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ON SOURCE TERMS IN MULTILANE TRAFFIC MODELS

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Abstract.

We propose in this paper some source terms for continuum models of multilane traffic flow, aiming at modeling entries, exits or inhomogeneities of the road. Nonlocal terms are considered as well. Rigorous results are given.

Keywords: Continuum traffic flow models, Balance Laws, Temple Systems.

1. Introduction

Many continuum models for multilane traffic flows are based on systems of conservation laws with source terms. Consider for instance the case of flows along homogeneous roads without entries or exits. In each equation the convective part describes the *intra–lane* dynamics, while the right hand side models the *inter-lane* coupling between adjacent lanes. In the simplest case the convective term is scalar, and it is given for instance by a LWR (Lighthill-Whitham[23] and Richards[24]) equation; the related multilane model was proposed by Herty and Klar[17]. More generally, the dynamics is each lane can be usefully described by a *pair* of equations; the flows can be given either through the Aw-Rascle[1,17,21,22] model, or through the Colombo[4,8] model. A combination of scalar and vector flows is also possible (for lanes with light or heavy traffic, respectively)[8], as well as models with phase transitions[5].

In addition to the papers quoted above about multilane flows we quote also those in Refs. 15,19. For general surveys about mathematical models for traffic flows the reader is referred to Refs. 20,14. However, we point out that most of the papers quoted above focus either on the introduction and the derivation of a particular model or on its numerical simulations. Rigorous analytical results as those of the authors[8] are less common[10,12].

We provide a unique framework to describe most of the models above. In each of these models, the dynamics of the lane ℓ is described by

$$\partial_t u_\ell + \partial_x \left[f_\ell(u_\ell)\right] = 0\,, \tag{1}$$

independently from the other lanes. In (1), the vectors of the conserved quantities u_ℓ

may either reduce to the only vehicle density or have two (or more) components. The interaction occurring among different lanes and the necessity of modelling entries, exits or, more generally, inhomogeneities of the road, lead to consider the system

$$\partial_t u_\ell + \partial_x \left[f_\ell(u_\ell)\right] = g_\ell(t,x,u) \qquad \ell = 1,\dots n\,. \tag{2}$$

For flows in homogeneous roads without entries or exits the term g_ℓ usually depend on u only through $u_{\ell-1}$, u_ℓ, $u_{\ell+1}$. If u_ℓ represents the vehicle density in lane ℓ, then the conservation of the number of vehicles is obtained by

$$\sum_{\ell=1}^{n} g_\ell = 0\,. \tag{3}$$

The conservation of the momentum variable is not required: in the literature[2] suitable source terms in the momentum equation simulate specific patterns of realistic situations.

Below, we first state a result by the authors[8] on the well posedness of the initial value problem for (2). A key assumption is that all conservation laws (1) are of Temple type, which is satisfied in all the models above. Then we consider additional source terms as in the single-lane case[2], taking into account the presence of inhomogeneities in the road. We consider as well nonlocal terms, modelled by convolutions, and make use of recent analytical results[9].

2. Main Results

Let Ω_ℓ be a closed connected subset of $\mathbb{R}^{n_\ell}$ and consider system (2), where $u_\ell = (u_{l,1},\dots,u_{\ell,n_l}) \in \Omega_\ell$. Define $N = \sum_{\ell=1}^{n} n_\ell$ and $\Omega = \prod_{\ell=1}^{n} \Omega_\ell \subset \mathbb{R}^N$. Denote $u = (u_1,\dots,u_n) \in \Omega$, and $f(u) = (f_1(u_1),\dots,f_n(u_n))$. On the convective part (1) we assume

(H1) for $\ell = 1,\dots,n$ the flow $f_\ell \in \mathbf{C}^3(\Omega_\ell;\mathbb{R}^{n_\ell})$ and

- *(i)* Df_ℓ is strictly hyperbolic and for every compact subset K of Ω_ℓ its eigenvalues $\lambda_1^\ell,\dots,\lambda_{n_\ell}^\ell$ satisfy $\sup_K \lambda_i^\ell < \inf_K \lambda_{i+1}^\ell$;
- *(ii)* shock and rarefaction curves coincide;
- *(iii)* there exist a system w_ℓ of smooth Riemann coordinates in Ω_ℓ.

Define $w = (w_1(u_1),\dots,w_n(u_n))$. Assumption **(H1)** *(ii)* can be rephrased saying that each system $(1)_\ell$ is a Temple[25] system. All the traffic models mentioned in the introduction are of Temple type. This property is obvious for the LWR system, being scalar, and holds also for the Aw-Rascle[1] and the Colombo[4] model. Indeed, they both fit in the same generale framework[2]. Moreover, also the hyperbolic phase transition model[5] is known to share all the properties of Temple systems[10]. Note that for each ℓ system (1) is strictly hyperbolic, while the full system (2) may loose strict hyperbolicity.

