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Computational procedure for a time-dependent Walrasian price equilibrium problem

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Abstract.

We consider a Walrasian economic equilibrium problem when the data are time-dependent. The equilibrium conditions that describe this pure exchange economic model are expressed in terms of an evolutionary Variational Inequality. With the variational inequality theory we are able to give computational procedures.

Keywords: Walrasian price equilibrium; evolutionary variational inequality; direct method; subgradient method.

1. Introduction

In this paper we are concerned with a general economic equilibrium problem when the data are time-dependent: we consider a dynamic Walrasian price equilibrium problem. It was Leon Walras[16] who, in 1874, laid the fundamental ideas for the study of the general equilibrium theory, providing a succession of models, each taking into account more aspects of a real economy. In this theory all prices, which balance both the demand and the supply of each goods, are determined simultaneously on the market. Even better the complete process examined by this theory is the following: each consumer maximizes his own utility and each producer maximizes his own profit. The rigorous mathematical formulation of general equilibrium problem was elaborated by Arrow and Debreu[1] in the 1954 (see e.g. Ref. 2). Furthermore, Walras introduced a price-adjustement mechanism, called tatonnement, for the calculation of the equilibrium. As formalized by Samuelson,[14] the tatonnement process is governed by a system of differential equations over price-representative variables. Contrary to Walras' intuition and hopes, tatonnement does not converge in general, but only for markets satisfying certain restrictions. Scarf[15] was the first to apply the fixed-point theory for the calculation of the equilibrium. Indeed, following Debreu,[8] he defined a continuous mapping from the price simplex into itself, derived from the excess demand function for the given economy. Scarf's method[15] computes the fixed-point of this mapping, that are the equilibrium prices. General fixed point algorithms are notoriously slow and unpromising in anything but simple, low-dimensional situations. Alternative approaches have been opening up, however, formulating the general economic equilibrium problems as non linear complementary problems (see e.g. Ref. 10, 11). Although this approach has been proved to be more effective than fixed point methods, its convergence has not been proved theoretically.

In 1985, Border[3] elaborated a variational inequality formulation of Walrasian price equilibrium. Nagurney[12] , Dafermos and Nagurney[4] , Nagurney and Zhao[13] considered a Walrasian price equilibrium in the static case and used a variational inequality approach for the study and analysis of the equilibrium. In particular Zhao and Nagurney[13] used this last approach to develop several algorithms for the calculation of the equilibrium. In Ref. 7, we generalize to dynamic case some of results shown in Ref. 4 and 13: we consider a model in which the prices of goods evolve in the time. The motivation for this time-dependent approach is that we can't but consider that each phenomenon of our economic and physical world is not stable related to the time and that our static models of equilibria are a first useful abstract approach (see e.g. Ref. 5 and its complete bibliography). In this paper we consider the evolution in time of a Walrasian price equilibrium problem in which the excess demand function depends on time $t \in [0, T]$. This dynamic problem can be formulated in terms of an evolutionary variational inequality and, by means of variational theory, we are able to give an existence result. In order to support the theoretical results, we provide a numerical example: we find the dynamic Walrasian price equilibrium by solving the evolutionary variational inequality by the direct method (see e.g. Ref. 9). In addition we give the subgradient method in order to solve the evolutionary variational inequality.

2. Dynamic Walrasian price equilibrium problem

During a period of time $[0, T]$, $T > 0$, we consider a pure exchange economy with l different commodities and with column price vector $p(t) = (p_1(t), p_2(t), ..., p_l(t))^T$, where:

$$p_i : [0, T] \to \mathbb{R}, \quad i = 1, 2, .., l$$

$$p_i \in L^2([0, T], \mathbb{R}).$$

Hence:

$$p \in \prod_{i=1}^{l} L^2([0, T], \mathbb{R}) = L.$$

Denote the induced aggregate excess demand function $z(p(t))$, which is a row vector with components $(z_1(p(t)), z_2(p(t)), ..., z_l(p(t))$, where:

$$z_i : L \to L, \quad i = 1, 2, .., l.$$

Assume that $z(p(t))$ is generally defined on the subcone $\widetilde{C}$ of L such that $C_+ \subset \widetilde{C} \subset C$, where C is the cone of L:

$$C = \{p \in L : \, p(t) \geq 0 \, a.\, e.\, in \, [0, T]\},$$

and C_+ is q.r.i. C:

