Communications to SIMAI Congress,
ISSN 1827-9015, Vol. 3 (2009) 203 (7pp)

DOI: 10.1685/CSC09203

Buffon's Problem with a Star of Needles and a Lattice of Rectangles II

Uwe Bäsel[1], Vittoria Bonanzinga[2], Lucia Fiorino[2]

[1] *Department of Mechanical and Energy Engineering*
Leipzig University of Applied Sciences, Germany
baesel@me.htwk-leipzig.de

[2] *Faculty of Engineering-DIMET*
University of Reggio Calabria, Italy
vittoria.bonanzinga@unirc.it, vittoria.bonanzinga@tin.it
lfiorino@iol.it

Abstract

In this paper we study Buffon type problems with multiple intersections for lattices of rectangles and a star consisting of four or six needles as test body.

Keywords: `Geometric probability, stochastic geometry, random sets and random convex sets.`

1. Introduction.

Stars $\mathcal{S}_{n,\ell}$ with n needles and plane lattices of parallel lines are considered in [1]. Stars $\mathcal{S}_{3,\ell}$ and $\mathcal{S}_{5,\ell}$ and the lattice of rectangles are considered in [2]. Among other test elements stars $\mathcal{S}_{3,\ell}$ and lattices with obstacles are considered in [3].

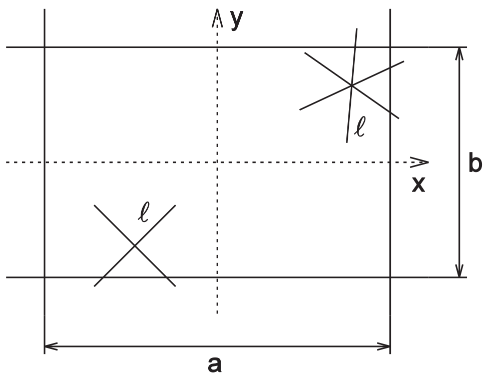

Fig. 1. Stars $\mathcal{S}_{4,\ell}$, $\mathcal{S}_{6,\ell}$ and lattice $\mathcal{R}_{a,b} = \mathcal{R}_a \cup \mathcal{R}_b$

Received 10/02/2009, in final form 10/06/2009
Published 31/07/2009

We denote by $\mathcal{R}_{a,b}$ the lattice of rectangles of sides a and b, $\lambda := \ell/a$ and $\mu := \ell/b$, where ℓ is the length of the needles of the star (see figure 1). $\mathcal{R}_{a,b}$ can be considered as union of the lattice $\mathcal{R}_a$ of vertical lines distance a apart and the lattice $\mathcal{R}_b$ of horizontal lines distance b apart.

The coordinates of the center point (x, y) of the star are random variables uniformly distributed in $[-\frac{a}{2}, \frac{a}{2}] \times [-\frac{b}{2}, \frac{b}{2}]$. The angle ϕ between one needle of the star and the x-axis is a random variable uniformly distributed in $[0, 2\pi]$ (see figure 2). All random variables are stochastically independent.

We assume $2\ell \leq \min(a, b)$, so that the stars can intersect only one line of $\mathcal{R}_a$ and one of $\mathcal{R}_b$ (except sets of stars with measure zero).

2. Stars with four needles

Theorem 2.1. *A star $\mathcal{S}_{4,\ell}$ is thrown at random onto the lattice $\mathcal{R}_{a,b}$. If $2\ell \leq \min(a, b)$, $\mathcal{S}_{4,\ell}$ intersects at most one of the vertical lines and at most one of the horizontal lines of $\mathcal{R}_{a,b}$. The maximum number of intersections is 4. The probabilities $p(i, 4, \lambda, \mu)$ of exactly i intersections are given by*

$$p(0, 4, \lambda, \mu) = 1 - \frac{4\sqrt{2}}{\pi}(\lambda + \mu) + \left(2 + \frac{4}{\pi}\right) \lambda\, \mu\,,$$