On the source part we assume

(H2) $g\colon\ [0,+\infty[\, \times \mathbb{R} \times \Omega \mapsto \mathbb{R}^N$ is measurable in (t,x,u) and smooth in u; moreover

- *(i)* there exists a positive finite measure μ on $\mathbb{R}$ such that for a.e. $x_1 \leq x_2 \in \mathbb{R}$

$$\left|g(t,x_1^-,u) - g(t,x_2^+,u)\right| \leq \mu\left([x_1,x_2]\right)\,; \tag{1}$$

(ii) there exist functions $A, B \in \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}([0,+\infty[\,;\mathbb{R})$ and for every compact set $K \subseteq \mathcal{U}$ there exist $L_K \in \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}([0,+\infty[\,;\mathbb{R})$ such that for a.e. $t \in [0,+\infty[$, a.e. $x \in \mathbb{R}$

$$\left|g(t,x,u) - g(t,x,u')\right| \leq L_K(t)\left|u-u'\right| \quad \forall\, u,u' \in K, \tag{2}$$

$$|g(t,x,u)| \leq A(t) + B(t)|u| \qquad \forall\, u \in \Omega\,. \tag{3}$$

Assumption **(H2)** ensures the well posedness of the differential system

$$\partial_t u = g(t,x,u)\,. \tag{4}$$

Finally we require a compatibility condition:

(H3) for all ℓ, there exists a set $\mathcal{U}_\ell \subset \Omega_\ell$ which is invariant with respect to (1) and the set $\mathcal{U} = \prod_{\ell=1}^{n} \mathcal{U}_\ell$ is invariant with respect to (4).

Here, a set $\mathcal{U} \subseteq \mathbb{R}^n$ is said to be *(positively) invariant* with respect to (1), resp. (4), if any admissible initial data u_o attaining values in $\mathcal{U}$, i.e. $u_o(x) \in \mathcal{U}$ for all $x \in \mathbb{R}$, leads to a solution $u = u(t,x)$ to (1), resp. (4), attaining values in $\mathcal{U}$, i.e. $u(t,x) \in \mathcal{U}$ for all $t \geq 0$ and $x \in \mathbb{R}$. By *admissible* initial data we mean that (1), resp. (4), with initial data u_o has a weak entropic, resp. Carathéodory, solution defined for all positive times.

The set $\mathcal{U}_\ell$ is invariant with respect to $(1)_\ell$ if and only if any Riemann problem with data in $\mathcal{U}_\ell$ yields a solution attaining values in $\mathcal{U}_\ell$[18]. In the case of (4) the classical Nagumo condition is, for instance, a sufficient condition for invariance. Remark that $\mathcal{U}_\ell$ and $\mathcal{U}$ need neither be convex nor compact in the u_ℓ (or u) coordinates.

$\mathbf{BV}(\mathbb{R},\mathcal{U})$ denotes the space of functions defined on $\mathbb{R}$ valued in $\mathcal{U}$ with bounded total variation. By $\mathrm{TV}(u)$ we denote the total variation of $u \in \mathbf{BV}(\mathbb{R},\mathcal{U})$ computed by means of the Riemann invariants[6]. We say that a function u belongs to the space $\mathbf{L}^{\mathbf{1}}_{*}(\mathbb{R},\mathcal{U})$ if $u \in \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}(\mathbb{R},\mathcal{U})$ and there exist u_+, u_- in $\mathcal{U}$ such that $u - u_+$ (respectively $u - u_-$) is in $\mathbf{L}^{\mathbf{1}}([0,+\infty[)$ (in $\mathbf{L}^{\mathbf{1}}(]-\infty,0])$, respectively).Let $\mathbf{X}(\mathbb{R},\mathcal{U}) = \mathbf{L}^{\mathbf{1}}_{*}(\mathbb{R},\mathcal{U}) \cap \mathbf{BV}(\mathbb{R},\mathcal{U})$.

