Communications to SIMAI Congress,
ISSN 1827-9015, Vol. 3 (2009)229 (12pp)
DOI:10.1685/CSC09229

On the Modelling of the Time Dependent Walras Equilibrium Problem

Antonio Causa[1], Fabio Raciti[1]

[1] *Dipartimento di Matematica e Informatica, Università di Catania*
Viale A. Doria 6–I, 95125, Catania, Italy
causa@dmi.unict.it
fraciti@dmi.unict.it

Abstract

Two ways of extending the classical Walras equilibrium problem to the case of time varying data are discussed. Existence theorems, as well as remarks on computational issues are provided.

Keywords: `Variational Inequalities, Monotonicity, Walras Equilibrium, monotone operator.`

1. Introduction

The purpose of this note is to study the time dependent extension of the classical Walras equilibrium problem. Although the idea of modelling a pure exchange economy is quite old (see e.g. [26]), the mathematical study of the Walras equilibrium is still an active research topic. In fact, the first rigorous results were achieved after the development of nonlinear analysis, (see e.g. [1], [3], [2]) and since then many scholars have attacked Walras problem with various techniques.
After the development of the theory of Variational Inequalities (V.I. for short) pioneered by G. Stampacchia and J. Lions in connection with partial differential equations (see e.g. [18]), a new mathematical tool was available. In fact, it was soon recognized the effectiveness of the new theory in the study of equilibrium problems both from the theoretical and the algorithmic point of view ([21], [15], [23], [17], [16], [12], [20]).
The idea of applying the variational inequality approach to Walras problem dates to S. Dafermos ([7], [8]). In the recent paper [11], as well as in this note, the authors consider that the data in Walras problem depend on time and introduce an integral variational inequality model of the new problem. The idea of treating equilibrium problems by time dependent V.I. in Lebesgue spaces has been introduced to model the traffic equilibrium problem in [14] and put on rigorous grounds in [9]. This approach has

Received 30/01/2009, in final form 23/06/2009
Published 31/07/2009

been subsequently exploited for studying other time dependent equilibrium problems arising from economics, finance or transportation science and the interested reader can find several applications in [10].
However, the method of passing from the finite dimensional formulation of a stationary equilibrium problem to an infinite dimensional, $L^p([0,T],\mathbb{R}^n)$ formulation, cannot be applied straightforwardly to Walras problem. In particular, the existence theorem in [8] cannot be extended to $L^2([0,T],\mathbb{R}^n)$ because the convex set under consideration is not closed. Another problem is that the classical Cobb-Douglas functions do not give rise to a Cobb-Douglas operator between Lebesgue spaces unless their domain is restricted. In this note, after presenting the stationary model (Sect. 2), we introduce time as a parameter, give existence and continuity results and introduce the parametric Cobb-Douglas functions (Sect. 3). In Sect. 4 we provide a rigorous L^p formulation of the time dependent model and consider the Cobb-Douglas operator between Lebesgue spaces. We close the paper with some considerations on the numerical solution of the problem.

2. The Stationary Model

The pure exchange economy model introduced by L. Walras long ago (see [26]) was the first attempt to analyze on mathematical grounds one of the oldest institution of the human society: the exchange of scarce goods. The most elementary situation under consideration involves a commodity space $X \subset \mathbb{R}^n$ and a set of m economic agents. Each agent i has to be considered in this context as consumer, trader and resource owner and we denote by $e_i \in X$ the resources endowment of consumer i. If some components of e_i are small, then the consumer under consideration will probably start to perform exchanges of goods in order to improve his welfare, i.e. to reach his desired consumption basket.
To cast the problem in an optimization framework it is stipulated that the preferences of consumer i are represented by a so called *utility function* $u_i : X \to \mathbb{R}$ which the consumer wishes to maximize under his budget constraints (which are represented by its resources endowment). A key assumption in Walras model is that all exchanges are mediated by a price system common to all parties and all consumers are considered as *price takers*, that is they regard prices as given and beyond their influence. To progress in our study of Walras equilibrium, we have to define the price-dependent demand of consumer i, which we denote by x_i:

$$x_i(p) := \operatorname{argmax}\{u_i(y) : \langle p, y\rangle \leq \langle p, e_i\rangle\}. \tag{1}$$

Here and in the sequel we shall assume that the correspondence $p \mapsto x_i(p)$ is well defined and unique.

