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Existence of three solutions for a Neumann Boundary Value Problem

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Abstract

The mathematical modeling of important questions are often governed by nonlinear differential problems. In this paper, existence results of three solutions for a Neumann boundary value problem are established by using a very recent three critical points theorem.

Keywords: `Neumann problem, multiple solutions.`

1. Introduction.

In different fields of research, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, economics and many others, the mathematical modeling of important questions leads naturally to consider nonlinear differential boundary value problems. For instance, a steady state temperature distribution in a rod (identified with a closed interval), or a semi-infinity porous medium initially filled with gas at a uniform pressure, are governed by nonlinear second order differential equations with suitable boundary conditions, as in particular, Neumann boundary conditions.
In this paper, we study the following Neumann problem:

$$(GS) \qquad \begin{cases} -(\overline{p}u')' + \overline{r}u' + \overline{q}u = \lambda g(x,u) \\ u'(0) = u'(1) = 0, \end{cases}$$

where $\overline{p} \in C^1([0,1])$, $\overline{q}, \overline{r} \in C^0([0,1])$, with $\overline{p}$ and $\overline{q}$ positive functions, and λ is a positive real parameter.
The aim of this paper is to establish three solutions for the Neumann boundary value problem (GS).

Multiple solutions for Neumann Problems have been obtained in several papers, as [1], [3], [6], [11], [12], and references therein.

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Our pricipal result, Theorem 2.1 improves Theorem 3.3 of [2]. Moreover a consequence is established (Theorem 3.1) and a special case is presented (Theorem 2.2). Our main tool is a very recent theorem of three critical points obtained by Bonanno and Marano in [5], that here we recall.

Theorem 1.1. *(Theorem 2.6 of [5]) Let X be a reflexive real Banach space, let $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous and continuously Gâteaux differentiable functional whose Gâteaux derivative is compact.*
Assume that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$ and there exists $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that

(j) $\displaystyle\sup_{\Phi(x)<r} \Psi(x) < r\frac{\Psi(\bar{x})}{\Phi(\bar{x})}$

(jj) for each $\lambda \in \Lambda_r := \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)<r}\Psi(x)}\right[$, *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

2. Main result

Consider the following problem

$$(S) \qquad \begin{cases} -(pu')' + qu = \lambda f(x,u) \\ u'(0) = u'(1) = 0 \end{cases}$$

where $p \in C^1([0,1])$, $q \in C^0([0,1])$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and λ is a positive real parameter.

Put $F(x,t) = \displaystyle\int_0^t f(x,\xi)d\xi$ for all $(x,t) \in [0,1] \times \mathbb{R}$,

$$p_0 = \min_{[0,1]} p(x) > 0,\ q_0 = \min_{[0,1]} q(x) > 0,$$

$$(1) \qquad m = \min\{p_0, q_0\}, \quad k = \frac{m}{\|q\|_1}$$

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where, as usual, $\|q\|_1 = \int_0^1 q(x)dx$.

Now, we point out our main result.

Theorem 2.1. *Assume that:*

(*i*) *there exist two positive constants c, d, with $c < d$, such that:*

$$\frac{\int_0^1 \max_{t\in[-c,c]} F(x,t)dx}{c^2} < \frac{k}{2}\frac{\int_0^1 F(x,d)dx}{d^2}$$

when k is given by (1);

(*ii*) *there exist two positive constants a, s, with $s < 2$, such that:*

$$F(x,t) \le a(1 + |t|^s)$$

for all $(x,t) \in [0,1]\times\mathbb{R}$.

Then, for each $\lambda \in \left]\frac{d^2\|q\|_1}{2\int_0^1 F(x,d)dx}, \frac{mc^2}{4\int_0^1 \max_{t\in[-c,c]} F(x,t)dx}\right[$, *problem* (S) *admits at least three classical solutions.*

Proof. Take as X the Sobolev space $W^{1,2}([0,1])$ endowed with the norm

$$\|u\| = \left(\int_0^1 p(x)|u'(x)|^2dx + \int_0^1 q(x)|u(x)|^2dx\right)^{\frac{1}{2}}$$

which is equivalent to the usual one.
Put

$$\Phi(u) := \frac{1}{2}\|u\|^2, \qquad \Psi(u) := \int_0^1 F(x,u(x))dx$$

for each $u \in X$.

Fix λ as in the conclusion. Since the critical points of the functional $\Phi - \lambda\Psi$ in X are exactly the classical solutions of the problem (S), in order to obtain our assertion, it's enough to apply Theorem 1.1 to the functionals Φ and Ψ.
Clearly, regularity assumptions on Φ and Ψ requested in Theorem 1.1 are satisfied.

Moreover, hypothesis (jj) of Theorem 1.1 follows easily from (ii).

In order to prove (j), put $r = m\left(\frac{c}{2}\right)^2$.
Taking into account that

$$|u(t)| \leq \sqrt{\frac{2}{m}}\,\|u\| \qquad \text{for all } u \in X$$

we obtain

$$\sup_{\Phi(u)\leq r} \Psi(u) = \sup_{\Phi(u)\leq r} \int_0^1 F(x,u(x))dx \leq$$

$$\leq \int_0^1 \max_{|\xi|\leq 2\sqrt{\frac{r}{m}}} F(x,\xi)dx = \int_0^1 \max_{t\in[-c,c]} F(x,t)dx.$$