Theorem 2.1. *Assume conditions* (**H1**)*,* (**H2**) *and* (**H3**)*. Then, for every initial data* $u_o \in \mathbf{X}(\mathbb{R},\mathcal{U})$ *the Cauchy problem*

$$\begin{cases} \partial_t u + \partial_x f(u) = g(t,x,u) \\ u(0,x) = u_o(x) \end{cases} \tag{5}$$

admits a global solution $u\colon [0,+\infty[\, \times \mathbb{R} \mapsto \mathcal{U}$ *with* $u(t) \in \mathbf{X}(\mathbb{R},\mathcal{U})$ *for a.e.* t*. Moreover, for all* $M, T > 0$ *there exists a positive constant* L *such that if* $u_o, u_o' \in \mathbf{X}(\mathbb{R},\mathcal{U})$ *with* $\mathrm{TV}(u_o), \mathrm{TV}(u_o') \leq M$*, then the corresponding solutions* u, u' *satisfy for all* $t \in [0,T]$

$$\left\|u(t) - u'(t)\right\|_{\mathbf{L}^1} \leq L \cdot \left\|u_o - u_o'\right\|_{\mathbf{L}^1}\,. \tag{6}$$

3. Applications to Traffic Modelling

We present in this section several applications of Theorem 2.1 to multilane flows. Rewrite system (2) as

$$\partial_t u_\ell + \partial_x\left[f_\ell(u_\ell)\right] = g_\ell(t,x,u) + h_\ell(t,x,u) \qquad \ell = 1,\ldots n\,. \tag{1}$$

If a set $\mathcal{U}$ is invariant both with respect to $\partial_t u = g$ and to $\partial_t u = h$, then of course it is invariant with respect to $\partial_t u = g + h$.

3.1. *First Order Multilane Models*

3.1.1. *A Prototype*

The following system is a general prototype of first order multilane models:

$$\left\{\begin{array}{rcl}\partial_t\rho_1+\partial_x[f_1(\rho_1)] &=& g_1(t,x,\rho_1,\ldots,\rho_n)\\ \vdots & & \vdots\\ \partial_t\rho_n+\partial_x[f_n(\rho_n)] &=& g_n(t,x,\rho_1,\ldots,\rho_n)\,.\end{array}\right. \tag{2}$$

Here ρ_ℓ is the vehicle density in lane ℓ, where the maximal vehicle density is R_ℓ, so $L_\ell = 1/R_\ell$ is the mean vehicle length. We assume that for $\ell = 1,\ldots,n$ the maps $f_\ell : [r_\ell, R_\ell] \to \mathbb{R}$ are smooth, for some $r_\ell < R_\ell$. Then, **(H1)** holds. Denote $\Omega = \prod_{\ell=1}^n [r_\ell, R_\ell]$. We assume that the function $g = (g_1,\ldots,g_n) : [0,+\infty[\times\mathbb{R}\times\Omega \to \mathbb{R}^n$ satisfies **(H2)**. Invariant domains for the homogeneous part of (2) are $\prod_{\ell=1}^n [a_\ell, A_\ell]$, for $r_\ell \le a_\ell \le A_\ell \le R_\ell$. They are invariant also for the right-hand side of (2) provided

$$g_{\ell_{|\rho_\ell=a_\ell}} \ge 0 \quad\text{and}\quad g_{\ell_{|\rho_\ell=A_\ell}} \le 0 \quad \text{for } \ell = 1,\ldots,n\,. \tag{3}$$

If this condition holds, then **(H3)** is satisfied and Theorem 2.1 applies.

3.1.2. *An LWR-based Model*

A particular case of (2) is when the flows are governed by LWR equations:

$$\left\{\begin{array}{rcl}\partial_t\rho_1+\partial_x[\rho_1 v_1(\rho_1)] &=& g_1(t,x,\rho_1,\ldots,\rho_n)\\ \vdots & & \vdots\\ \partial_t\rho_n+\partial_x[\rho_n v_n(\rho_n)] &=& g_n(t,x,\rho_1,\ldots,\rho_n)\,.\end{array}\right. \tag{4}$$

Here $v_\ell = v_\ell(\rho_\ell)$ is the speed law in the lane $\ell = 1,\ldots,n$; we assume that $v_\ell : [0, R_\ell] \to \mathbb{R}$ are smooth (positive and decreasing) functions satisfying $v_\ell(R_\ell) = 0$. Consider first the case of a homogeneous road without neither entries nor exits; then we require moreover that

$$\sum_{\ell=1}^n g_\ell(t,x,\rho_1,\ldots,\rho_n) \equiv 0 \tag{5}$$

so that $\rho = \sum_{\ell=1}^n \rho_\ell$ is conserved. This model was introduced by Klar and Wegener[21,22] with specific terms g_ℓ. The assumptions $R_\ell \le R_{\ell+1}$ and $v_\ell(\rho) \le v_{\ell+1}(\rho)$ for $\rho \le R_1$ describe the case where trucks mostly occupy right lanes and cars the left ones. The choice $R_\ell = R$ for any ℓ models traffic flow when the length of vehicles is uniformly distributed over the different lanes, allowing velocities to be different. On the other hand, the case $v_\ell = v$ for every ℓ (but possibly $R_\ell \le R_{\ell+1}$) describes congested traffic flow[3].

