s$ and $\pi := B / (1 + u') (\eta_f^{\circledS} / m_f (1 + u') \dot{\phi}_s)'$ be the only non-vanishing components of Darcy and Brinkman friction forces (see [2] for their extended formulation); the coefficients K and B are known as Darcy and Brinkman coefficients. We introduce a Rayleigh-like functional $\mathcal{R}$ accounting for Darcy-Brinkman dissipation as

$$2\mathcal{R} := \int_{B_s} \left[\frac{\eta_f^{\circledS}}{m_f} (1 + u') \dot{\phi}_s \kappa + \frac{1}{(1 + u')} \left(\frac{\eta_f^{\circledS}}{m_f} (1 + u') \dot{\phi}_s \right)' \pi \right] (1 + u'). \tag{5}$$

The Hamilton-Rayleigh variational principle can then be formulated as

$$\delta \mathcal{A} := \frac{\partial \mathcal{A}}{\partial \chi_s} \delta \chi_s + \frac{\partial \mathcal{A}}{\partial \phi_s} \delta \phi_s + \frac{\partial \mathcal{A}}{\partial \chi_s'} \delta \chi_s' + \frac{\partial \mathcal{A}}{\partial \phi_s'} \delta \phi_s' = \int_0^T \frac{\partial \mathcal{R}}{\partial \dot{\phi}_s} \delta \phi_s + \frac{\partial \mathcal{R}}{\partial \dot{\phi}_s'} \delta \phi_s' =: \delta \mathcal{R}, \tag{6}$$

where all the differentiation operations appearing must be understood in the sense of Frechet derivatives.

Recalling the definition of Ψ it is easy to recover that the variation $\delta \mathcal{A}$ of the action functional can be evaluated as

$$\delta \mathcal{A} = - \int_{B_s \times (0,T)} \frac{\partial \Psi}{\partial \varepsilon} \delta \varepsilon + \frac{\partial \Psi}{\partial m_f} \delta m_f + \frac{\partial \Psi}{\partial \varepsilon'} \delta \varepsilon' + \frac{\partial \Psi}{\partial m_f'} \delta m_f',$$

where, recalling the definitions of ε and m_f, it is clear that $\delta\varepsilon = \chi_s' \delta\chi_s'$, $\delta m_f = \eta_f^{ⓢ} \delta\phi_s'$, $\delta\varepsilon' = (\delta\varepsilon)' = \chi_s'' \delta\chi_s' + \chi_s' \delta\chi_s''$ and $\delta m_f' = (\delta m_f)' = \eta_f^{ⓢ} \delta\phi_s''$.

We now assume that the test functions $\delta\chi_s$, $\delta\phi_s$, $\delta\chi_s'$ and $\delta\phi_s'$ have compact support $K \subset B_s$ such that the point S_s belongs to K. This means that, for any compact set $K \subset B_s$ these functions vanish on the boundary and outside K. Integrating by parts in space, recalling that η_f is constant and simplifying, the variation of the action functional gives

$$\begin{aligned}
\delta\mathcal{A} = &\int_{B_s \cap K \times (0,T)} \left\{ \left[(1+u') \left(\frac{\partial \Psi}{\partial \varepsilon} - \left(\frac{\partial \Psi}{\partial \varepsilon'} \right)' \right) \right]' \delta\chi_s + \eta_f^{ⓢ} \left[\frac{\partial \Psi}{\partial m_f} - \left(\frac{\partial \Psi}{\partial m_f'} \right)' \right]' \delta\phi_s \right\} \\
&- \left[\left| (1+u') \left(\frac{\partial \Psi}{\partial \varepsilon} - \left(\frac{\partial \Psi}{\partial \varepsilon'} \right)' \right) \right|\right]_{S_s \times (0,T)} \delta\chi_s - \left[\left| \eta_f^{ⓢ} \frac{\partial \Psi}{\partial m_f} - \left(\frac{\partial \Psi}{\partial m_f'} \right)' \right|\right]_{S_s \times (0,T)} \delta\phi_s \\
&- \left[\left| (1+u') \left(\frac{\partial \Psi}{\partial \varepsilon'} \right)' \right|\right]_{S_s \times (0,T)} \delta\chi_s' - \left[\left| \eta_f^{ⓢ} \frac{\partial \Psi}{\partial m_f'} \right|\right]_{S_s \times (0,T)} \delta\phi_s' \,.
\end{aligned} \tag{7}$$

Here and in the sequel the symbol $[|f|]$ indicates the jump of the quantity f through the discontinuity S_s.