$$C_+ = \{p \in L : \, p(t) > 0 \, a.\, e.\, in \, [0, T]\}.$$

Hence, the possibility that the aggregate excess demand function may become unbounded for some $t \in [0, T]$ when the price of a certain commodity vanished is allowed. As usual, assume that $z(p(t))$ satisfies Walras' law: $\langle z(p(t)), p(t) \rangle = 0$ a. e. in $[0, T]$ $\forall p(t) \in C$, and that $z(p)$ is homogenous of degree zero in $p(t)$ on C: $z(\alpha p(t)) = z(p(t))$ for all $p(t) \in C$, $\alpha > 0$

a.e. in $[0,T]$. Because of homogeneity, one may normalize the prices so that they take values in the simplex:

$$S = \{p \in L : \, p(t) \geq 0, \; \sum_{i=1}^{l} p_i(t) = 1 \, a.\, e.\, in \, [0,T]\},$$

and, therefore, one may restrict the aggregate excess demand function on the intersection $D = S \cap \widetilde{C}$. Let

$$S_+ = \{p \in L : \, p(t) > 0, \; p \in S\},$$

and note that $S_+ \subset D \subset S$.
In general, economic equilibrium theory assume that the function $z(p(t))$ satisfies Walras' law:

$$(1) \qquad \langle z(p(t)), p(t)\rangle = \sum_{i=1}^{l} z_i(p(t)) \cdot p_i(t) = 0 \quad \forall p \in D \quad \text{a. e. in } [0,T].$$

The definition of a Walrasian equilibrium is now stated.

Definition 1. A price vector $p^* \in D$ is a Walrasian equilibrium price vector if

$$(2) \qquad z(p^*(t)) \leq 0 \quad a.\, e.\, in \, [0,T]$$

besides

$$\langle z(p^*(t)), p^*(t)\rangle = 0 \quad a.\, e.\, in \, [0,T]$$

Equilibrium conditions states that if the price of equilibrium of the goods j at the time t is equal to zero, then the aggregate excess demand $z^j(p^*(t))$, related to the goods j at the same time t, must not be greater than zero, that is the supply must be over the demand. If the price of equilibrium of the goods j at the time t is greater than zero, then $z^j(p^*(t))$, related to the goods j at the same time t, must be equal to zero, that is the demand and the supply must be equals.

The following theorem establishes that the dynamic Walrasian price equilibrium can be characterized as a solution of an evolutionary variational inequality.

Theorem 1. *A price vector $p^* \in D$ is a dynamic Walrasian equilibrium if and only if it satisfies the evolutionary variational inequality:*

$$(3) \qquad \langle z(p^*(t)), p(t) - p^*(t)\rangle_L = \int_0^T \langle z(p^*(t)), p(t) - p^*(t)\rangle dt \leq 0 \quad \forall p \in S.$$

Proof. See e.g. Ref. 7

Remark 1. We observe that equation(3) is equivalent to

$$(4) \qquad \langle z(p^*(t)), p(t) - p^*(t)\rangle = \sum_{i=1}^{l} z_i(p^*(t)) \cdot (p_i(t) - p_i^*(t)) \leq 0 \quad \forall p \in S \quad \text{a. e. in } [0,T].$$

3. Existence theorems

Let us recall some concepts that will be useful in the following. The function $z : D \to L$ is said to be:

1. *pseudomonotone* if and only if

$$\forall p_1, p_2 \in D \quad \langle z(p_1), p_2 - p_1 \rangle_L \geq 0 \Rightarrow \langle z(p_2), p_1 - p_2 \rangle_L \leq 0;$$

2. *hemicontinuous* if and only if

$$\forall p \in D \quad \text{the function} \quad q \to \langle z(q), p - q \rangle_L$$

   is upper semicontinuous on D;
3. *hemicontinuous along line segments* if and only if

$$\forall p_1, p_2 \in D \quad \text{the function} \quad q \to \langle z(q), p_2 - p_1 \rangle_L$$

   is upper semicontinuous on the line segment $[p_1, p_2]$.

Being D a convex and bounded set, we can adapt a classical existence theorem for the solution of a variational inequality to our problem (see e. g. Ref. 6, Theorem 5.1); then we will have the following theorem, which provides the existence with or without the pseudomonotonicity assumptions.

