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Asymptotic behavior of Ginzburg-Landau equations of superfluidity

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Abstract

The asymptotic behavior of the solutions for a non-isothermal model in superfluidity is studied. The model describes the transition between the normal and the superfluid phase in liquid ^{4}He by means of a non-linear differential system, where the concentration of the superfluid phase satisfies a non-isothermal Ginzburg-Landau equation. Starting from an existence and uniqueness result known for this problem, the system is proved to admit a Lyapunov functional. This allows to obtain existence of the global attractor which consists of the unstable manifold of the stationary solutions.

Keywords: `Superfluids, Ginzburg-Landau equations, Global attractor`

1. Introduction.

When the temperature overcomes a critical value of about $2.2K$, liquid helium II undergoes a phase transition between the normal and the superfluid state. This is apparent in the ability of helium to flow through narrow channels with no viscosity.

The phenomenon of superfluidity can be interpreted as a second-order phase transition and accordingly set into the framework of the Ginzburg-Landau theory (see *e.g.* [4,6]). In particular, the order parameter f related to this transition is identified with the concentration of superfluid phase. The other state variables describing the behavior of the superfluid are the velocity $\mathbf{v}_s$ of the superfluid component and the ratio u between the absolute temperature and the transition temperature. The derivation of this model and the interpretation of some physical aspects related to superfluidity can be found in [7].

The evolution of f is ruled by the Ginzburg-Landau equation typical of second order phase transitions (see [6]), *i.e.*

$$\gamma f_t = \frac{1}{\kappa^2}\Delta f - f(f^2 - 1 + u + \mathbf{v}_s^2), \tag{1.1}$$

where γ, κ are positive constants. This equation allows to prove that if $\mathbf{v}_s$ overcomes a threshold value, the unique solution satisfying the conditions

$$\nabla f \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad f(x,0) = f_0(x),$$

is $f = 0$. In other words, there exists a critical velocity above which superfluid properties disappear so that the material is in the normal phase.

In agreement with Landau's viewpoint, each particle of the superfluid is considered as a pair endowed with two different excitations, normal and superfluid, represented respectively by two components $\mathbf{v}_n$ and $\mathbf{v}_s$ of the velocity. The normal component $\mathbf{v}_n$ is supposed to be expressed in terms of the superfluid velocity through the constitutive equation (see [7])

$$\mathbf{v}_n = \nabla \times \mathbf{v}_s.$$

The superfluid component is assumed to solve the equation

$$(\mathbf{v}_s)_t = -\nabla\phi_s - \mu\nabla\times\nabla\times\mathbf{v}_s - f^2\mathbf{v}_s + \nabla u + \mathbf{g}, \tag{1.2}$$

where μ is a positive constant, $\mathbf{g}$ is a known function related to the body force and ϕ_s is a suitable scalar function satisfying

$$\nabla\cdot(f^2\mathbf{v}_s) = -\kappa^2\gamma f^2\phi_s. \tag{1.3}$$

We associate to equations (1.2) and (1.3) the following boundary and initial conditions

$$\mathbf{v}_s\cdot\mathbf{n}|_{\partial\Omega} = 0, \qquad (\nabla\times\mathbf{v}_s)\times\mathbf{n}|_{\partial\Omega} = \boldsymbol{\omega}, \qquad \mathbf{v}_s(x,0) = \mathbf{v}_{s0}(x).$$

We notice that (1.2) and (1.3) are similar to equations ruling the evolution of the velocity of the superconducting electrons in the framework of superconductivity [11], when f^2 is the concentration of the superconducting electrons. The main difference with superconductivity is the occurence of the term ∇u introduced in order to explain the thermomechanical effect. Indeed, since ∇u has the same sign of the acceleration, an increase of the temperature yields a superfluid flow in the direction of the heat flux.