As for the variation of the Rayleigh dissipation functional it can be recovered starting from expression (5) that its variation $\delta\mathcal{R}$ gives

$$\begin{aligned}
\delta\mathcal{R} = &\int_{B_s \cap K \times (0,T)} \left[\frac{\eta_f^{ⓢ}}{m_f} (1+u')\, \kappa + \frac{1}{(1+u')} \left(\frac{\eta_f^{ⓢ}}{m_f} (1+u') \right)' \pi \right] (1+u')\, \delta\phi_s + \\
&+ \int_{B_s \cap K \times (0,T)} \frac{\eta_f^{ⓢ}}{m_f} (1+u')\, \pi\, \delta\phi_s',
\end{aligned}$$

which integrating by parts and simplifying reduces to

$$\delta\mathcal{R} = \int_{B_s \cap K \times (0,T)} (1+u') \left[\frac{\eta_f^{ⓢ}}{m_f} (1+u')\, \kappa - \frac{\eta_f^{ⓢ}}{m_f} \pi' \right] \delta\phi_s + \left[\left| (1+u') \frac{\eta_f^{ⓢ}}{m_f} \pi \right|\right]_{S_s \times (0,T)} \delta\phi_s. \tag{8}$$

Replacing in the Hamilton-Rayleigh principle (6) expressions (7) and (8) for $\delta\mathcal{A}$ and $\delta\mathcal{R}$ respectively, noticing that these expressions hold for any compact set K included in B_s and assuming the arbitrariness of the test functions $\delta\chi_s$, $\delta\phi_s$, $\delta\chi_s'$, and $\delta\phi_s'$ we obtain the following set of equations of motions and jump conditions valid in $B_s \times (0,T)$ and in $S_s \times (0,T)$ respectively

$$(9) \qquad \left[(1+u')\left(\frac{\partial \Psi}{\partial \varepsilon} - \left(\frac{\partial \Psi}{\partial \varepsilon'}\right)'\right)\right]' = 0,$$

$$(10) \qquad \eta_f^{ⓢ}\left[\frac{\partial \Psi}{\partial m_f} - \left(\frac{\partial \Psi}{\partial m_f'}\right)'\right]' = (1+u')\left[\frac{\eta_f^{ⓢ}}{m_f}(1+u')\,\kappa - \frac{\eta_f^{ⓢ}}{m_f}\pi'\right],$$

$$(11) \qquad \left[\left|\,(1+u')\left(\frac{\partial \Psi}{\partial \varepsilon} - \left(\frac{\partial \Psi}{\partial \varepsilon'}\right)'\right)\right|\right]_{S_s\times(0,T)} = 0$$

$$(12) \qquad \left[\left|\,\eta_f^{ⓢ}\frac{\partial \Psi}{\partial m_f} - \left(\frac{\partial \Psi}{\partial m_f'}\right)'\right|\right]_{S_s\times(0,T)} = -\left[\left|\,(1+u')\frac{\eta_f^{ⓢ}}{m_f}\pi\,\right|\right]_{S_s\times(0,T)},$$

$$(13) \qquad \left[\left|\,(1+u')\left(\frac{\partial \Psi}{\partial \varepsilon'}\right)'\right|\right]_{S_s\times(0,T)} = 0, \qquad \left[\left|\,\eta_f^{ⓢ}\frac{\partial \Psi}{\partial m_f'}\right|\right]_{S_s\times(0,T)} = 0.$$

4. Porous Medium-Pure Fluid Interface

When the discontinuity S_s is such that it separates a porous medium and a pure fluid, the differential system (9)-(13) must be rewritten for this particular case. In order to do so, we denote by Ψ^+ the potential energy of the porous medium, by Ψ^- the potential energy of the pure fluid and we notice that in the pure fluid region $\kappa = \pi = 0$. Since Ψ^- is the solid-Lagrangian free energy of a pure fluid, it must be such that

$$(14) \qquad \Psi^-(\varepsilon, m_f) = (1+u')\,\psi_f(\rho_f^{ⓢ}) = \sqrt{1+2\varepsilon}\;\psi_f\left(\frac{m_f}{\sqrt{1+2\varepsilon}}\right).$$