Theorem 2. *Each of the following conditions is sufficient to ensure the existence of the solution of (3):*

1. *$z(p(t))$ is hemicontinuous with respect to the strong topology and there exist $A \subseteq D$ compact and there exist $B \subseteq D$ compact, convex with respect to the strong topology such that*

$$\forall p_1 \in D \setminus A \quad \exists p_2 \in B : \langle z(p_1), p_2 - p_1 \rangle_L < 0;$$

2. *z is pseudomonotone, z is hemicontinuous along line segments and there exist $A \subseteq D$ compact and $B \subseteq D$ compact, convex with respect to the weak topology such that*

$$\forall p \in D \setminus A \quad \exists \tilde{p} \in B : \langle z(p), \tilde{p} - p \rangle_L < 0;$$

3. *z is hemicontinuous on D with respect to the weak topology, there exist $A \subseteq D$ compact and $B \subseteq D$ compact, convex with respect to the weak topology such that*

$$\forall p \in D \setminus A \quad \exists \tilde{p} \in B : \langle z(p), \tilde{p} - p \rangle_L < 0.$$

Remark 2. When $D = S$, being S weakly compact, the conditions of theorem (2) become:

1. $z(p(t))$ is hemicontinuous with respect to the strong topology on D and there exists $A \subseteq S$ nonempty, compact and there exists $B \subseteq S$ compact, convex with respect to the strong topology such that

$$\forall p_1 \in S \setminus A \quad \exists p_2 \in B : \langle z(p_1), p_2 - p_1 \rangle_L < 0;$$

2. z is pseudomonotone, z is hemicontinuous along line segments;
3. z is hemicontinuous on S with respect to the weak topology.

4. Example

During a period of time $[0,1]$, let us consider a pure exchange economy with two goods, $j=1,2$, and three agents, $a=1,2,3$. Let us assume that, at the time t, the agents 1, 2 and 3 have, respectively, the endowment vectors:

$$e_1(t)=(e_1^1(t),e_1^2(t))=(\beta,2t+5),\qquad e_2(t)=(e_2^1(t),e_2^2(t))=(t,0),$$

$$e_3(t)=(e_3^1(t),e_3^2(t))=(2\beta+3,2t+1)$$

and the demand vectors:

$$x_1(t)=(x_1^1(t),x_1^2(t))=$$

$$=\left(\frac{t(p^1(t)\beta(t)+(1-p^1(t))(2t+5))}{p^1(t)(t+5\beta(t))},\frac{5\beta(t)(p^1(t)\beta(t)+(1-p^1(t))(2t+5))}{(1-p^1(t))(t+5\beta(t))}\right),$$

$$x_2(t)=(x_2^1(t),x_2^2(t))=\left(\frac{\beta(t)t}{\beta(t)+5t+1},\frac{tp^1(t)(5t+1)}{(1-p^1(t))(\beta(t)+5t+1)}\right).$$

$$x_3(t)=(x_3^1(t),x_3^2(t))=\left(0,\frac{p^1(t)(2\beta(t)+3)+(1-p^1(t))(2t+1)}{1-p^1(t)}\right).$$

The demand relative to each agent depends on one non negative function $\beta(t)$, by means of which it is possible to control the demand in the market. We can observe that the demand function derives from the Cobb-Douglas utility function and it is of the form:

$$x_a^j(t)=\frac{\alpha(t)\sum_{j=1}^2 p^j(t)e_a^j(t)}{p^j(t)}\qquad \forall\ a=1,2,3,\quad j=1,2,$$

with $p^j(t)\neq 0$ a. e. in $[0,1]$ and for all $j=1,2$.
Since in this economy there is only pure exchange, the excess demand function relative to the commodity j, is so definite:

$$z^j(p(t))=\sum_{a=1}^3 x_a^j(t)-\sum_{a=1}^3 e_a^j(t)\quad \forall p\in P,\quad \forall j=1,2;$$

and in our case it becomes:

$$z^1(p(t))=\frac{t\beta(t)}{t+5\beta(t)}+\frac{t(1-p^1(t))(2t+5)}{(t+5\beta(t))p^1(t)}+\frac{t\beta(t)}{\beta(t)+5t+1}-(3\beta(t)+t+3),$$

$$z^2(p(t))=\frac{5\beta^2(t)p^1(t)}{(1-p^1(t))(t+5\beta(t))}+\frac{5(2t+5)\beta(t)}{t+5\beta(t)}+\frac{t(5t+1)p^1(t)}{(1-p^1(t))(\beta(t)+5t+1)}+$$