In order to give explicit expressions to the terms g_ℓ, normalize the densities in (4) by setting $\tilde\rho_\ell = \rho_\ell/R_\ell$ so system (4) becomes

$$\partial_t\tilde\rho_\ell + \partial_x[\tilde\rho_\ell \tilde v_\ell(\tilde\rho_\ell)] = \tilde g_\ell(t,x,\tilde\rho_1,\ldots,\tilde\rho_n), \quad \ell = 1,\ldots,n \tag{6}$$

for $\tilde v_\ell(\tilde\rho_\ell) = v_\ell(R_\ell\cdot\tilde\rho_\ell)$ and $\tilde g_\ell(t,x,\tilde\rho_1,\ldots,\tilde\rho_n) = g_\ell(t,x,R_1\tilde\rho_1,\ldots,R_n\tilde\rho_n)/R_\ell$. We can assume that in the system (4) we have $R_\ell = 1$ for every $\ell = 1,\ldots,n$ and then drop the tildas in (6). The model of Klar and Wegener[21] sets then

$$g_\ell(\rho) = \left(g_{\ell-1}^{\ell}\rho_{\ell-1} - g_\ell^{\ell-1}\rho_\ell\right)\cdot(1-\delta_{1,\ell}) - \left(g_\ell^{\ell+1}\rho_\ell - g_{\ell+1}^{\ell}\rho_{\ell+1}\right)\cdot(1-\delta_{\ell,n}) \tag{7}$$

where δ_{ij} is Kronecker's delta (i.e. $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$) and

$$g_\ell^{\ell+1}(\rho_\ell, \rho_{\ell+1}) = P^L(\rho_{\ell+1}) \cdot \nu(\rho_\ell) = 1/T_\ell^L \tag{8}$$

$$g_\ell^{\ell-1}(\rho_{\ell-1}, \rho_\ell, \rho_{\ell+1}) = P^R(\rho_{\ell-1}) \cdot \left(1 - P^L(\rho_{\ell+1})\right) \cdot \nu(\rho_\ell) = 1/T_\ell^R\,. \tag{9}$$

Here P^L, P^R are the lane changing probabilities (to the left or the right), ν the interaction frequency, T_ℓ^L, T_ℓ^R the transition rates from lane ℓ to the left, resp. right. The functions P^L, P^R and ν are positive and satisfy $P^L(0) = P^R(0) = 1$, $P^L(1) = P^R(1) = 0$, $\nu(0) = \nu(1) = 0$. Condition (5) holds for g_ℓ given by (7).

Remark the asymmetry in the definitions of $g_\ell^{\ell+1}$ and $g_\ell^{\ell-1}$ in (8), (9); this is due to the following. A lane change to the left happens if a braking point is reached and a lane change (to the left, probability P^L) is possible. A lane change to the right occurs if the follower reaches the braking point and cannot overtake (change to the left, probability P^L); in addition a lane change to the right must be possible (probability P^R). In case of light traffic one can drop the term $1 - P^L(\rho_{\ell+1})$ in (9) and deduce that $g_\ell^{\ell+1} = g_\ell^{\ell+1}(\rho_\ell, \rho_{\ell+1})$, $g_\ell^{\ell-1} = g_\ell^{\ell-1}(\rho_{\ell-1}, \rho_\ell)$.

Assumptions **(H1)** and **(H2)** hold for (4)-(7) and the set $[0,1]^n$ is invariant for the homogeneous part. Moreover $g_{\ell|\rho_\ell=0} = \nu(\rho_{\ell-1})\rho_{\ell-1}\,(1-\delta_{1,\ell}) + \left(1 - P^L(\rho_{\ell+2})\right)\nu(\rho_{\ell+1})\rho_{\ell+1}\,(1-\delta_{\ell,n}) \geq 0$ and $g_{\ell|\rho_\ell=1} = 0$. Assumption **(H3)** holds for the domain $[0,1]^n$, then Theorem 2.1 applies.