This expression for Ψ^- is simply the pull-back of the energy ψ_f of a pure fluid which, by definition, must depend only on the Eulerian fluid density $\rho_f^{ⓢ} := m_f(1+u')^{-1}$. Notice that the energy of a pure fluid does not depend on second gradient fields so that $\partial\Psi^-/\partial\varepsilon' = \partial\Psi^-/\partial m_f' = 0$.

We now introduce the pressure p_f and the chemical potential μ_f of a pure fluid as $p_f := -\psi_f + \rho_f^{ⓢ}\,\partial\psi_f/\partial\rho_f^{ⓢ}$ and $\mu_f := \partial\psi_f/\partial\rho_f^{ⓢ}$. Deriving with respect to X_s it is easy to recognize that $p_f' := \rho_f^{ⓢ}\,\mu_f'$.
Starting from expression (14) for Ψ^- and using these definitions for p_f and μ_f it can be recovered that $\partial\Psi^-/\partial\varepsilon = -1/(1+u')\,p_f$ and $\partial\Psi^-/\partial m_f = \mu_f$. We can conclude that in the pure fluid region the equations of motion (9) and (10) simplify into $p_f' = 0$ and $\mu_f' = 0$ respectively. These two

equations are clearly equivalent in virtue of the relationship between p'_f and μ'_f. On the other hand, the motion of the porous region is still ruled by equations (9) and (10) simply setting $\Psi = \Psi^+$, while the jump conditions (11)-(13)become

(15)
$$(1+u')\left(\frac{\partial\Psi^+}{\partial\varepsilon}-\left(\frac{\partial\Psi^+}{\partial\varepsilon'}\right)'\right)=-p_f,\quad (\eta_f^{ⓢ})^+\,\frac{\partial\Psi^+}{\partial m_f}-\left(\frac{\partial\Psi^+}{\partial m'_f}\right)'=(\eta_f^{ⓢ})^-\,\mu_f,$$

(16)
$$(1+u')\left(\frac{\partial\Psi^+}{\partial\varepsilon'}\right)'=0,\quad \eta_f^{ⓢ}\,\frac{\partial\Psi^+}{\partial m'_f}=0,$$

where clearly all the quantities on the left hand sides are relative to the porous medium and those on the right hand side are relative to the pure fluid. Nevertheless, in order to lighten notation + and − superscripts are omitted if no confusion can arise.

5. Linearization of the differential system

We now consider the problem of the consolidation of a soil layer originally studied by Terzaghi. In particular, we consider a semi-infinite soil layer of initial depth L and we assume that both deformation of the solid matrix and fluid flow happen only in the vertical direction which we label by x [b]. Moreover, we assume that the surface of the soil layer is in contact with a thin layer of fluid, so that Eqs. (9), (10) are suitable to describe the motion of the porous system, while jump conditions (15)-(16) have to be used at the discontinuity point. We are not interested here to study the motion of the external fluid. Notice that if an external load p^{ext} is applied to the surface of the layer (e.g. construction of a building), a proportional increasing of the fluid pressure appearing in Eq. (15) is expected, while we can assume that the variation of the chemical potential μ_f is negligible.

We now study the motion of this porous layer linearizing the differential system obtained in the previous section around the equilibrium configuration $m_f = m_f^0$, $\varepsilon = 0$. As a first step we linearize the balance of mass (3) around this equilibrium configuration which, recalling that $\eta_s^{ⓢ}$ is constant gives

(17)
$$\dot{m}_f-\eta_f^{ⓢ}\,\dot{\phi}'_s=0,$$

where, with abuse of notation, we still indicate with m_f the perturbations Δm_f around m_f^0. This abuse of notation will be kept in the sequel.