$$p(1, 4, \lambda, \mu) = \frac{8}{\pi}\left(\sqrt{2} - 1\right)(\lambda + \mu) - 4\,\lambda\,\mu\,,$$

$$p(2, 4, \lambda, \mu) = \frac{4}{\pi}\left(2 - \sqrt{2}\right)(\lambda + \mu) + 4\left(1 - \frac{4}{\pi}\right)\lambda\,\mu\,,$$

$$p(3, 4, \lambda, \mu) = 4\left(\frac{4}{\pi} - 1\right)\lambda\,\mu\,, \quad p(4, 4, \lambda, \mu) = 2\left(1 - \frac{2}{\pi}\right)\lambda\,\mu\,.$$

Proof. There are i intersections between $\mathcal{S}_{4,\ell}$ and $\mathcal{R}_{a,b}$ ($i \in \{0, 1, 2, 3, 4\}$), if $\mathcal{S}_{4,\ell}$ intersects $\mathcal{R}_a$ in j points and at the same time $\mathcal{R}_b$ in k points with $j + k = i$ ($j, k \in \{0, 1, 2\}$). E. g. there are 2 intersections between $\mathcal{S}_{4,\ell}$ and $\mathcal{R}_{a,b}$, if

$$1)\ j + k = 2 + 0\,, \quad 2)\ j + k = 1 + 1\,, \quad 3)\ j + k = 0 + 2.$$

We consider the situation for fixed angle ϕ (see figure 2). $\mathcal{S}_{4,\ell}$ intersects $\mathcal{R}_a$ in j points and at the same time $\mathcal{R}_b$ in k points, if the center point (x, y) of $\mathcal{S}_{4,\ell}$ lies in a rectangle with side lengths a_j and b_k. We denote by $p(i, 4, \lambda, \mu \,|\, \phi)$ the conditional probability of exactly i intersections of $\mathcal{S}_{4,\ell}$ and $\mathcal{R}_{a,b}$ for fixed angle ϕ. The probability $p(i, 4, \lambda, \mu)$, that $\mathcal{S}_{4,\ell}$ intersects the sides of the lattice $\mathcal{R}_{a,b}$ i-times, is computed with the formula

$$p(i, 4, \lambda, \mu) = \int_{\phi=0}^{2\pi} p(i, 4, \lambda, \mu \,|\, \phi)\, f(\phi)\, \mathrm{d}\phi$$

DOI: 10.1685/CSC09203

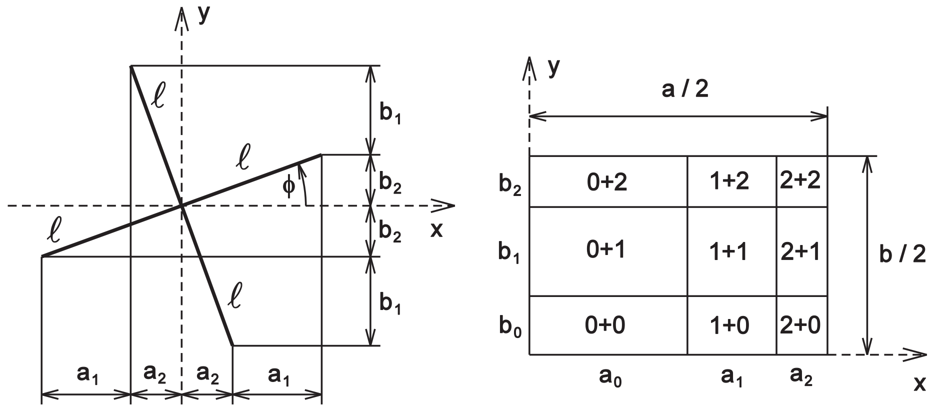

Fig. 2. Star $\mathcal{S}_{4,\,\ell}$ and numbers of intersections

where $f(\phi)$ is the density function of the random variable ϕ. Since ϕ is uniformly distributed in $[0, 2\pi]$, we have

$$f(\phi) = \begin{cases} \dfrac{1}{2\pi} \text{ for } \quad 0 \le \phi \le 2\pi\,, \\ 0 \ \text{ elsewhere} \end{cases}$$

and

$$p(i, 4, \lambda, \mu) = \frac{1}{2\pi} \int_{\phi=0}^{2\pi} p(i, 4, \lambda, \mu \,|\, \phi)\, \mathrm{d}\phi\,.$$