In this model, we assume that the heat flux $\mathbf{q}$ satisfies the constitutive equation

$$\mathbf{q} = -k_0 u \nabla u,$$

namely, $\mathbf{q}$ obeys the Fourier law where the thermal conductivity depends linearly on the temperature. With this choice and by means of the thermal balance law and the first principle of Thermodynamics, it has been proved in [7] that the evolution equation for the temperature is given by

$$c_0 u_t - f f_t = k_0 \Delta u + \nabla \cdot (f^2 \mathbf{v}_s) + r, \tag{1.4}$$

where $c_0 > 0$ is related to the specific heat and r is the heat supply. The temperature is required to verify the boundary and initial conditions

$$u|_{\partial\Omega} = u_b \qquad\qquad u(x,0) = u_0(x)\,.$$

2. Well-posedeness

In order to deal with a more convenient functional setting for the differential equations (1.1)-(1.4), we introduce the following decomposition of the variables $\mathbf{v}_s, \phi_s$

$$\mathbf{v}_s = -\mathbf{A} + \frac{1}{\kappa}\nabla\varphi, \qquad \phi_s = \nabla \cdot \mathbf{A} - \frac{1}{\kappa}\varphi_t, \tag{2.1}$$

where $\mathbf{A}$ and φ satisfy

$$\mathbf{A} \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad\qquad \nabla\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{2.2}$$

and the complex valued function ψ defined as

$$\psi = f e^{i\varphi}.$$

Furthermore, we assume that $\mathbf{g}, r, \boldsymbol{\omega}, u_b$ are time independent.

In addition, we consider the new variables

$$\hat{\mathbf{A}} = \mathbf{A} - \mathbf{A}_{\mathcal{H}}, \qquad \hat{u} = u - u_{\mathcal{H}},$$

where $\mathbf{A}_{\mathcal{H}}$ and $u_{\mathcal{H}}$ are solutions of the problems

$$\begin{cases} \nabla \times \nabla \times \mathbf{A}_{\mathcal{H}} = 0 \\ \nabla \cdot \mathbf{A}_{\mathcal{H}} = 0 \\ \mathbf{A}_{\mathcal{H}} \cdot \mathbf{n}|_{\partial\Omega} = 0 \\ (\nabla \times \mathbf{A}_{\mathcal{H}}) \times \mathbf{n}|_{\partial\Omega} = -\boldsymbol{\omega} \end{cases} \qquad\qquad \begin{cases} \Delta u_{\mathcal{H}} = 0 \\ u_{\mathcal{H}}|_{\partial\Omega} = u_b \end{cases}$$

Accordingly, the differential system can be written in terms of the variables $\psi, \hat{\mathbf{A}}, \hat{u}$ as

$$\gamma\psi_t = \frac{1}{\kappa^2}\Delta\psi - \frac{2i}{\kappa}(\hat{\mathbf{A}} + \mathbf{A}_{\mathcal{H}}) \cdot \nabla\psi - \psi|\hat{\mathbf{A}} + \mathbf{A}_{\mathcal{H}}|^2$$

$$+i\beta(\nabla \cdot \hat{\mathbf{A}})\psi - \psi(|\psi|^2 - 1 + \hat{u} + u_{\mathcal{H}}) \tag{2.3}$$

$$\hat{\mathbf{A}}_t = \nabla(\nabla \cdot \hat{\mathbf{A}}) - \mu\nabla \times \nabla \times \hat{\mathbf{A}} - |\psi|^2(\hat{\mathbf{A}} + \mathbf{A}_{\mathcal{H}}) + \frac{i}{2\kappa}(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi)$$

$$-\nabla\hat{u} - \nabla u_{\mathcal{H}} - \mathbf{g} \tag{2.4}$$

$$c_0\hat{u}_t = \frac{1}{2}(\psi_t\bar{\psi} + \psi\bar{\psi}_t) + k_0\Delta\hat{u}$$

$$+\nabla \cdot \left[-|\psi|^2(\hat{\mathbf{A}} + \mathbf{A}_{\mathcal{H}}) + \frac{i}{2\kappa}(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi)\right] + r \tag{2.5}$$

with homogeneous boundary conditions

$$\hat{\mathbf{A}} \cdot \mathbf{n}|_{\partial\Omega} = 0 \qquad (\nabla \times \hat{\mathbf{A}}) \times \mathbf{n}|_{\partial\Omega} = \mathbf{0} \tag{2.6}$$

$$\nabla\psi \cdot \mathbf{n}|_{\partial\Omega} = 0 \qquad \hat{u}|_{\partial\Omega} = 0 \tag{2.7}$$

and initial data

$$\psi(x,0) = \psi_0(x) \qquad \hat{\mathbf{A}}(x,0) = \hat{\mathbf{A}}_0(x) \qquad \hat{u}(x,0) = \hat{u}_0(x), \tag{2.8}$$

where $\hat{\mathbf{A}}_0(x) = \mathbf{A}_0(x) - \mathbf{A}_{\mathcal{H}}(x)$ and $\hat{u}_0(x) = u_0(x) - u_{\mathcal{H}}(x)$.