3.1.3. *The switch curve*

Consider again system (4). Any domain $\mathcal{A} = \prod_{\ell=1}^n [a_\ell, A_\ell]$ with $0 \leq a_\ell \leq A_\ell \leq 1$ is invariant for the homogeneous part of (4). Such domains are invariant also for the right-hand side if conditions (3) are imposed. If we denote by $g_{\ell,\ell+1} = g_\ell^{\ell+1}\rho_\ell - g_{\ell+1}^\ell \rho_{\ell+1}$ the total inflow/outflow balance between lanes ℓ-$\ell+1$, then conditions (3) imply suitable sign changes for $g_{\ell,\ell+1}$ and in particular $g_{\ell,\ell+1}(a) = g_{\ell,\ell+1}(A) = 0$ for $\ell = 1, \dots, n-1$. Here $a = (a_1, \dots, a_n)$ and analogously A. Roughly speaking, in $\mathcal{A}$ there are n hypersurfaces of dimension $n-1$ passing through the points a and A prescribing the switches from a lane to the others. We consider the case $n = 2$. In this case system (4) reduces to

$$\begin{cases} \partial_t \rho_1 + \partial_x[\rho_1 v_1(\rho_1)] = g(t,x,\rho_1,\rho_2) \\ \partial_t \rho_2 + \partial_x[\rho_2 v_2(\rho_2)] = -g(t,x,\rho_1,\rho_2)\,. \end{cases} \tag{10}$$

The domain $[a_1, A_1] \times [a_2, A_2]$ is invariant if

$$g(t,x,a_1,\rho_2) \geq 0,\; g(t,x,A_1,\rho_2) \leq 0 \quad \text{for} \quad \rho_2 \in [a_2, A_2]\,, \tag{11}$$

$$g(t,x,\rho_1,a_2) \leq 0,\; g(t,x,\rho_1,A_2) \geq 0 \quad \text{for} \quad \rho_1 \in [a_1, A_2]\,. \tag{12}$$

Then $g(t,x,a_1,a_2) = g(t,x,A_1,A_2) = 0$. Condition **(H3)** holds if we choose any function g vanishing on a monotone curve (the switch curve) joining (a_1, a_2) with (A_1, A_2), g being positive (negative) above (below) this curve. See Ref. 13 for a related definition of the switch curve. In the case the right-hand terms in (10) are given by (7) we have

$$g = g_1 = -(g_1^2 \rho_1 - g_2^1 \rho_2) = -g_2\,. \tag{13}$$

Denote $G^L(\rho) = P^L(\rho)/(\rho \cdot \nu(\rho))$ and analogously G^R. Then the invariance assumption **(H3)** for the domain $[a_1, A_1] \times [a_2, A_2]$ is satisfied when

$$G^L(A_2) \leq G^R(\rho_1) \leq G^L(a_2) \quad \text{if } a_1 \leq \rho_1 \leq A_1\,,$$
$$G^R(A_1) \leq G^L(\rho_2) \leq G^R(a_1) \quad \text{if } a_2 \leq \rho_2 \leq A_2\,.$$

3.1.4. *Entries and exits for an LWR-based model*

Consider now the case of an entry or an exit for the (normalized) model introduced in (4), where the inter-lane dynamics is given by terms (7). Then only the first lane is responsible for the inflow or the outflow of vehicles.

The vehicles may exit (resp. enter) the considered road along the interval $[a, b]$ (resp. $[c, d]$). Let $g_{\text{out}}(t)$ be the fraction of the vehicle density in the first lane, per unit time, that exits along $[a, b]$ and $a_{\text{out}}(t, x) = g_{\text{out}}(t)\chi_{[a,b]}(x)$. Then the outflow is modeled through the source term $a_{\text{out}}(t, x)\rho_1$. Analogously let $g_{\text{in}}(t)$ be the fraction of the available vehicle density in the first lane (per unit time) used by the vehicles entering the road and $a_{\text{in}}(t, x) = g_{\text{in}}(t)\chi_{[c,d]}(x)$; the entry is modeled through $a_{\text{in}}(t, x)\,(1 - \rho)$. The term to be added on the right hand side of the first equation is then $h_1(t, x, \rho_1) = a_{\text{in}}(t, x) \cdot (1 - \rho_1) - a_{\text{out}}(t, x) \cdot \rho_1$. Taking into account the inter-lane terms the full system becomes

$$\left\{\begin{array}{l} \partial_t\rho_1 + \partial_x[\rho_1 v_1(\rho_1)] = g_1(\rho_1, \ldots, \rho_n) + h_1(x, \rho_1) \\ \partial_t\rho_2 + \partial_x[\rho_2 v_1(\rho_2)] = g_2(\rho_2, \ldots, \rho_n) \\ \qquad\vdots \qquad\qquad\qquad \vdots \\ \partial_t\rho_n + \partial_x[\rho_n v_n(\rho_n)] = g_n(\rho_1, \ldots, \rho_n) \end{array}\right. \tag{14}$$

with the terms g_ℓ given as in (7). Since both $a_{\text{in}}(t, x)$ and $a_{\text{out}}(t, x)$ are non-negative, then Nagumo condition applies to the equation $\partial_t\rho_1 = h_1(x, \rho_1)$ and to the interval $[0, 1]$. Therefore the set $[0, 1]^n$ is still invariant and Theorem 2.1 applies also to (14). In the case of system (10) the addition of the term h_1 to the right hand side of the first equation leaves the domain $[a_1, A_1] \times [a_2, A_2]$ still invariant if for any t, x we have $a_1 \le \frac{a_{\text{in}}}{a_{\text{in}}+a_{\text{out}}} \le A_1$.