[b] From now on x labels the solid-Lagrangian space variable previously indicated by X_s

As a second step we introduce the following quadratic form for the second gradient deformation energy Ψ^+

$$\Psi^+ = -p_0^{ext}\varepsilon + \frac{1}{2}\left(\lambda + 2\mu + b^2 M\right)\varepsilon^2 + \frac{1}{2}M\left(\frac{m_f}{m_f^0}\right)^2 - bM\,\varepsilon\frac{m_f}{m_f^0} + \frac{1}{2}\left(K_{ss} + \mathbb{M}K_{sf}^2\right)\left(\varepsilon'\right)^2 + \mathbb{M}K_{sf}\,\frac{m_f'}{m_f^0}\varepsilon' + \frac{1}{2}\mathbb{M}\left(\frac{m_f'}{m_f^0}\right)^2. \tag{18}$$

The constant coefficient p_0^{ext} accounts for the pressure of the external fluid in the equilibrium configuration (before the application of the external load p^{ext}), λ and μ are the classical Lamé coefficients, b and M the Biot parameters and $\mathbb{M}$, K_{ss} and K_{sf} are the second gradient constitutive parameters. Using this expression for Ψ^+, equations of motion (9) and (10) are linearized as

$$\left[\left(-p_0^{ext} + \lambda + 2\mu + b^2 M\right)\varepsilon - bM\frac{m_f}{m_f^0} - \left(K_{ss} + \mathbb{M}K_{sf}^2\right)\varepsilon'' - \mathbb{M}K_{sf}\,\frac{m_f''}{m_f^0}\right]' = 0, \tag{19}$$

and

$$m_f^0\left[M\frac{m_f'}{\left(m_f^0\right)^2} - \frac{bM}{m_f^0}\,\varepsilon' - \frac{\mathbb{M}K_{sf}}{m_f^0}\,\varepsilon''' - \frac{\mathbb{M}}{\left(m_f^0\right)^2}m_f'''\right] = K\frac{\eta_f^{s}}{m_f^0}\dot{\phi}_s - B\frac{\eta_f^{s}}{m_f^0}\dot{\phi}_s'', \tag{20}$$

where Eq. (10) has been multiplied on both sides by ϕ_s' and where, with abuse of notation, we also indicate by ε the perturbation $\Delta\varepsilon$ around the undeformed equilibrium configuration. Notice that the fields m_f and ε depend on the variables x and t. If we now integrate the first equation of motion and integrate the second one (with respect to x) and if we introduce the dimensionless quantities

$$\xi = \frac{x}{L}, \quad \tilde{m}_f = \frac{m_f}{m_f^0}, \quad \tilde{t} = \frac{t}{\tau}, \quad \text{with} \quad \tau = \frac{KL^2}{M},$$

L being the depth of the layer, we get the following dimensionless differential equations

$$\frac{\left(\lambda + 2\mu + b^2 M - p_0^{ext}\right)}{\lambda + 2\mu}\varepsilon - \frac{bM}{\lambda + 2\mu}\tilde{m}_f - \frac{\left(K_{ss} + \mathbb{M}K_{sf}^2\right)}{\left(\lambda + 2\mu\right)L^2}\varepsilon'' + - \frac{\mathbb{M}K_{sf}}{\left(\lambda + 2\mu\right)L^2}\,\tilde{m}_f'' = \frac{c_0(t)}{\lambda + 2\mu} \tag{22}$$

and

$$\frac{\mathbb{M}}{ML^2}\tilde{m}_f'''' + \frac{\mathbb{M}}{ML^2}K_{sf}\,\varepsilon'''' - \tilde{m}_f'' + b\varepsilon'' - \frac{B}{KL^2}\dot{\tilde{m}}_f'' + \dot{\tilde{m}}_f = 0. \tag{23}$$

In the firs equation $c_0(t)$ is clearly an integration constant with respect to x, while in the second equation we used the balance of mass (17). In the sequel, with abuse of notation, we do not distinguish m_f from $\tilde{m}_f$ and, if not specified, m_f indicates the dimensionless quantity. Moreover, the dimensionless variables ξ and $\tilde{t}$ are also indicated by x and t if no confusion can arise.