Due to the symmetry of $S_{4,\ell}$ it is sufficient to consider only ϕ with $0 \le \phi \le \pi/4$ and coordinates of the center point (x, y) with $0 \le x \le a/2$ and $0 \le y \le b/2$. The conditional probabilities $p(i, 4, \lambda, \mu \,|\, \phi)$ are given by

$$p(i, 4, \lambda, \mu \,|\, \phi) = \frac{F_i(\phi)}{F} \quad \text{for} \quad i \in \{0, 1, 2, 3, 4\}$$

with $F = \frac{1}{4}ab$ and

$$\begin{aligned} F_0(\phi) &= a_0 b_0\,, \\ F_1(\phi) &= a_0 b_1 + a_1 b_0\,, \\ F_2(\phi) &= a_0 b_2 + a_1 b_1 + a_2 b_0\,, \\ F_3(\phi) &= a_1 b_2 + a_2 b_1\,, \\ F_4(\phi) &= a_2 b_2\,, \end{aligned}$$

where

$$a_0 = a_0(\phi) = \tfrac{a}{2} - \ell \cos\phi\,,\ a_1 = a_1(\phi) = \ell(\cos\phi - \sin\phi)\,,\ a_2 = a_2(\phi) = \ell \sin\phi\,,$$
$$b_0 = b_0(\phi) = \tfrac{b}{2} - \ell \cos\phi\,,\ b_1 = b_1(\phi) = a_1(\phi)\,, \qquad b_2 = b_2(\phi) = a_2(\phi)$$

(see figure 2). Calculating

$$p(i,4,\lambda,\mu) = \frac{16}{\pi ab}\int_{\phi=0}^{\pi/4} F_i(\phi)\,\mathrm{d}\phi$$

for $i \in \{0,1,2,3,4\}$ we obtain the desired formulas. $\square$

From the formulas in theorem 2.1 one gets the following approximate expression:

$$\begin{aligned}
p(0,4,\lambda,\mu) &\approx 1 - 1{,}80063\,(\lambda+\mu) + 3{,}27324\,\lambda\,\mu\,,\\
p(1,4,\lambda,\mu) &\approx 1{,}05479\,(\lambda+\mu) - 4\,\lambda\,\mu\,,\\
p(2,4,\lambda,\mu) &\approx 0{,}745846\,(\lambda+\mu) - 1{,}09296\,\lambda\,\mu\,,\\
p(3,4,\lambda,\mu) &\approx 1{,}09296\,\lambda\,\mu\,,\\
p(4,4,\lambda,\mu) &\approx 0{,}72676\,\lambda\,\mu\,.
\end{aligned}$$

3. Stars with six needles

Theorem 3.1. *A star $\mathcal{S}_{6,\ell}$ is thrown at random onto the lattice $\mathcal{R}_{a,b}$. If $2\ell \le \min(a,b)$, $\mathcal{S}_{6,\ell}$ intersects at most one of the vertical lines and at most one of the horizontal lines of $\mathcal{R}_{a,b}$. The maximum number of intersections is 6. The probabilities $p(i,6,\lambda,\mu)$ of exactly i intersections are given by*