3.1.5. *Nonlocal source terms*

We introduce in this section some models including *nonlocal* source terms. Consider for example model (2). There, all the densities ρ_ℓ are computed at the same point x and time t, and the model evolves exploiting only these punctual values. In some cases, however, drivers may change their driving attitudes after consideration of the traffic in wider stretches of the road. In this cases the introduction of some flow averaging seems necessary.

Rigorous results about the local existence of solution to balance laws with nonlocal source terms were recently established in the framework of strictly hyperbolic systems. In particular the case of small initial data was considered by Colombo and Guerra[11]; the extension of these results to the special case of Temple systems, for data with large total variation was recently accomplished by the authors in collaboration with Rosini[9]. Consider

$$\left\{\begin{array}{l} \partial_t\rho_1 + \partial_x[f_1(\rho_1)] = G_1(\rho_1, \ldots, \rho_n) \\ \qquad\vdots \qquad\qquad\qquad \vdots \\ \partial_t\rho_n + \partial_x[f_n(\rho_n)] = G_n(\rho_1, \ldots, \rho_n) \end{array}\right. \tag{15}$$

where the G_ℓ are operators in the variable ρ, namely, $G : \mathbf{L}^1(\mathbb{R}_x, \mathbb{R}^n) \to \mathbf{L}^1(\mathbb{R}_x, \mathbb{R}^n)$. On the operators G_ℓ it is required[9] that

$$\|G(\rho) - G(\sigma)\|_{\mathbf{L}^1} \le C\|\rho - \sigma\|_{\mathbf{L}^1} \tag{16}$$

$$\|G(\rho)\|_{\mathbf{L}^\infty} \;\le\; C_1 + C_2\|\rho\|_{\mathbf{L}^\infty}\,, \qquad \mathrm{TV}\,(G(\rho)) \;\le\; C_1 + C_2\mathrm{TV}(\rho) \tag{17}$$

for suitable constants C, C_1, C_2. Assumptions (16)-(17) hold if $G(\rho) = h(\rho) + \mathcal{K} * g(\rho)$ where g and h are bounded and locally Lipschitz functions and the kernel $\mathcal{K}$ is in $\mathbf{L^1}$. As an example consider system (4) with source terms g_ℓ given by (7). We can assume that g_ℓ are smooth functions; therefore these terms can be replaced in (4) by

$$G_\ell(\rho) = \mathcal{K}_\ell * g_\ell(\rho)$$

for any kernel $\mathcal{K}_\ell$ in $\mathbf{L^1}$. Formally we recover the model (4)-(7) if $\mathcal{K}_\ell = \delta$ is the delta measure. For $k_\ell \geq 0$ the choice $\mathcal{K}_\ell(x) = e^{-k_\ell x} \chi_{[0,+\infty)}(x)$, that is,

$$[G_\ell(\rho)](x) = \int_{-\infty}^{x} g_\ell(\rho(y))\, e^{-k_\ell(x-y)}\, dy\,, \tag{18}$$

allows to take into account the flow until the point x. An "anticipation" term is obtained for $h > 0$ choosing $\mathcal{K}_\ell(x) = e^{-k_\ell x} \chi_{[-h,+\infty)}(x)$; in this case

$$[G_\ell(\rho)](x) = \int_{-\infty}^{x+h} g_\ell(\rho(y))\, e^{-k_\ell(x-y)}\, dy\,.$$

To motivate these nonlocal terms, consider (10)-(13). Assume that the first lane is occupied by long groups of slow cars separated by stretches of much lighter traffic (strongly correlated behavior), while the second lane has a light and fast traffic; see Figure 1. Drivers in the second lane are not likely to occupy the first lane between two long groups, say at x_o. However, at x_o the density ρ_1 is small and then P^R (in g_2^1) is close to its maximal value 1: the term (13) pushes then the cars in the second lane to move to the first lane. A term like (18) takes into account also the high values of ρ_1 before x_o and models better the behavior of cars in the second lane.

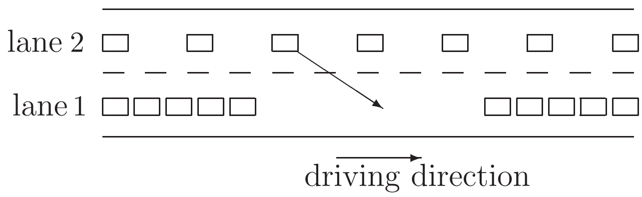

Fig. 1. Two lane traffic flow: a case study.