DOI: 10.1685/CSC09229

The *aggregate demand* and the *total supply* are defined as follows:

$$x(p) := \sum_{i=1}^{m} x_i(p), \quad S := \sum_{i=1}^{m} e_i$$

while the *aggregate excess demand* is given by:

$$z(p) := x(p) - S. \tag{2}$$

Definition 2.1. A price $p^* \in \mathbb{R}^n_+$ is a *Walras equilibrium price* iff

$$z(p^*) \leq 0. \tag{3}$$

Since in a pure exchange economy only relative prices are important, it follows that prices can be normalized to take values in the unitary simplex:

$$S^n = \{p \in \mathbb{R}^n : p_j \geq 0, \ \sum_{j=1}^{n} p_j = 1\}. \tag{4}$$

Finally we state the famous *Walras Law*:

$$\langle z(p), p \rangle = 0 \tag{5}$$

which describes the fact that the two market's sides always record the same economic volume of monetary transactions.

3. Time as a parameter

Let $z : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the time dependent aggregate excess demand function and consider the following problem.
For almost every $t \in [0,T]$, find $p^*(t) \in S^n$ such that

$$z(t, p^*(t)) \leq 0. \tag{6}$$

One can prove (see the proof of [21], for fixed t) that (6) is equivalent to the following parametric variational inequality (PVI).
For almost every $t \in [0,T]$, find $p^*(t) \in S^n$, such that:

$$\langle z(t, p^*(t), p - p^*(t) \rangle \leq 0, \ \forall p \in S^n. \tag{7}$$

The following theorem assures that the problem above is solvable.

Theorem 3.1. *For $t \in [0,T]$ let $z(t,\cdot)$ be continuous. Then problem (7) is solvable.*

Proof. Since $z(t,\cdot)$ is continuous, for each $t \in [0,T]$, and becausa of the compactness of S^n, the existence of a solution follows directly from the classical Stampacchia Theorem (see e.g. [18]). □

When studying the stationary Walras equilibrium problem, it is possible that the aggregate excess demand becomes unbounded when the price of a certain commodity approaches zero. In this case $z(p)$ can be continuous only on a subset D such that: $S^n_+ \subset D \subset S^n$ where S^n_+ denotes the intersection of S^n with $\mathbb{R}^n_{++}$. In this case, we extend a theorem proved in [8] for the discontinuous stationary case and obtain the following:

Theorem 3.2. *For each fixed $t \in [0,T]$ assume that the aggregate excess demand function $z(t,p)$ satisfies the following assumption (B):*

If $S^n \setminus D$ is nonempty then to any sequence $\{p_n\} \in S^n_+$ which converges to a point of $S^n \setminus D$, there is associated a point $\overline{p}(t) \in S^n_+$ such that the sequence $\{\langle z(t,p_n), \overline{p}(t)\rangle\}$ contains infinitely many positive terms.
Then there is a Walrasian price vector $p^*(t) \in D$.

Proof. Let us consider the following family of closed convex and bounded sets:

$$K_m := \{p \in \mathbb{R}^n : p_i \geq 1/m,\ \Sigma^n_{j=1} p_j = 1,\ i = 1\ldots,n\}$$

Let us observe that, for m large enough the sets just defined are nonempty. Moreover the increasing sequence K_m converges to S^n_+ in the set theoretical sense, and converges to S^n in the sense of Kuratowsky. The latter kind of convergence means that:

i) For each $p \in S^n$ it exists a sequence of points $p_m \in K_m$ such that $\lim_{m\to\infty} p_m = p$.
ii) If $p_m \in K_m$ and $p_m \to p$, when $m \to \infty$ then $p \in S^n$.

Let us fix $t \in [0,T]$, $m \in \mathbb{N}$ and consider the following variational inequality problem.
Find $p^*_m(t) \in K_m$ such that:

$$\langle z(t, p^*_m(t)), p - p^*_m(t)\rangle \leq 0,\ \forall p \in K_m. \tag{8}$$