$$+\frac{p^1(t)}{1-p^1(t)}(2\beta(t)+3)-(2t+5).$$

From Theorem 1 we find the dynamic Walrasian price equilibrium of this economy, by solving the following variational inequality:

$$z^1(\overline{p}(t))(p^1(t)-\overline{p}^1(t))+z^2(\overline{p}(t))(p^2(t)-\overline{p}^2(t))\leq 0\quad \forall p\in P\quad \text{a. e. in } [0,1] \tag{1}$$

that is:

$$\left(\frac{t\beta(t)-5\beta(t)(2t-5)}{t+5\beta(t)}+\frac{t(1-p^1(t))(2t+5)}{p^1(t)(t+5\beta(t))}+\frac{t\beta(t)}{\beta(t)+5t+1}-\frac{5\beta^2(t)p^1(t)}{(1-p^1)(t+5\beta(t))}+\right.$$

$$\left.-\frac{t(5t+1)p^1(t)}{(1-p^1(t))(\beta(t)+5t+1)}-\frac{p^1(t)}{1-p^1(t)}(2\beta(t)+3)-3\beta(t)+t+2\right)(p^1(t)-\overline{p}^1(t))\leq 0 \tag{2}$$

$\forall p^1(t)\geq 0$ a. e. in $[0,1]$, with $p^2(t)=1-p^1(t)$ a. e. in $[0,1]$.

By applying the direct method, we obtain the exact solution to variational inequality (2). Hence the dynamic Walrasian price equilibrium is given by:

$$\overline{p}^1(t)=\frac{t(5+5\beta(t)+10t^2+27t+2\beta(t)t)}{90\beta(t)t+15\beta(t)+8t+77\beta^2(t)t+30\beta^2(t)+37\beta(t)t^2+43t^2+15t^3+15\beta^3(t)}$$

$$\overline{p}^2(t)=\frac{85\beta(t)t+15\beta(t)+3t+77\beta^2(t)t+30\beta^2(t)+35\beta(t)t^2+16t^2+5t^3+15\beta^3(t)}{90\beta(t)t+15\beta(t)+8t+77\beta^2(t)t+30\beta^2(t)+37\beta(t)t^2+43t^2+15t^3+15\beta^3(t)}.$$

5. Calculus of the solution

We consider the following variational inequality:

Find $p^*\in S$ such that:

$$\langle -z(p^*), p-p^*\rangle_L\geq 0\quad \forall p\in S. \tag{1}$$

In order to ensure the existence of the solution to our variational inequality, we suppose that the operator $-z(\cdot)$ is pseudomonotone and hemicontinuous. By means of these hypothesis on $-z(\cdot)$, variational inequality (1) is equivalent to the following Minty variational inequality:

Find $p^*(t)\in S$ such that:

$$\langle -z(p), p-p^*\rangle_L\geq 0\quad \forall p\in S. \tag{2}$$

Let us set, for all $p\in S$:

$$\psi(p)=\max_{q\in S}\langle -z(q), p-q\rangle_L.$$

We observe that ψ is well defined, because the operator

$$q\rightarrow\langle -z(q), p-q\rangle_L$$

is weakly upper semicontinuous, S is a bounded convex subset of L and hence weakly compact set. Moreover $\psi(p)$, being the maximum of a family of continuous affine functions, is convex and weakly lower semicontinuous. It results that:

$$\psi(p)\geq 0\quad\text{and}\quad\psi(p)=0\ \Leftrightarrow\ p\ \text{is a solution to (1).}$$

We consider the subdifferential of $\psi(p)$, for all $p \in S$:

$$\partial\psi(p) = \{\tau \in L : \ \psi(q) - \psi(p) \geq \langle \tau, q - p \rangle_L, \ \forall q \in S\}.$$

We will show that $\psi(p)$ has nonempty subdifferential for all $p \in S$. The subgradient method for finding a solution to variational inequality (1) runs as follows: choose $p_0 \in S$ arbitrary, given $p_n \in S$, $\psi(p_n) > 0$, let

$$p_{n+1} = Proj_S(p_n - \rho_n \tau_n),$$

where:

$$\tau_n \in \partial\psi(p_n) \ \Big(\tau_n \neq 0 \text{ because } \psi(p_n) > 0\Big), \qquad \rho_n = \frac{\psi(p_n)}{||\tau_n||^2}\,.$$