3.2. *Second order Multilane Models*

Many cases considered in the previous section are now extended to second order models. For brevity we omit here the case of nonlocal source terms.

3.2.1. *A model*

We present a model where the flow in each lane is controlled by two equations. The model considered by Klar and Wegener[21] is based on the Aw-Rascle[1] convective system; we focus instead on the model proposed by Colombo[4] for the convective part. Our model is

$$\left\{\begin{array}{rcl} \partial_t \rho_1 + \partial_x[\rho_1 v_1(\rho_1, q_1)] &=& s_1(t, x, \rho_1, \ldots, \rho_n, q_1, \ldots,, q_n) \\ \partial_t q_1 + \partial_x[(q_1 - q_1^*) v_1(\rho_1, q_1)] &=& m_1(t, x, \rho_1, \ldots, \rho_n, q_1, \ldots, q_n) \\ \vdots & & \vdots \\ \partial_t \rho_n + \partial_x[\rho_n v_n(\rho_n, q_n)] &=& s_n(t, x, \rho_1, \ldots, \rho_n, q_1, \ldots, q_n) \\ \partial_t q_n + \partial_x[(q_n - q_n^*) v_n(\rho_n, q_n)] &=& m_n(t, x, \rho_1, \ldots, \rho_n, q_1, \ldots, q_n)\,. \end{array}\right. \tag{19}$$

We assume as in the previous subsection that $R_\ell = 1$, so that the densities ρ_ℓ are valued in $[0,1]$. With q_ℓ we denote a weighted momentum variable which takes values in $\mathbb{R}$, q^*_ℓ is a constant that can be determined for instance as in Ref. 7. At last $v_\ell(\rho_\ell, q_\ell) = (1-\rho_\ell)q_\ell/\rho_\ell$. Assume $\sum_{\ell=1}^n s_\ell \equiv 0$.

Invariant domains for the convective part are of the type $\mathcal{U}_1 \times \ldots \times \mathcal{U}_n$, where for $(\rho_\ell, q_\ell) \in [0,1] \times \mathbb{R}$

$$\mathcal{U}_\ell = \left\{(\rho_\ell, q_\ell) \colon v_\ell(\rho_\ell, q_\ell) \in [V^1_\ell, V^2_\ell],\ q^*_\ell + \frac{q_\ell - q^*_\ell}{\rho_\ell} \in [Q^1_\ell, Q^2_\ell]\right\} \tag{20}$$

for some constants $0 \le V^1_\ell < V^2_\ell$ and $Q^1_\ell < Q^2_\ell$. They are invariant also for (19) if Nagumo condition holds. For many single lane models where this condition is satisfied we refer to Ref. 2. We consider now a case of source terms inspired by Klar and Wegener[21]. It is

$$\begin{aligned} s_\ell = {} & \left(g^\ell_{\ell-1}\rho_{\ell-1} - g^{\ell-1}_\ell \rho_\ell\right) \cdot (1-\delta_{1,\ell}) \\ & - \left(g^{\ell+1}_\ell \rho_\ell - g^\ell_{\ell+1}\rho_{\ell+1}\right) \cdot (1-\delta_{\ell,n}), \end{aligned} \tag{21}$$

$$\begin{aligned} m_\ell = {} & \left(g^\ell_{\ell-1}(q_{\ell-1} - q^*_{\ell-1}) - g^{\ell-1}_\ell (q_\ell - q^*_\ell)\right) \cdot (1-\delta_{1,\ell}) \\ & - \left(g^{\ell+1}_\ell (q_\ell - q^*_\ell) - g^\ell_{\ell+1}(q_{\ell+1} - q^*_{\ell+1})\right) \cdot (1-\delta_{\ell,n}), \end{aligned} \tag{22}$$

with the same notations of (8)-(9). Remark that both the total density and the total momentum are conserved. Denote $\mathcal{T} = \mathcal{T}_1 \times \ldots \times \mathcal{T}_n$ for

$$\mathcal{T}_\ell = \left\{(\rho_\ell, q_\ell) \in [0,1] \times \mathbb{R} \colon q^*_\ell + \frac{q_\ell - q^*_\ell}{\rho_\ell} \in [Q^1_\ell, Q^2_\ell]\right\}.$$

Lemma 3.1. *The set $\mathcal{T}$ is invariant both for the homogeneous part of (19) and for the ordinary differential equation associated to (21)–(22).*

Proof. The first assertion follows directly from (20). A straightforward calculation shows the second one: condition (3.2) in Ref. 2 holds. □

By Lemma 3.1 condition **(H3)** is satisfied and then Theorem 2.1 applies to system (19), with source terms given by (21)–(22) and invariant domain $\mathcal{T}$.