This differential system is of the sixth order in space and has an undetermined integration constant: we need seven boundary conditions to find a unique solution. We choose some of these boundary conditions between those arising from the variational principle (natural BCs) and we choose the remaining ones on the basis of the physics of the problem. In particular, we impose in $x = 0$ (i.e. on the surface of the layer) the B.C.s (15) which in their linearized, dimensionless form read

$$\begin{aligned}\frac{\left(\lambda + 2\mu + b^2 M - p_0^{ext}\right)}{\lambda + 2\mu}\varepsilon - \frac{bM}{\lambda + 2\mu}m_f - \frac{\left(K_{ss} + \mathbb{M}K_{sf}^2\right)}{(\lambda + 2\mu)\,L^2}\varepsilon''&\\ - \frac{\mathbb{M}K_{sf}}{(\lambda + 2\mu)\,L^2}\,m_f'' = -\frac{\Delta p^{ext}}{(\lambda + 2\mu)}&,\end{aligned} \tag{24}$$

and

$$m_f - b\varepsilon - \frac{\mathbb{M}K_{sf}}{ML^2}\varepsilon'' - \frac{\mathbb{M}}{ML^2}m_f'' + \frac{B}{KL^2}\dot{m}_f = \frac{m_f^0\Delta\mu^{ext}}{M} = 0, \tag{25}$$

where Δp^{ext} represents the incremental external pressure acting on the surface of the layer due to the application of an external load and $\Delta\mu^{ext}$ is the incremental chemical potential at the surface of the layer which we assume to be negligible.
Moreover, we use the natural BCs (16) both in $x = 0$ and $x = L$. Their linearized, dimensionless form read

$$m_f' + \frac{\left(K_{ss} + \mathbb{M}K_{sf}^2\right)}{\mathbb{M}K_{sf}}\varepsilon' = 0, \quad \text{and} \quad m_f' + K_{sf}\varepsilon' = 0. \tag{26}$$

For the last BC we assume that the soil layer is impermeable in $x = L$. This means that in $x = L$ the relative velocity $\dot{\phi}_s$ of the fluid with respect to the solid is vanishing. With this assumption Eq. (20), which is valid for any x in $(0, L)$, particularize in $x = L$ as

$$-m_f' + b\,\varepsilon' + \frac{\mathbb{M}K_{sf}}{ML^2}\varepsilon''' + \frac{\mathbb{M}}{ML^2}m_f''' - \frac{B}{KL^2}\dot{m}_f' = 0. \tag{27}$$

Comparing equation (22) with the boundary condition (24) it is immediate that $c_0(t) = -\Delta p^{ext}$. From now on we assume that the external applied load is constant with respect to time.

Finally, the initial condition (corresponding to the instant in which the external load is applied) is deduced assuming that the apparent Lagrangian fluid density is vanishing for $t = 0^+$, i.e.

$$m_f\left(x, t = 0^+\right) = 0. \tag{28}$$

This initial condition states that, correspondingly to the instant in which the external load is applied, there is no instantaneous variation of the fluid density m_f inside the soil layer.

6. Linearized Problem of Consolidation

In the previous section we derived an initial boundary value problem which is suitable for describing the evolution of a porous soil layer of depth L after the application of an external load. We now introduce an auxiliary function $V(x, t)$ which satisfies the following relationships [c]

$$\varepsilon = \frac{K_{sf}\mathbb{M}}{(\lambda + 2\mu)\, L^2} V^{II} + \frac{bM}{(\lambda + 2\mu)} V \tag{29}$$

and

$$m_f = \frac{\left(\lambda + 2\mu + b^2 M - p_0^{ext}\right)}{(\lambda + 2\mu)} V - \frac{K_{ss} + \mathbb{M} K_{sf}^2}{(\lambda + 2\mu)\, L^2} V^{II} + \frac{\Delta p^{ext}}{bM}. \tag{30}$$

In such a way, Eq. (22) is identically satisfied, while Eq. (23) can be rewritten, after some straightforward calculations, as

$$C_1 V^{VI} - C_2 V^{IV} - C_3 \dot{V}^{IV} + C_4 V^{II} + C_5 \dot{V}^{II} - C_6 \dot{V} = 0. \tag{31}$$