This problem admits at least one solution, $p^*_m(t)$, from Theorem (3.1) and because of Walras law (5) we get:

$$\langle z(t, p^*_m(t)), p\rangle \leq 0,\ \forall p \in K_m. \tag{9}$$

DOI: 10.1685/CSC09229

The sequence $\{p_m^*(t)\}$ contains a converging subsequence which with abuse of notation we still call $\{p_m^*(t)\}$, and since S^n is closed it follows that:

$$p_m^*(t) \longrightarrow p^*(t) \in S^n, \quad m \to \infty.$$

In order to prove that $p^*(t) \in D$, suppose by contradiction that $p^*(t) \in S^n \setminus D$. Then, due to assumption (B) we can find $\overline{p}(t) \in S_+^n$ such that the sequence $\{\langle z(t, p_m^*(t)), \overline{p}(t)\rangle\}$ has infinitely many positive terms. But, by construction, $\exists \nu \in \mathbb{N}$ such that for $m > \nu$ we get $\overline{p}(t) \in K_m$, which contradicts (9); hence, $p^*(t) \in D$.
Now, let us fix $p \in S^n$ arbitrarily. Because of i) $\exists q_m \in K_m$ such that $q_m \to p$ when $m \to \infty$ and, from (8), we get:

$$\langle z(t, p_m^*(t)), q_m - p_m^*(t)\rangle \leq 0.$$

At last, by the continuity of $z(t, \cdot)$ on D, taking the limit $m \to \infty$ we find:

$$\langle z(t, p^*(t)), p - p^*(t)\rangle \leq 0. \qquad \square$$

In what follows we shall turn to continuous functions $z(t, \cdot)$, in order to study the regularity of the time dependent equilibrium price. In particular we discuss the continuity of the solution map $t \mapsto p^*(t)$. The proof is done for a general case which includes a time dependent convex set. This could be useful for the study of equilibrium problems with perturbations in the constraints set. The interested reader can find in [6] a general analysis of continuity results. We recall the following definition:

Definition 3.1. For each $t \in [0, T]$, the operator $F(t, \cdot) = -z(t, \cdot)$ is called monotone, iff.

$$\langle F(t, x) - F(t, y), x - y\rangle \geq 0, \quad \forall x, y \in \mathbb{R}^n.$$

F is strictly monotone iff the equality holds only for $x = y$.

Theorem 3.3. *Let $z : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous function on $[0, T] \times \mathbb{R}^n$. For each $t \in [0, T]$, let $C(t)$ be a closed, convex and bounded subset of $\mathbb{R}^n$ and assume that if $t_n \to t$ the set sequence $C(t_n)$ converges to $C(t)$ in the sense of Kuratowsky (see theorem (3.2) for the definition of Kuratowsky convergence). Moreover, suppose that for each $t \in [0, T]$, $p^*(t)$ is the unique solution of the problem:*

$$\langle z(t, p^*(t)), p - p^*(t)\rangle \leq 0, \ \forall p \in C(t) \tag{10}$$

Then, the solution map $t \mapsto p^*(t)$ of (10) is continuous on $[0, T]$.

Proof. Let us first observe that, due to Theorem (3.1), the solution set of (10) is nonempty. Moreover let us remark that a sufficient condition for the uniqueness of the solution is the strict monotonicty of $-z(t,\cdot)$ on $C(t)$, nevertheless our theorem applies if one can ensure uniqueness without any monotonicity requirement as well.
Now, let us fix $t \in [0,T]$ and let $\{t_n\} \subset [0,T]$, such that $t_n \to t$ when $n \to \infty$. If p is a point of $C(t)$, because of Kuratowsky convergence we can find $p_m \in C(t_m)$ such that $p_m \to p$ and

$$\langle z(t_m, p^*(t_m)), p_m - p^*(t_m)\rangle \leq 0,\ \forall p \in C(t) \tag{11}$$

Since $\{p^*(t_m)\}$ is bounded, it admits a converging subsequence $\{p^*(t_{k_m})\}$, such that $p^*(t_{k_m}) \to u \in C(t)$. Thus:

$$\langle z(t_{k_m}, p^*(t_{k_m})), p_m - p^*(t_{k_m})\rangle \leq 0,\ \forall p \in C(t) \tag{12}$$

Passing to the limit for $m \to \infty$ (12) yields to

$$\langle z(t,u), p-u\rangle \leq 0$$

By uniqueness and by standard arguments we finally get that the whole sequence $p^*(t_m)$ converges to $p^*(t)$ as $n \to \infty$. □

In the example which follows we recall the definition of the classical (i.e. stationary) Cobb-Douglas functions, and consider the corresponding time-dependent, parametric, version.