We denote by $z = (\psi, \hat{\mathbf{A}}, \hat{u})$ and introduce the functional spaces

$$\mathcal{Z}^1(\Omega) = H^1(\Omega) \times \mathcal{H}^1(\Omega) \times L^2(\Omega),$$
$$\mathcal{Z}^2(\Omega) = H^2(\Omega) \times \mathcal{H}^2(\Omega) \times H^1_0(\Omega),$$

where $L^p(\Omega)$ and $H^p(\Omega)$ are the usual Lebesgue and Sobolev spaces with their standard norms and the spaces

$$\mathcal{H}^1(\Omega) = \left\{\mathbf{w} \in H^1(\Omega): \ \mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = 0\right\},$$
$$\mathcal{H}^2(\Omega) = \left\{\mathbf{w} \in H^2(\Omega): \ \mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ (\nabla \times \mathbf{w}) \times \mathbf{n}|_{\partial\Omega} = \mathbf{0}\right\}$$

endowed respectively with the norms

$$\|\mathbf{w}\|^2_{\mathcal{H}^1} = \|\nabla \cdot \mathbf{w}\|^2 + \|\nabla \times \mathbf{w}\|^2,$$
$$\|\mathbf{w}\|^2_{\mathcal{H}^2} = \|\nabla(\nabla \cdot \mathbf{w})\|^2 + \|\nabla \times \nabla \times \mathbf{w}\|^2.$$

Here and in the sequel the symbol $\|\cdot\|$ stands for the L^2-norm.

Theorem 2.1. *Let $z_0 = (\psi_0, \hat{\mathbf{A}}_0, \hat{u}_0) \in \mathcal{Z}^1(\Omega)$, $\mathbf{A}_{\mathcal{H}} \in \mathcal{H}^1(\Omega)$, $u_{\mathcal{H}} \in H^1(\Omega)$, $\mathbf{g}, r \in L^2(\Omega)$. Then, for every $T > 0$, there exists a unique solution z of the* *problem (2.3)-(2.8) such that*

$$\begin{aligned}\psi &\in L^2(0,T,H^1(\Omega)) \cap H^1(0,T,L^2(\Omega))\\ \hat{\mathbf{A}} &\in L^2(0,T,\mathcal{H}^1(\Omega)) \cap H^1(0,T,L^2(\Omega))\\ \hat{u} &\in L^2(0,T,H^1_0(\Omega)) \cap H^1(0,T,H^{-1}(\Omega)).\end{aligned}$$

Moreover $\psi \in L^2(0,T,H^2(\Omega)) \cap C(0,T,H^1(\Omega))$, $\hat{\mathbf{A}} \in L^2(0,T,\ \mathcal{H}^2(\Omega)) \cap C(0,T,\ \mathcal{H}^1(\Omega))$, $\hat{u} \in C(0,T,L^2(\Omega))$.

The proof of this result is given in [2].

In order to study the long-time behavior of the solution to problem (2.3)-(2.8), further assumptions on the sources are required. In particular, we assume that $\mathbf{g} \in H^1(\Omega)$ and

$$\nabla \cdot \mathbf{g} = 0, \qquad\qquad r = 0.$$

Moreover, since $\Delta u_{\mathcal{H}} = 0$, there exists a vector-valued function $\mathbf{G}$ such that

$$\nabla u_{\mathcal{H}} + \mathbf{g} = \nabla \times \mathbf{G}, \qquad \mathbf{G} \times \mathbf{n}|_{\partial\Omega} = \mathbf{0}.$$

Theorems 2.1 ensures that (2.3)-(2.8) generate a strongly continuous semigroup $S(t)$ on the phase space $\mathcal{Z}^1(\Omega)$ (see *e.g.* [10]), provived that the solution of the problem depends continuously on the initial data. To this aim, we need to prove some a-priori estimates of the solutions (see [1]).