Now, fix $u_1 = d$. Clearly, $u_1 \in X$ and one has

$$\Psi(u_1) = \int_0^1 F(x,u_1(x))dx = \int_0^1 F(x,d)dx.$$

$$\Phi(u_1) = \frac{1}{2}\|u_1\|^2 = \frac{1}{2}\int_0^1 p(t)|u_1'(t)|^2dt + \frac{1}{2}\int_0^1 q(t)|u_1(t)|^2dt = \frac{d^2}{2}\|q\|_{L_1}.$$

Since $c < d$, one has $0 < c < \sqrt{\frac{\|q\|_1}{m}}d$, that is $r < \Phi(u_1)$. Therefore,

$$\frac{\sup\limits_{\Phi(u)\leq r} \Psi(u)}{r} \leq \frac{\int_0^1 \max\limits_{t\in[-c,c]} F(x,t)dx}{m\left(\frac{c}{2}\right)^2} \tag{2}$$

and

$$\frac{\Psi(u_1)}{\Phi(u_1)} = \frac{\int_0^1 F(x,d)dx}{\frac{d^2}{2}\|q\|_1}. \tag{3}$$

Hence, from (i) we have

$$\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} < \frac{\Psi(u_1)}{\Phi(u_1)}$$

and (e_1) of Theorem 1.1 is satisfied.
From Theorem 1.1, for each $\lambda \in \left(\frac{\Phi(u_1)}{\Psi(u_1)}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right)$, the functional $\Phi-\lambda\Psi$ has at least three distinct critical points which are the classical solutions of (S) and the conclusion is achieved. $\square$

If $p = q = 1$, we have the following problem

$$(PS) \qquad \begin{cases} -u'' + u = \lambda f(x,u) \\ u'(0) = u'(1) = 0. \end{cases}$$

Under the same assumptions on f, Theorem 2.1 assumes the following form.

Theorem 2.2. *Assume that:*

(i) *there exist two positive constants c, d, with $c < d$, such that:*

$$\frac{\int_0^1 \max_{t\in[-c,c]} F(x,t)dx}{c^2} < \frac{1}{2}\frac{\int_0^1 F(x,d)dx}{d^2}$$

(ii) *there exist two positive constants a, s, with $s < 2$, such that:*

$$F(x,t) \le a(1 + |t|^s)$$

for all $(x,t) \in [0,1] \times \mathbb{R}$.

Then, for each $\lambda \in \left(\frac{d^2}{2\int_0^1 F(x,d)dx}, \frac{c^2}{4\int_0^1 \max_{t\in[-c,c]} F(x,t)dx} \right)$, *problem (PS) admits at least three classical solutions.*

3. Three Solutions for the complete problem

Consider the problem (GS)

$$\begin{cases} -(\overline{p}u')' + \overline{r}u' + \overline{q}u = \lambda g(x,u), \\ u'(0) = u'(1) = 0. \end{cases}$$

where $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\overline{p} \in C^1([0,1])$, $\overline{q}, \overline{r} \in C^0([0,1])$, and λ is a positive parameter. Moreover $\overline{p}$ and $\overline{q}$ are positive functions. Put

$$G(x,t)=\int_0^t g(x,\xi)d\xi \quad \text{for all } (x,t)\in[0,1]\times\mathbb{R}$$

$$k'=\frac{m'}{\|e^{-R}\overline{q}\|_1}, \qquad m'=\min\{\min_{[0,1]} e^{-R}\overline{p}, \min_{[0,1]} e^{-R}\overline{q}\} \tag{4}$$

where R is a primitive of $\dfrac{\overline{r}}{\overline{p}}$.

Here, we give the following result of three solutions for the Neumann problem (GS).

Theorem 3.1. *Assume that:*

(i') *there exist two positive constants c,d, with $c<d$, such that:*

$$\frac{\int_0^1 \max_{t\in[-c,c]} e^{-R}G(x,t)dx}{c^2} < \frac{k'}{2}\frac{\int_0^1 e^{-R}G(x,d)dx}{d^2}$$

with k' given by (4);

(ii') *there exist two positive constants a,s, with $s<2$, such that:*

$$G(x,t)\le a(1+|t|^s)$$

for all $(x,t)\in[0,1]\times\mathbb{R}$.

Then, for each $\lambda\in\left(\dfrac{d^2\|e^{-R}\overline{q}\|_1}{2\int_0^1 e^{-R}G(x,d)dx}, \dfrac{m'c^2}{4\int_0^1 \max_{t\in[-c,c]} e^{-R}G(x,t)dx}\right)$, *problem (GS) admits at least three classical solutions.*

Proof. Since the solutions of the problem

$$\begin{cases} -(e^{-R}\overline{p}u')'+e^{-R}\overline{q}u=\lambda e^{-R}g(x,u) \\ u'(0)=u'(1)=0 \end{cases}$$

are solutions of the problem (GS), from Theorem 2.1 the conclusion follows. □

Now we present an example of application of Theorem 3.1.

Example 3.1. *The problem*

$$\begin{cases} -u''+u'+u=2\lambda x u^{10}(11-u) \\ \\ u'(0)=u'(1)=0 \end{cases} \tag{5}$$

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admits at least three solutions for each $\lambda \in \left(\frac{3}{2^9 5}, \frac{3}{2(e-2)}\right)$.
In fact, if we choose, for instance, $c = 1$ *and* $d = 2$*, hypotheses of Theorem 2.1 are satisfied.*

Remark 3.1. Theorem 2.1 improves Theorem 3.3 of [2], and Theorem 3.1 improves Corollary 4.2 in [2] since conditions (i) and (i') in our results are more general than conditions (i) and (i') in Theorem 3.3 and Corollary 4.2 respectively in [2].
In the previous example, we obtain a larger interval of parameter λ than the interval insured by Corollary 4.2 in [2].

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