$$\begin{aligned}
p(0,6,\lambda,\mu) &= 1 - \frac{6}{\pi}(\lambda+\mu) + \left(\frac{6}{\pi} + \sqrt{3}\right)\lambda\,\mu\,,\\
p(1,6,\lambda,\mu) &= \left(\frac{12-6\sqrt{3}}{\pi}\right)(\lambda+\mu) - \frac{6}{\pi}\,\lambda\,\mu\,,\\
p(2,6,\lambda,\mu) &= \left(\frac{12\sqrt{3}-18}{\pi}\right)(\lambda+\mu) + \left(\frac{6}{\pi} - 3\sqrt{3}\right)\lambda\,\mu\,,\\
p(3,6,\lambda,\mu) &= \frac{6}{\pi}\left(2-\sqrt{3}\right)(\lambda+\mu) - 4\left(\frac{6}{\pi} - \sqrt{3}\right)\lambda\,\mu\,,\\
p(4,6,\lambda,\mu) &= 5\left(\frac{6}{\pi} - \sqrt{3}\right)\lambda\,\mu\,, \quad p(5,6,\lambda,\mu) \;=\; 2\left(2\sqrt{3} - \frac{9}{\pi}\right)\lambda\,\mu\,,\\
p(6,6,\lambda,\mu) &= \left(\frac{6}{\pi} - \sqrt{3}\right)\lambda\,\mu\,.
\end{aligned}$$

Proof. We denote by $p(i,6,\lambda,\mu\,|\,\phi)$ the conditional probability of exactly i intersections of $\mathcal{S}_{6,\ell}$ and $\mathcal{R}_{a,b}$ for fixed angle ϕ. The probability, that the

DOI: 10.1685/CSC09203

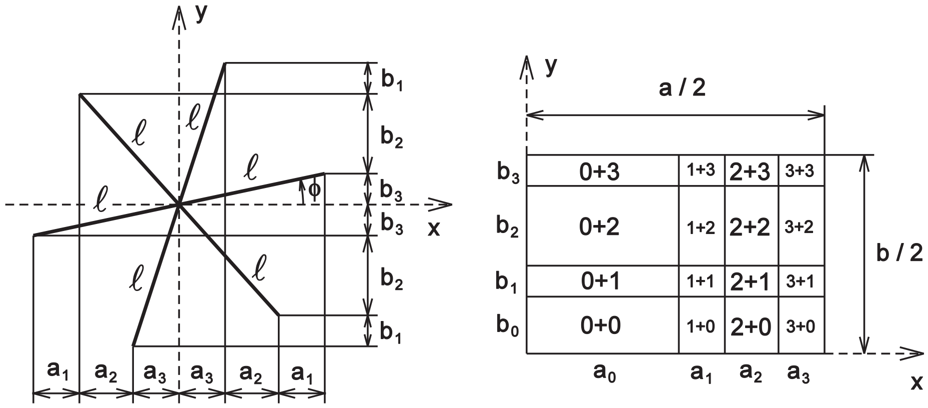

Fig. 3. Star $\mathcal{S}_{6,\,\ell}$ and numbers of intersections

test body intersects the sides of the lattice $\mathcal{R}_{a,\,b}$ i-times, is computed with the formula

$$p(i,6,\lambda,\mu)=\frac{1}{2\pi}\int_{\phi=0}^{2\pi}p(i,6,\lambda,\mu\,|\,\phi)\,\mathrm{d}\phi\,.$$

Due to the symmetry of $S_{6,\ell}$ it is sufficient to consider only ϕ with $0\le\phi\le\frac{\pi}{6}$ and coordinates of the center point (x,y) with $0\le x\le a/2$ and $0\le y\le b/2$. The conditional probabilities $p(i,6,\lambda,\mu\,|\,\phi)$ are given by

$$p(i,6,\lambda,\mu\,|\,\phi)=\frac{F_i(\phi)}{F}\quad\text{for}\quad i\in\{0,1,\ldots,6\}$$

with $F=\frac{1}{4}ab$ and

$$\begin{aligned}
F_0(\phi)&=a_0b_0\,,\\
F_1(\phi)&=a_0b_1+a_1b_0\,,\\
F_2(\phi)&=a_0b_2+a_1b_1+a_2b_0\,,\\
F_3(\phi)&=a_0b_3+a_1b_2+a_2b_1+a_3b_0\,,\\
F_4(\phi)&=a_1b_3+a_2b_2+a_3b_1\,,\\
F_5(\phi)&=a_2b_3+a_3b_2\,,\\
F_6(\phi)&=a_3b_3,
\end{aligned}$$