Lemma 2.1. *The solution of (2.3)-(2.8) with initial datum* $z_0 \in \mathcal{Z}^1(\Omega)$ *such that* $\|z_0\|_{\mathcal{Z}^1} \le R$*, satisfies the following a-priori estimates*

$$\sup_{t\ge 0}(\|\psi(t)\|_{H^1} + \|\hat{\mathbf{A}}(t)\|_{\mathcal{H}^1} + \|\hat{u}(t)\|) \le C_R, \tag{2.9}$$

$$\sup_{t\ge 0}\int_0^t \left(\|\psi_t\|^2 + \|\hat{\mathbf{A}}_t\|^2\right) ds \le C_R, \tag{2.10}$$

$$\int_0^t \left(\|\psi\|^2_{H^2} + \|\hat{\mathbf{A}}\|^2_{\mathcal{H}^2} + \|\hat{u}\|^2_{H^1_0}\right) ds \le C_R(1+t), \qquad t>0, \tag{2.11}$$

$$\int_t^{t+1} \left(\|\psi\|^2_{H^2} + \|\hat{\mathbf{A}}\|^2_{\mathcal{H}^2} + \|\hat{u}\|^2_{H^1_0}\right) ds \le C_R, \qquad t>0, \tag{2.12}$$

where C_R *depends increasingly on* R*.*

The previous lemma allows to prove the continuous dependence of the solutions to (2.3)-(2.8) on the initial data.

Theorem 2.2. *Let* $z_i = (\psi_i, \hat{\mathbf{A}}_i, \hat{u}_i)$*,* $i = 1, 2$ *be two solutions of (2.3)-(2.8) with data* $(\mathbf{A}_{\mathcal{H}}, u_{\mathcal{H}}, \mathbf{G}) \in \mathcal{H}^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ *and* $z_{0i} = (\psi_{0i}, \mathbf{A}_{0i}, u_{0i}) \in \mathcal{Z}^1(\Omega)$*,* $i = 1, 2$*. Then, there exists a constant* C_R *such that*

$$\|z_1(t) - z_2(t)\|^2_{\mathcal{Z}^1} \leq C_R e^{C_R t} \|z_{01} - z_{02}\|^2_{\mathcal{Z}^1}.$$

Moreover, inequality

$$\int_0^t \|z_1(t) - z_2(t)\|^2_{\mathcal{Z}^2} \, ds \leq C(t) \|z_{01} - z_{02}\|^2_{\mathcal{Z}^1} \tag{2.13}$$

holds, where $C(t)$ *is a suitable function depending on* t*.*

In the sequel, we denote by $S(t)z_0$ the unique solution to (2.3)-(2.7) with initial datum z_0, *i.e.* $S(t)z_0 = z(t), z(0) = z_0$.

3. The global attractor

In this section we prove existence of the global attractor for the semigroup $S(t)$ and of an absorging set, whose definitions are given below.

Definition 3.1. The global attractor $\mathcal{A} \subset \mathcal{Z}^1(\Omega)$ is the unique compact set which is

(i) fully invariant, *i.e.*

$$S(t)\mathcal{A} = \mathcal{A}, \qquad \forall t \geq 0;$$

(ii) attracting, *i.e.*

$$\lim_{t \to \infty} \text{dist}_{\mathcal{Z}^1}(S(t)\mathcal{B}, \mathcal{A}) = 0$$

for every bounded set $\mathcal{B} \subset \mathcal{Z}^1(\Omega)$, where $\text{dist}_{\mathcal{Z}^1}$ denotes the usual Hausdorff semidistance in $\mathcal{Z}^1(\Omega)$.

Definition 3.2. A bounded subset $\mathcal{B}_1 \subset \mathcal{Z}^1(\Omega)$ is an absorbing set for the semigroup $S(t)$ if for every bounded set $\mathcal{B} \subset \mathcal{Z}^1(\Omega)$ there exists $t_1(\mathcal{B})$ such that

$$S(t)\mathcal{B} \subset \mathcal{B}_1 \qquad \forall t \geq t_1(\mathcal{B}).$$

Usually, the first step to prove existence of the global attractor consists in showing that the semigroup admits a bounded absorbing set and that the operators $S(t)$ are uniformly compact for large values of t ([10, Theor. 1.1]). This approach has been followed in [3] in the framework of superconductivity where the authors analyzed the asymptotic behavior of the solutions by taking advantage of a maximum theorem which guarantees the bound $f^2 \leq 1$ almost everywhere. Here, we are unable to obtain directly the estimates that guarantee the dissipativity of the semigroup, due to the presence of the absolute temperature. More precisely, the boundness of $|\psi|$ could be established by exploiting the positivity of the absolute temperature. Unfortunately, even if from a physical point of view $u > 0$, such a bound cannot be proved a-priori from equations (2.3)-(2.5).