Example 3.1 Let us consider a pure exchange economy with m consumers and n goods. The consumers act in order to maximize their wealth, and in economic theory this is modeled by introducing for each consumer a utility function which each consumer wants to maximize under his/her budget constraint. According to the Cobb-Douglas model, the demand for the good j made by the consumer i, under the price system $(p_1, \ldots, p_n)$ is given by:

$$x_{ij} = \alpha_{ij} \frac{\langle e_i, p\rangle}{p_j} \quad i = 1, \ldots, m;\ j = 1, \ldots, n$$

where $e_i = (e_{i1}, \ldots, e_{in})$ represents the vector of all the goods owned by the consumer i before starting to trade. Thus, $w_i = \langle e_i, p\rangle$ represents the (initial) budget of consumer i under the price system $(p_1, \ldots, p_n)$. The price is considered as a column vector and with the row vector $e_i = (e_{i1}, \ldots, e_{in})$ one can build the matrix of the initial endowments (e_{ij}), $i = 1, \ldots, m;\ j = 1, \ldots, n$. Moreover, it is common practice to set $\sum_{j=1}^{n} \alpha_{ij} = 1,\ i = 1, \ldots, m$.

DOI: 10.1685/CSC09229

The relevant function to study the Walras equilibrium is the excess aggregate demand, which is a vector function whose component j (corresponding to the good j) is given by:

$$z_j = \sum_{i=1}^{m} x_{ij} - \sum_{i=1}^{m} e_{ij} = \sum_{i=1}^{m} \alpha_{ij} \frac{\langle e_i, p \rangle}{p_j} - \sum_{i=1}^{m} e_{ij} \tag{13}$$

Thus, the Cobb-Douglas excess aggregate demand $z : \mathbb{R}^n \to \mathbb{R}^n$ is actually defined in $\mathbb{R}^n_{++}$ and because prices can be normalized we shall study it on the (relative) interior of the price simplex S^n. Consider now the parametric time-dependent version of (13):

$$z_j(t,p) = \sum_{i=1}^{m} \alpha_{ij}(t) \frac{\langle e_i(t), p \rangle}{p_j} - \sum_{i=1}^{m} e_{ij}(t) \tag{14}$$

where:

$$\alpha_{ij} : [0,T] \to \mathbb{R},\ e_{ij} : [0,T] \to \mathbb{R},\ \forall i = 1, \cdots, m, j = 1, \cdots, n$$
$$\sum_{j=1}^{n} \alpha_{ij}(t) = 1,\ \forall i = 1, \cdots, m,\ t \in [0,T]$$

Here and in the sequel the functions defined above are nonnegative measurable and bounded functions on $[0,T]$.

Remark 3.1. It is easy to verify that the parametric Cobb-Douglas excess demand functions verify the hypothesis of Theorem (3.2). Roughly speaking for each $t \in [0,T]$ at least one component of $z(t,p)$ becomes arbitrarily large as p approaches the boundary of S^n.

4. A Class of Integral Variational Inequality Models

In this section we propose an operator formulation of the Walras Equilibrium Problem. As already noticed, in a pure exchange economy prices can be normalized to take values in the unitary simplex, and this fact remains true if the data are time dependent. Thus, looking for an integral formulation of the time dependent Walras equibrium problem, the natural functional setting for the feasible set is $L^\infty([0,T], \mathbb{R}^n)$. Hence, let us consider:

$$K = \{p \in L^\infty([0,T], \mathbb{R}^n) : p_j(t) \geq 0, \sum_{j=1}^{n} p_j(t) = 1, \} \tag{15}$$

Now, let $Z : L^\infty([0,T], \mathbb{R}^n) \to L^1([0,T], \mathbb{R}^n)$ the aggregate excess demand operator, which associates to each price function $p(t)$ the corresponding

demand $Z(p(t))$. Thus we can consider the following integral variational inequality.
Find $p^* \in K$ such that

$$\int_0^T \langle Z(p^*(t)), p(t) - p^*(t)\rangle \, dt \leq 0, \ \forall p \in K. \tag{16}$$

Thus, the hypothesis of price normalization leads naturally to an (L^∞, L^1) formulation of the Walras Equilibrium Problem. However, for technical reasons we shall introduce, $\forall\, r > 1$, the following subsets $K_r \subseteq L^r([0,T], \mathbb{R}^n)$:

$$K_r = \{p \in L^r([0,T], \mathbb{R}^n) : p_j(t) \geq 0, \sum_{j=1}^{n} p_j(t) = 1, \} \tag{17}$$

and the corresponding operators:

$$Z : L^r([0,T], \mathbb{R}^n) \to L^s([0,T], \mathbb{R}^n), \ \ 1/r + 1/s = 1$$

Thus, we are led to the following variational inequality problem.
Find $p^* \in K_r$ such that

$$\int_0^T \langle Z(p^*(t)), p(t) - p^*(t)\rangle \, dt \leq 0, \ \forall p \in K_r \tag{18}$$

In this way we shall have access to various existence theorems for variational inequalities in the literature. Moreover, for computational purposes, the case $r = 2$ is of particular interest. We first recall two definitions which will be used in the sequel.