On the other hand, the boundary conditions given in Eqs. (25)-(27) read

$$\mathbb{C}_1 V^{IV} - \mathbb{C}_2 V^{II} - \mathbb{C}_3 \dot{V}^{II} + \mathbb{C}_4 V + +\mathbb{C}_5 \dot{V} + \mathbb{C}_6 = 0 \quad \text{in} \;\; x = 0, \tag{32}$$

$$V^{I} = 0, \quad V^{III} = 0 \quad \text{in} \;\; x = 0, L, \qquad V^{V} = 0 \quad \text{in} \;\; x = L. \tag{33}$$

All the coefficients appearing in these equations are given in the Appendix in terms of the constitutive parameters. Finally, the initial condition (28) becomes

$$V(x, 0^+) := V_{in} = -\frac{\Delta p^{ext}}{bM} \frac{1}{(C_4 + k_6)}, \tag{34}$$

[c] In order to use a compact notation, from now on the Roman figure superscripts indicate the order of the derivative with respect to x

Considering the linearity of the differential problem and the non homogeneity appearing in b.c. (32) we look for a solution $V(x,t)$ in the form

$$V(x,t) = \bar{V}(x) + W(x,t) \tag{35}$$

where $\bar{V}(x)$ is the solution of the stationary part of the problem (the terms in which appears the time derivative are neglected) with non-homogeneous BCs, while the deviation $W(x,t)$ is the solution of the non-stationary part of the problem with homogeneous BCs and non-trivial initial condition $W(x,0^+) := W_{in} = V_{in} - \bar{V}(x)$.

It can be proven that the solution of the stationary part of the problem when $\mathbb{C}_4 \neq 0$ is given by $\bar{V}(x) = -\mathbb{C}_6/\mathbb{C}_4$while if $\mathbb{C}_4 = 0$ the stationary solution $\bar{V}(x)$ exists if and only if $\mathbb{C}_6 = 0$; in this case an infinite family of constant solutions of the stationary problem arises.

As for the deviation $W(x,t)$, we use the separation of variables method setting $W(x,t) = X(x)T(t)$ and looking for a Fourier series solution. In particular, we find the following formal expression for W

$$W(x,t) = \mathbb{C}_4 C_6 W_{in} \sum_{k=1}^{+\infty} \left[\frac{1}{\|X_k\|^2} \int_0^L X_k(x)\, dx \right] X_k(x)\, e^{\lambda_k t}, \tag{36}$$

where λ_k and X_k are the eigenvalues and eigenfunctions of the space problem respectively. Moreover, $\|X_k\|^2 := \alpha_0 \int_0^L X_k X_k dx + \alpha_1 \int_0^L X_k^I X_k^I dx + \alpha_2 \int_0^L X_k^{II} X_k^{II} dx + \alpha_3 \int_0^L X_k^{III} X_k^{III} dx$, where the coefficients α_i are defined in the Appendix.

As a final step, we find a numerical solution for this problem choosing values of the first gradient constitutive parameters relative to a normally-consolidated clay filled of water and trial values of the second gradient dimensionless parameters (see table 1).

Table 1. Values of first and second gradient parameters.

$M\ (GPa)$	$\lambda\ (GPa)$	$\mu\ (GPa)$	k_1	k_2	k_3	k_4
5	2.3	1.5	10^{-2}	10^{-2}	10^{-2}	10^{-2}

Figure (a) shows second gradient solutions for instants close to the application of the external load ($t = 0^+$). It is immediate to note a perfect convergence of the solution to the initial datum $m_f(x,0^+) = 0$. Moreover, an over-pressurization phenomenon is visible close to $x = L$ which is not predicted by the classical theory. This means that in the deeper regions of the layer the fluid remains entrapped in the pores which are deforming and this results in a pressure of the fluid increased with respect to the initial value. In the superficial regions of the layer the fluid is flowing out resulting

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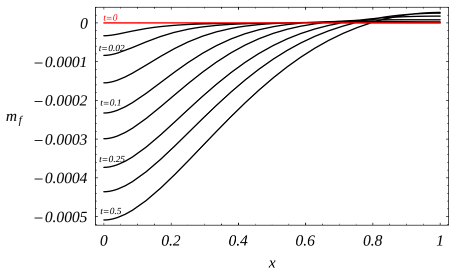
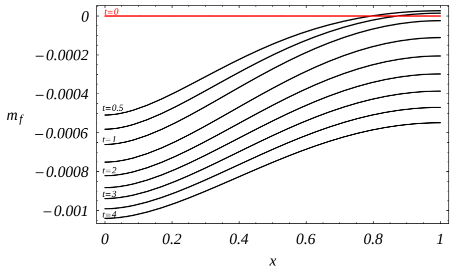
in a decreasing pressure. Figure (b) shows the behavior of m_f for further times. We note that the fluid start flowing also in the deeper regions and for infinite times m_f reaches a constant value which is the same as the classical theory.