Thus we adopt a different strategy showing that the semigroup $S(t)$ possesses a global attractor by means of a Lyapunov functional.

We denote by $\mathcal{S}$ the set of stationary solutions of problem (2.3)-(2.7), namely every steady solution satisfies

$$0 = \frac{1}{\kappa^2}\Delta\psi - \frac{2i}{\kappa}(\hat{\mathbf{A}} + \mathbf{A}_{\mathcal{H}}) \cdot \nabla\psi - \psi|\hat{\mathbf{A}} + \mathbf{A}_{\mathcal{H}}|^2 + i\beta(\nabla \cdot \hat{\mathbf{A}})\psi$$

$$-\psi(|\psi|^2 - 1 + \hat{u} + u_{\mathcal{H}}) \tag{3.1}$$

$$\mathbf{0} = \nabla(\nabla \cdot \hat{\mathbf{A}}) - \mu\nabla \times \nabla \times \hat{\mathbf{A}} - |\psi|^2(\hat{\mathbf{A}} + \mathbf{A}_{\mathcal{H}}) + \frac{i}{2\kappa}(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi)$$

$$-\nabla\hat{u} - \nabla \times \mathbf{G} \tag{3.2}$$

$$0 = k_0\Delta\hat{u} + \nabla \cdot \left[-|\psi|^2(\hat{\mathbf{A}} + \mathbf{A}_{\mathcal{H}}) + \frac{i}{2\kappa}(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi)\right] \tag{3.3}$$

$$\hat{\mathbf{A}} \cdot \mathbf{n}|_{\partial\Omega} = 0 \qquad (\nabla \times \hat{\mathbf{A}}) \times \mathbf{n}|_{\partial\Omega} = \mathbf{0} \tag{3.4}$$

$$\nabla\psi \cdot \mathbf{n}|_{\partial\Omega} = 0 \qquad \hat{u}|_{\partial\Omega} = 0 \tag{3.5}$$

Definition 3.3. A continuous function $\mathcal{L} : \mathcal{Z}^1(\Omega) \to \mathbb{R}$ is said a Lyapunov functional if

(i) $t \to \mathcal{L}(S(t)z)$ is non-increasing for any $z \in \mathcal{Z}^1(\Omega)$;
(ii) $\mathcal{L}(z) \to \infty \Leftrightarrow \|z\|_{\mathcal{Z}^1} \to \infty$;
(iii) $\mathcal{L}(S(t)z) = \mathcal{L}(z),\ \forall t > 0 \Rightarrow z \in \mathcal{S}$.

Prop 3.1. The function

$$\mathcal{L}(\psi, \hat{\mathbf{A}}, \hat{u}) = \frac{1}{2}\int_\Omega \Bigg\{ \left|\frac{i}{\kappa}\nabla\psi + \psi\hat{\mathbf{A}} + \psi\mathbf{A}_{\mathcal{H}}\right|^2 + \frac{1}{2}(|\psi|^2 - 1)^2 + |\psi|^2 u_{\mathcal{H}}$$

$$+\mu|\nabla \times \hat{\mathbf{A}}|^2 + \eta(\nabla \cdot \hat{\mathbf{A}})^2 + 2\nabla \times \mathbf{G} \cdot \hat{\mathbf{A}} + c_0\hat{u}^2 \Bigg\} dv,$$

where $\eta = 2k_0/(k_0 + 1)$, is a Lyapunov functional.

Proof. By means of (2.3)-(2.7), a direct check leads to the inequality

$$\frac{d\mathcal{L}}{dt} = \int_\Omega \left[-\gamma|\psi_t - i\kappa\psi\nabla \cdot \hat{\mathbf{A}}|^2 - q(\hat{\mathbf{A}}_t, \nabla(\nabla \cdot \hat{\mathbf{A}}), \nabla\hat{u})\right] dv, \tag{3.6}$$

where

$$q(\hat{\mathbf{A}}_t, \nabla(\nabla\cdot\hat{\mathbf{A}}), \nabla\hat{u}) = |\hat{\mathbf{A}}_t|^2 + |\nabla(\nabla\cdot\hat{\mathbf{A}})|^2 + (k_0+1)|\nabla\hat{u}|^2 + 2\hat{\mathbf{A}}_t\cdot\nabla\hat{u} \\ + (\eta-2)\hat{\mathbf{A}}_t\cdot\nabla(\nabla\cdot\hat{\mathbf{A}}) - 2\nabla(\nabla\cdot\hat{\mathbf{A}})\cdot\nabla\hat{u}.$$

Since $\eta = 2k_0/(k_0+1)$, q is a positive definite quadratic form. Therefore $\mathcal{L}$ is non-increasing.