Definition 4.1. Let X be a reflexive Banach space, and X^* its topological dual space. $A : X \longrightarrow X^*$ is said to be hemicontinuous iff the function

$$t \mapsto \langle A(u + tv), w\rangle_{X,X^*}$$

is continuous on [0,1] for all $u, v, w \in X$, where $\langle \cdot, \cdot \rangle_{X,X^*}$ denotes the duality pairing between X and X^*.

Definition 4.2. Let X be a reflexive Banach space, and X^* its topological dual space. $A : X \longrightarrow X^*$ is said to be monotone iff

$$\langle Ax - Ay, x - y\rangle_{X,X^*} \geq 0, \ \ \forall x, y \in X$$

DOI: 10.1685/CSC09229

The following theorem is due to Stampacchia ([24]).

Theorem 4.1. *Let X be a reflexive Banach space and $A : X \longrightarrow X^*$ be monotone and hemicontinuous. Then, for each closed convex and bounded set $K \subset X$, the following variational inequality*

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in K$$

is solvable.

We conclude by explaining under which conditions it is possible to define a Cobb-Douglas operator between Lebesgue spaces. In fact, it is evident that the parametric functions defined previously cannot belong to any Lebesgue space unless some restriction on their domains is performed.

Remark 4.1. Under the assumptions that $\alpha_{ij}(t)$ and $e_{ij}(t)$ have positive infimum on $[0, T]$ there exists a closed subset C of S^n_+, independent on t, such that for each $t \in [0, T]$ at least one component of $z(t, p)$ is strictly positive. This can be proved first by fixing $t \in [0, T]$ and observing that $\lim_{p \to \overline{p}} z_j = +\infty$, where $\overline{p}$ is a point of the boundary of S^n whose *j-th* component is zero, and the limit is uniform with respect to the choice of the point. Moreover this is true for the time-dependent Cobb-Douglas excess aggregate demand functions. We also underline that the determination of the set C can be a task as difficult as the determination of the equilibrium points, but its mere existence will allow the use of the Lebesgue theory outlined previously.

It follows from the previous remark that all the equilibrium points $p^*(t)$ are contained in C. Then, when dealing with the Cobb-Douglas functions, instead of the set K of (15) we can consider the subset K_C of the functions $p \in K$ such that $p(t) \in C$, for a.e. $t \in [0, T]$.

5. Some Remarks on Computational Issues

In this section, we briefly comment on the numerical solution of problems (7) and (18). First of all we point out that in [9] (as well as in [11]) the authors start with the integral formulation and connect the solution of the integral problem to that of a pointwise problem, which is close to our parametric one. In a forthcoming paper ([5]), we shall show how starting from the parametric problem one can obtain an integral problem under natural assumptions.
For the solution of the parametric problem, we remark that if a continuity result (such as Theorem (3.3)) holds, we can apply one of the several algorithms available for solving finite dimensional inequalities (see e.g. [21]) for

each t where the solution is needed. When the solution is Lipschitz continuous (see e.g. [4], [25]), it is interesting to construct global approximations to the solution by sampling its values in $[0, T]$, because we expect that it is possible to obtain error estimates.
Numerical methods for the integral problem (18), without regularity results for the solution, are much less developed. For istance, in the paper [9] the authors propose a subgradient method in order to find an approximate solution of an integral variational inequality describing the traffic equilibrium problem. Unfortunately, when this method has been tested on concrete examples ([22]), it has proved to be unpractical even for very simple cases. We do not expect that the subgradient method can give better results in the case of the Walras Equilibrium Problem, because the constraints set is of the same type of the constraints set of the traffic problem (a simplex instead of a product of simplices).
A discretization procedure for integral variational inequalities of the same type of (18) has been proposed in [23] and further developed in [17]. Although the first tests performed in [16] on small scale problems are satisfactory, the method cannot be applied to large scale problems due to the curse of dimension, although parallelization should increase its range of application. Thus, we can conclude that although the theory of variational inequalities in Lebesgue spaces is well developed, much efforts have to be done to find suitable methods for their numerical solution in concrete applications.