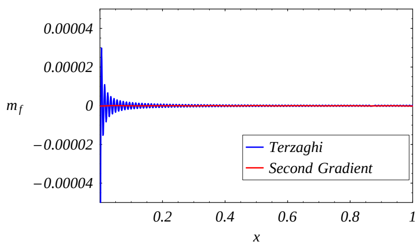


(a) 2^{nd} gradient solutions for small times

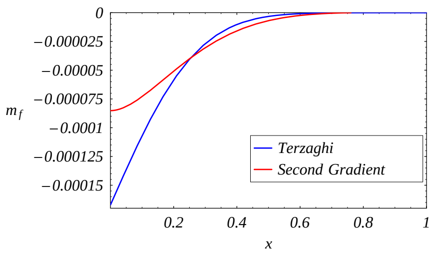


(b) 2^{nd} gradient solutions for further times

Figures (c) and (d) show a comparison between Terzaghi's and second gradient solutions for two different instants. In particular Fig.1 is the solution relative to $t = 0^+$. We note an incompatibility between initial datum and the boundary conditions in $x = 0$ for the Terzaghi solution. This incompatibility is cured by the second gradient solution which smoothly converges to the initial datum.



(c) 1^{st} and 2^{nd} gradient solutions at $t = 0$



(d) 1^{st} and 2^{nd} gradient solutions at $t > 0$

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A. Appendix: Coefficients of the Differential Problem

The constant coefficients C_i and $\mathbb{C}_i$ appearing in the differential problem given by Eqs. (31) and (32)-(33) are defined as

$$C_1 = k_1 k_3, \quad C_2 = k_1 + k_3 k_5 (k_2 + b), \quad C_3 = k_4 \left(k_1 + k_3 k_5 k_2^2\right)^2 + k_3 C_4$$
$$C_4 = 1 - \frac{p_0^{ext}}{\lambda + 2\mu}, \quad C_5 = k_4 [C_4 + k_6] + k_1 + k_3 k_5 k_2^2, \quad C_6 = C_4 + k_6,$$

with

$$k_1 = \frac{K_{ss}}{(\lambda + 2\mu) L^2}, \quad k_2 = K_{sf}, \quad k_3 = \frac{\mathbb{M}}{ML^2}, \quad k_4 = \frac{B}{KL^2}, \quad k_5 = \frac{M}{\lambda + 2\mu}, \quad k_6 = b^2 k_5.$$

Moreover, the coefficients appearing in the boundary conditions (32) are defined as $\mathbb{C}_1 = C_1$, $\mathbb{C}_2 = C_2$, $\mathbb{C}_3 = C_3$, $\mathbb{C}_4 = C_4$ and

$$\mathbb{C}_5 = C_5 - \left(k_1 + k_3 k_5 k_2^2\right), \qquad \mathbb{C}_6 = \frac{\Delta p^{ext}}{bM}.$$

Finally the coefficients of the inner product appearing in (36) are given by

$$\alpha_0 = \mathbb{C}_4 C_6, \qquad \alpha_1 = \mathbb{C}_4 C_5 - C_4 \mathbb{C}_5 + \mathbb{C}_2 C_6,$$
$$\alpha_2 = \mathbb{C}_4 C_3 - \mathbb{C}_3 C_4 + \mathbb{C}_2 C_5 - C_2 \mathbb{C}_5 + \mathbb{C}_1 C_6, \quad \alpha_3 = \mathbb{C}_2 C_3 - C_2 \mathbb{C}_3 + \mathbb{C}_1 C_5 - C_1 \mathbb{C}_5.$$

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