Moreover, by letting

$$\mathcal{F}_1(\psi,\hat{\mathbf{A}},\hat{u}) = \left\| \frac{i}{\kappa}\nabla\psi + \psi\hat{\mathbf{A}} + \psi\mathbf{A}_{\mathcal{H}} \right\|^2 + \frac{1}{2}\|\,|\psi|^2 - 1\,\|^2 + \int_\Omega |\psi|^2 u_{\mathcal{H}} dv \\ + \|\hat{\mathbf{A}}\|^2_{\mathcal{H}_1} + \|\hat{u}\|^2,$$

the following inequalities hold

$$\begin{aligned} &c_1\mathcal{F}_1 - c_2 \le \mathcal{L} \le c_3\mathcal{F}_1 + c_2 \\ &\|z\|^2_{\mathcal{Z}^1} \le C\left[1 + \mathcal{F}_1(z) + \mathcal{F}_1^2(z) + \mathcal{F}_1^3(z)\right] \\ &\mathcal{F}_1(z(t)) \le C(1 + \|z(t)\|^2_{\mathcal{Z}^1} + \|z(t)\|^4_{\mathcal{Z}^1}). \end{aligned}$$

Hence, we deduce that

$$\mathcal{L}(z)\to\infty \quad \Leftrightarrow \quad \mathcal{F}_1(z)\to\infty \quad \Leftrightarrow \quad \|z\|_{\mathcal{Z}^1}\to\infty.$$

Finally, if $\mathcal{L}(S(t)z) = \mathcal{L}(z)$ for every $t>0$, then from (3.6) and the positive definitess of q we deduce that

$$\psi_t - i\kappa\psi\nabla\cdot\hat{\mathbf{A}} = 0 \tag{3.7}$$

$$\nabla(\nabla\cdot\hat{\mathbf{A}}) = \mathbf{0} \tag{3.8}$$

$$\nabla\hat{u} = \mathbf{0} \tag{3.9}$$

$$\hat{\mathbf{A}}_t = \mathbf{0} \tag{3.10}$$

Hence $\nabla\cdot\hat{\mathbf{A}} = 0$, which, in view of (3.7) implies $\psi_t = 0$. By substituting the previous relations into (2.4) and (2.5), we obtain $\hat{u}_t = 0$. Thus, $z\in\mathcal{S}$□

Existence of the global attractor is based on the following abstract result (see *e.g.* [9]).

Lemma 3.1. *Let the semigroup $S(t)$, $t>0$ satisfy the following conditions*

- *(a) $S(t)$ admits a continuous Lyapunov functional $\mathcal{L}$;*
- *(b) the set $\mathcal{S}$ of the stationary solutions is bounded in $\mathcal{Z}^1(\Omega)$;*
- *(c) for any bounded set $\mathcal{B}\subset\mathcal{Z}^1(\Omega)$, there exists a compact set $\mathcal{K}_\mathcal{B}\subset\mathcal{Z}^1(\Omega)$ such that $S(t)\mathcal{B}\subset\mathcal{K}_\mathcal{B}$, $t>0$.*

Then, $S(t)$ possesses a connected global attractor $\mathcal{A}$ which coincides with the unstable manifold of $\mathcal{S}$, namely

$$\mathcal{A} = \{z \in \mathcal{Z}^1(\Omega) : z \text{ belongs to a complete trajectory } S(t)z,\ t \in \mathbb{R}, \lim_{t\to-\infty} \mathrm{dist}_{\mathcal{Z}^1}(S(t)z, \mathcal{S}) = 0\}.$$

Theorem 3.1. *The semigroup $S(t)$ possesses a connected global attractor $\mathcal{A} \subset \mathcal{Z}^1(\Omega)$. In addition, $S(t)$ possesses a bounded absorbing set $\mathcal{B}_1 \subset \mathcal{Z}^1(\Omega)$ of radius*

$$R_1 = 1 + \sup\{\|z\|_{\mathcal{Z}^1},\ \mathcal{L}(z) \le K\}, \tag{3.11}$$

where $K = 1 + \sup_{z\in\mathcal{S}} \mathcal{L}(z)$.