where

$$a_0 = a_0(\phi) = \tfrac{a}{2} - \ell\cos\phi\,, \qquad a_1 = a_1(\phi) = \tfrac{1}{2}\,\ell(\cos\phi - \sqrt{3}\sin\phi)\,,$$
$$a_2 = a_2(\phi) = \sqrt{3}\,\ell\sin\phi\,, \qquad a_3 = a_3(\phi) = a_1(\phi)\,,$$
$$b_0 = b_0(\phi) = \tfrac{b}{2} - \tfrac{\ell}{2}(\sin\phi + \sqrt{3}\cos\phi)\,, \quad b_1 = b_1(\phi) = \ell\sin\phi\,,$$
$$b_2 = b_2(\phi) = \tfrac{\ell}{2}(\sqrt{3}\cos\phi - 3\sin\phi)\,, \qquad b_3 = b_3(\phi) = b_1(\phi)$$

(see figure 3). Calculating

$$p(i,6,\lambda,\mu) = \frac{24}{\pi ab}\int_{\phi=0}^{\pi/6} F_i(\phi)\,\mathrm{d}\phi$$

for $i \in \{0,1,\ldots,6\}$ we obtain the desired formulas. □

From the formulas in theorem 3.1 one gets the following approximate expression:

$$p(0,6,\lambda,\mu) \approx 1 - 1{,}90986\,(\lambda+\mu) + 3{,}64191\,\lambda\,\mu\,,$$
$$p(1,6,\lambda,\mu) \approx 0{,}511745\,(\lambda+\mu) - 1{,}90986\,\lambda\,\mu\,,$$
$$p(2,6,\lambda,\mu) \approx 0{,}886369\,(\lambda+\mu) - 3{,}28629\,\lambda\,\mu\,,$$
$$p(3,6,\lambda,\mu) \approx 0{,}511745\,(\lambda+\mu) - 0{,}711234\,\lambda\,\mu\,,$$
$$p(4,6,\lambda,\mu) \approx 0{,}889043\,\lambda\,\mu\,,$$
$$p(5,6,\lambda,\mu) \approx 1{,}19863\,\lambda\,\mu\,,$$
$$p(6,6,\lambda,\mu) \approx 0{,}177809\,\lambda\,\mu\,.$$

Corrections to [2]

Page 2: The correct denominators in the formulas of $p(2,3,\lambda,\mu)$ and $p(4,3,\lambda,\mu)$ are 2π and 4π resp. instead of π.

Page 3: In line 8 the formulas of the cases a and b must be exchanged.

In line 18 the correct text for case d is "…, needle 3 does'nt cut any line."

In line 20 "d)" must be replaced by "e)".

In line 25 the formulas of the cases d and e must be exchanged.

The right number of expressions in line 26 is "six" instead of "five".

Page 5: The correct formula for $p(6,5,\lambda,\mu)$ is

$$p(6,5,\lambda,\mu) = \frac{10\left(\sqrt{5}-1\right) - \sqrt{2\left(5+\sqrt{5}\right)}\,\pi}{8\pi}\,\lambda\,\mu\,.$$

DOI: 10.1685/CSC09203

REFERENCES

1. U. Bäsel, Buffon's problem with a star of needles. *VI International Conference "Stochastic geometry, convex bodies, empirical measures and applications to mechanics and engineering of train-transport", Milazzo, 2007.*

2. U. Bäsel, Buffon's problem with a star of needles and a lattice of rectangles. *Fernuniversität Hagen: Seminarberichte aus der Fakultät für Mathematik und Informatik.* **79** (2008) 1-11.

3. A. Duma, M. Stoka, Geometrische Wahrscheinlichkeiten für Hindernisse und nicht-konvexe Testkörper in der Ebene. *Fernuniversität Hagen: Seminarberichte aus der Fakultät für Mathematik und Informatik.* **78** (2007), 37-51.