3.2.2. *Entries and Exits*

An entry or an exit can be easily modelled for system (19), for instance in the case the source terms are given by (21), (22). It is sufficient to show only how the dynamics of the first lane is changed; the others are as in (19). We consider then, with the same notations of section 3.1.4,

$$\begin{cases} \partial_t \rho_1 + \partial_x[\rho_1 v_1(\rho_1, q_1)] = s_1 + a_{\text{in}}(t,x)\,(1-\rho_1) - a_{\text{out}}(t,x)\rho_1 \\ \partial_t q_1 + \partial_x[(q_1 - q^*_1)v_1(\rho_1, q_1)] = m_1 - (a_{\text{in}}(t,x) + a_{\text{out}}(t,x))\,(q_1 - q^*_1). \end{cases} \tag{23}$$

We refer to Ref. 2 for some motivations to the introduction of the source terms in the second equation. From Lemma 3.1 in Ref. 2 we know that the set $\mathcal{T}_1$ is invariant for system (23) if $s_1 = m_1 = 0$. Therefore Theorem 2.1 applies to the system consisting of (23) and the last $2n-2$ equations in (19), for the set $\mathcal{T}$.

3.2.3. *Relaxations*

Relaxation terms involve the "momentum" equation[16]. Consider model (19) with source terms defined as in (21), (22). Relaxation effexts can be included as well[2]:

$$\tag{24} \begin{cases} \partial_t \rho_1 + \partial_x [\rho_1 v_1(\rho_1, q_1)] & = & s_1 \\ \partial_t q_1 + \partial_x [(q_1 - q_1^*) v_1(\rho_1, q_1)] & = & m_1 + \dfrac{\rho_1}{1-\rho_1} \dfrac{v_1^e(\rho_1) - v_1(\rho_1, q_1)}{\tau_1} \\ \vdots & & \vdots \\ \partial_t \rho_n + \partial_x [\rho_n v_n(\rho_n, q_n)] & = & s_n \\ \partial_t q_n + \partial_x [(q_n - q_n^*) v_n(\rho_n, q_n)] & = & m_n + \dfrac{\rho_n}{1-\rho_n} \dfrac{v_n^e(\rho_n) - v_n(\rho_n, q_n)}{\tau_n} . \end{cases}$$

Here above v_ℓ^e is an equilibrium velocity while τ_ℓ is a relaxation time, which is related to the average acceleration time. Both depend on the lane considered. One can argue again as in the preceding sections, namely by considering the invariance of the domain for the dynamical systems associated to the new source terms. It is then easy to verify by Nagumo condition that the set $\mathcal{T}$ satisfies again condition (**H3**) if for $\ell = 1, \ldots, n$

$$\tag{25} \left(q_\ell^* + (Q_\ell^1 - q_\ell^*)\rho_\ell\right) \cdot \left(\frac{1}{\rho_\ell} - 1\right) \le v_\ell^e(\rho_\ell) \le \left(q_\ell^* + (Q_\ell^2 - q_\ell^*)\rho_\ell\right) \cdot \left(\frac{1}{\rho_\ell} - 1\right) .$$

The velocity $v_\ell^*(\rho_\ell) = \frac{1-\rho_\ell}{\rho_\ell} q_\ell^*$ can then be taken as an equilibrium velocity for lane ℓ satisfying (25).

3.2.4. *Inhomogeneities of the road*

As in the case of single-lane traffic flows[2], one can model inhomogeneities occurring in the road, such as ascents or descents. This is done adding to each ℓ-"momentum" equation the term $h_\ell(t, x, \rho_\ell, q_\ell) = \rho_\ell \cdot a_\ell(t, \rho_\ell, q_\ell) \cdot \chi_{[a,b]}$. Here the ascent or descent is localized along the stretch $[a, b]$ and a_ℓ is the mean acceleration in lane ℓ. In particular then $a_l \ge 0$ in the case of a descent, while $a_\ell \le 0$ in the case of an ascent.

Consider again (19) with terms as in (21), (22) and add either an acceleration or a deceleration term as here above. In the case of a descent one finds that the set $\mathcal{T}$ satisfies again condition (**H3**) if a_ℓ vanishes on the upper segment $q_\ell = q_\ell^* + (Q_\ell^2 - q_\ell^*)\rho_l$ for any ℓ. In the case of an ascent it is needed that a_ℓ vanishes on the lower segment $q_\ell = q_\ell^* + (Q_\ell^1 - q_\ell^*)\rho_l$.

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