Proof. Existence of the global attractor for the semigroup $S(t)$ is established once we prove conditions (a), (b), (c) of Lemma 3.1.

Proposition 3.1 assures that $S(t)$ satisfies (a). Let $z \in \mathcal{S}$. Thus, $\frac{d\mathcal{L}}{dt} = 0$ and, in view of the positive definiteness of q,

$$\nabla \hat{u} = \mathbf{0}, \qquad \nabla(\nabla \cdot \hat{\mathbf{A}}) = \mathbf{0}.$$

Moreover, (2.6)-(2.7) lead to

$$\hat{u} = 0, \qquad \nabla \cdot \hat{\mathbf{A}} = 0.$$

By multiplying (3.1) by $1/2\bar{\psi}$, its conjugate by $1/2\psi$, (3.2) by $\hat{\mathbf{A}}$ and integrating over Ω, we deduce

$$\|\psi\|_{H^1} \le C, \qquad \|\hat{\mathbf{A}}\|_{\mathcal{H}^1} \le C,$$

namely, condition (b).

Now we prove condition (c). Let $S(t)z$, $t > 0$, be a solution to problem (2.3)-(2.8) with initial datum $z \in \mathcal{Z}^1(\Omega)$ such that $\|z\|_{\mathcal{Z}^1} \le R$. In view of the compact embedding $\mathcal{Z}^2(\Omega) \hookrightarrow \mathcal{Z}^1(\Omega)$, our goal consists in proving the existence of a positive constant C_R depending on R, $\mathbf{A}_{\mathcal{H}}, u_{\mathcal{H}}, \mathbf{G}$ such that

$$\|z(t)\|_{\mathcal{Z}^2} \le C_R, \qquad t > 0. \tag{3.12}$$

Let us multiply (2.3) by $1/2\,\Delta\bar{\psi}_t$ and its conjugate by $1/2\,\Delta\psi_t$. Adding the resulting equations and integrating over Ω, thanks to the boundary condition $(2.7)_1$, we obtain

$$\frac{1}{2\kappa^2}\frac{d}{dt}\|\Delta\psi\|^2 + \gamma\|\nabla\psi_t\|^2 \le 4\varepsilon\|\nabla\psi_t\|^2 + C_R\Big[\|\hat{\mathbf{A}} + \mathbf{A}_{\mathcal{H}}\|^2_{\mathcal{H}^2}\|\psi\|^2_{H^2} + \|\hat{\mathbf{A}} + \mathbf{A}_{\mathcal{H}}\|^4_{\mathcal{H}^2} + \|\psi\|^4_{H^2} + \|\psi\|^2_{H^2}\|\hat{u}\|^2_{H^1_0} + 1\Big], \tag{3.13}$$

where ε is a suitable constant.

Multiplying (2.4) by $\nabla\times\nabla\times\hat{\mathbf{A}}_t$ and keeping $(2.6)_2$ into account, an integration over Ω provides

$$\begin{aligned}&\frac{\mu}{2}\frac{d}{dt}\|\nabla\times\nabla\times\hat{\mathbf{A}}\|^2+\|\nabla\times\hat{\mathbf{A}}_t\|^2\\ &\le 3\varepsilon\|\nabla\times\hat{\mathbf{A}}_t\|^2+C_R\left(\|\psi\|^4_{H^2}+\|\hat{\mathbf{A}}+\mathbf{A}_{\mathcal{H}}\|^2_{\mathcal{H}^2}+1\right).\end{aligned}\tag{3.14}$$

Let us multiply (2.4) by $\nabla(\nabla\cdot\hat{\mathbf{A}}_t)-\nabla\hat{u}_t$. An integration by parts and boundary conditions (2.6)-(2.7) lead to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla(\nabla\cdot\hat{\mathbf{A}})-\nabla\hat{u}-\nabla\times\mathbf{G}\|^2+\|\nabla\cdot\hat{\mathbf{A}}_t\|^2\\ &\quad\le 3\varepsilon\|\nabla\cdot\hat{\mathbf{A}}_t\|^2+\left(\varepsilon+\frac{1}{4\varepsilon}\right)\|\hat{u}_t\|^2+\\ &\qquad+C_R\left[1+\|\hat{\mathbf{A}}+\mathbf{A}_{\mathcal{H}}\|^2_{\mathcal{H}^2}+\|\psi\|^4_{H^2}\right].\end{aligned}\tag{3.15}$$