Acknowledgments. The authors wish to thank A. Villani for useful discussions.

REFERENCES

1. K. J. Arrow and G. Debrew, Existence of an equilibrium for a competitive economy, *Econometrica*, **22** (1954), pp. 265–290.
2. K. J. Arrow and F. H. Hahn, General Competitive Analysis. In C.J. Bliss and M. D. Intriligator editors, *Advanced textbooks in economics* 1991.
3. K. C. Border, *Fixed point theorems with applications to economics and game theory* Cambridge University Press, Cambridge, 1985.
4. A. Causa and F. Raciti, Lipschitz continuity results for a class of variational inequalities and applications: a geometrical approach, preprint.
5. A. Causa and F. Raciti, Some remarks on the time dependent Walras Equilibrium Problem, preprint.
6. A. O. Caruso, A. A. Khan, and F. Raciti, Continuity results for a class of variational inequalities with applications to time-dependent network

DOI: 10.1685/CSC09229

problems, accepted for the publication on: *Numerical Functional Analysis and Optimization.*

7. S. Dafermos, Traffic equilibrium and variational inequalities, *Transportation Science*, **14** (1980), pp. 42–54.
8. S. Dafermos, Exchange price equilibria and variational inequalities, *Math. Program.*, **46** (1990), pp. 391–402.
9. P. Daniele, A. Maugeri, and W. Oettli, Time-Dependent Traffic Equilibria, *Journal Optim.Th.Appl.*, **103** December (1999), No. 3, pp. 543–555.
10. P. Daniele, *Dynamic Networks and Evolutionary Variational Inequalities*, E. Elagar Publishing Limited, Cheltenhan, 2006.
11. M. B. Donato, M. Milasi, and C. Vitanza, Dynamic Walrasian price equilibrium problem: evolutionary variational approach with sensitivy analysis, *Optim. Letters*, **2** (2008), pp. 113–126.
12. P. Falsaperla and F. Raciti, An improved, non-iterative algorithm for the calculation of the equilibrium in the traffic network problem, *J. Optim. Theory. Appl.*, **133** (2007), pp. 401–411.
13. F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Vol. 1,2, Springer, New York, 2003.
14. T. L. Friesz, D. Bernstein, T. E. Smith, R. L. Tobin, and B.-W. Wie, A Variational inequality formulation of the dynamic network user equilibrium problem, *Oper. Res.*, **41** 1 (1993), pp. 179–191.
15. F. Giannessi and A. Maugeri editors, *Variational Inequalities and Network Equilibrium Problems, Erice 1994*, Plenum Press, New York, 1995.
16. J. Gwinner and F. Raciti, Random equilibrium problems on networks, *Mathematical and Computer Modelling*, **43** (2006), pp.880–891.
17. J. Gwinner and F. Raciti, On a class of random variational inequalities on random sets, *Numerical Functional Analysis and Optimization*, **2** 5-6 (2006), pp. 618–636.
18. D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*, Academic Press, New York, 1980.
19. A. Maugeri and F. Raciti, On existence theorems for monotone and nonmonotone variational inequalities, submitted to the *Journal of Convex Analysis*.
20. A. Maugeri and F. Raciti, On General Infinite Dimensional Complementarity Problems, *Optimization Letters*, **2** n.1 Jan. (2008), pp. 71–90.

21. A. Nagurney, *Network Economics: A Variational Inequality Approach*, Kluwer Academic Publishers, Dordrecht, Holland, 1993.
22. On the Calculation of Equilibrium in Time Dependent Traffic Networks, in P. Daniele, F. Giannessi and A. Maugeri, editors *Equilibrium Models and Variational Models*, pp. 369–377, Kluwer Academic Publishers, Norwell, MA, Nonconvex Optim. Appl., 68, 2003.
23. F. Raciti, Bicriterion Weight Varying Spatial Price Networks, *Journal of Optimization Theory and Applications*, **122**, n. 2 August (2004), pp. 387–403.
24. G. Stampacchia, Formes bilineaires sur les ensembles convexes, *Compt. Rend. Acad. Sci. Paris*, **258** (1964), pp. 4413–4416.
25. N. D. Yen, Lipschitz continuity of solutions of variational inequalities with a parametric polyhedral constraint, *Mathematics of Operations Research*, **20** (1995), pp. 695-208.
26. L. Walras, *Elements d'Economie Politique Pure*, Corbaz, Lausanne, Switzerland, 1874.