Finally, let us multiply (2.5) by $1/(2\varepsilon c_0)\hat{u}_t$ and integrate over Ω, thus obtaining

$$\begin{aligned}&\frac{k_0}{4\varepsilon c_0}\frac{d}{dt}\|\nabla\hat{u}\|^2+\frac{1}{2\varepsilon}\|\hat{u}_t\|^2\\ &\le\frac{1}{c_0}\|\hat{u}_t\|^2+C_R\left[\|\psi\|^2_{H^2}\|\psi_t\|^2+\|\psi\|^2_{H^2}+\|\hat{\mathbf{A}}+\mathbf{A}_{\mathcal{H}}\|^2_{\mathcal{H}^2}\right].\end{aligned}\tag{3.16}$$

By adding inequalities (3.13)-(3.16) and choosing properly ε, we obtain

$$\frac{d}{dt}\mathcal{F}_2\le\xi(t)\mathcal{F}_2+\xi(t),\tag{3.17}$$

where

$$\begin{aligned}\mathcal{F}_2&=\frac{1}{\kappa^2}\|\Delta\psi\|^2+\mu\|\nabla\times\nabla\times\hat{\mathbf{A}}\|^2+\|\nabla(\nabla\cdot\hat{\mathbf{A}})-\nabla\hat{u}-\nabla\times\mathbf{G}\|^2\\ &\quad+\frac{k_0}{2\varepsilon c_0}\|\nabla\hat{u}\|^2\\ \xi(t)&=C_R\left[1+\|\psi\|^2_{H^2}+\|\psi_t\|^2+\|\hat{\mathbf{A}}\|^2_{\mathcal{H}^2}\right].\end{aligned}$$

A-priori estimates (2.9)-(2.11) and Gronwall's uniform lemma (see [10]) guarantee that $\mathcal{F}_2$ is bounded. Thus, $\|z(t)\|_{\mathcal{Z}^2}<C_R$ and existence of a connected global attractor is shown. As a consequence, (see [5]), $S(t)$ possesses a bounded absorbing set $\mathcal{B}_1\subset\mathcal{Z}^1(\Omega)$ of radius given in (3.11) and the proof of the theorem is completed. □

By exploiting the semigroup properties and the previous theorem, it is easy to prove the following result.

Corollary 3.1. *The semigroup* $S(t)$ *possesses a bounded absorbing set* $\mathcal{B}_2 \subset \mathcal{Z}^2(\Omega)$ *of radius* R_2.

The global attractor uniformly attracts every bounded set of $\mathcal{Z}^1(\Omega)$. However, it could attract the trajectories in an arbitrary long time and consequentlty it can be sensitive to perturbations. In order to overcome this drawback one proves the existence of a regular set which attracts exponentially fast bounded subsets of $\mathcal{Z}^1(\Omega)$. To this aim, we recall the following definition.

Definition 3.4. A compact subset $\mathcal{E} \subset \mathcal{Z}^2(\Omega)$ of finite fractal dimension is an exponential attractor for the semigroup $S(t)$ if

(i) $\mathcal{E}$ is positively invariant, *i.e.* $S(t)\mathcal{E} \subset \mathcal{E}$ for all $t \geq 0$;
(ii) there exist $\omega > 0$ and a positive increasing function J such that

$$\mathrm{dist}_{\mathcal{Z}^1}(S(t)\mathcal{B}, \mathcal{E}) \leq J(R)e^{-\omega t}$$

for any bounded $\mathcal{B} \subset \mathcal{Z}^1(\Omega)$ with $R = \sup\{\|z\|_{\mathcal{Z}^1(\Omega)}, z \in \mathcal{B}\}$.

By means of some techniques devised in [8], it has been shown the following result (see [1]).

Theorem 3.2. *The semigroup* $S(t)$ *possesses an exponential attractor* $\mathcal{E} \subset \mathcal{Z}^2(\Omega)$.

Since the global attractor is the minimal compact attracting set, we have $\mathcal{A} \subset \mathcal{E}$. Thus, Theorem 3.2 gives as a byproduct the finite fractal dimension of the global attractor.

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