We shall see below that τ_n can be chosen so that its norm remains bounded. Then we have the following result:

Theorem 3. *If $\|\tau_n\|$ remains bounded and p_n is such that $\psi(p_n) > 0$ for all n, then there holds that $\psi(p_n) \to 0$. The sequence $\{p_n\}$ has weak cluster points, and every weak cluster point $\widetilde{p}$ is such that $\psi(\widetilde{p}) = 0$.*

Proof of Theorem 3. Let us p^* a solution to variational inequality (1); by inequality:

$$\langle \tau_n, p^* - p_n \rangle_L \leq \psi(p^*) - \psi(p_n) = -\psi(p_n),$$

and by nonexpansivity of the projection mapping, we have:

$$||p_{n+1} - p^*||^2 = ||Proj_S(p_n - \rho_n\tau_n) - Proj_S\, p^*||^2 \leq ||p_n - \rho_n\tau_n - p^*||^2 =$$

$$= ||p_n - p^*||^2 + \rho_n^2||\tau_n||^2 + 2\rho_n\langle \tau_n, p^* - p_n \rangle_L \leq$$

$$\leq ||p_n - p^*||^2 + \rho_n^2||\tau_n||^2 - 2\rho_n\psi(p_n) =$$

$$= ||p_n - p^*||^2 + \frac{\psi(p_n)^2}{||\tau_n||^4}||\tau_n||^2 - 2\frac{\psi(p_n)}{||\tau_n||^2}\psi(p_n) = ||p_n - p^*||^2 - \frac{\psi(p_n)^2}{||\tau_n||^2}\,.$$

Since the sequence $\{||p_n - p^*||\}$ is decreasing and bounded, and since $||\tau_n||$ are bounded from above, it follows that:

$$\lim_{n\to\infty} \psi(p_n) = 0\,.$$

In particular

$$||p_n - p^*|| \leq ||p_0 - p^*||, \qquad \forall n \in \mathbb{N}.$$

This show that $\{p_n\}$ is bounded and therefore has a weak cluster point. Since $\psi(p_n) \to 0$ and ψ is weakly lower semicontinuous, it follows that, if p is a weak cluster point:

$$0 = \liminf_{k\to\infty} \psi(p_{n_k}) \geq \psi(p) \geq 0,$$

namely $\psi(p) = 0$. □

We show that $\psi(p)$ has a nonempty subdifferential for all $p \in S$:
we call p^* the element of S such that:

$$\max_{q\in S}\langle -z(q), p - q \rangle_L = \langle -z(p^*), p - p^* \rangle_L = \psi(p),$$

and we pose $\tau = -z(p^*)$. We have that $\tau = -z(p^*)$ belongs to $\partial\psi(p)$; in fact, for all $q \in S$:

$$\langle -z(p^*), q-p\rangle_L = \langle -z(p^*), q-p^*+p^*-p\rangle_L = \langle -z(p^*), q-p^*\rangle_L +$$

$$-\langle -z(p^*), p-p^*\rangle_L \leq \max_{q\in S}\langle -z(q), p-q\rangle_L - \psi(p) = \psi(q) - \psi(p).$$

That is:

$$\tau = -z(p^*) \in \partial\psi(p) \Rightarrow \qquad \partial\psi(p) \neq \emptyset.$$

For all $n \in \mathbb{N}$, we consider $p_n^* \in S$ such that:

$$\max_{q\in S}\langle -z(q), p_n - q\rangle_L = \langle -z(p_n^*), p_n - p_n^*\rangle_L = \psi(p_n)$$

and we choose $\tau_n = -z(p_n^*) \in \partial\psi(p_n)$.

Then the approximating sequence is so defined:

$$p_{n+1} = Proj_S\Big(p_n + \frac{\langle -z(p_n^*), p_n - p_n^*\rangle_L}{||z(p_n^*)||^2}\, z(p_n^*)\Big).$$

6. Summary and conclusions

In this work it has been shown how a time-dependent Walrasian price equilibrium problem can be characterized with an evolutionary variational inequality. By means of this mathematics formulation we are able to investigate the existence of the equilibrium solution and the behavior of the Walrasian pure exchange economy (see e. g. Ref. 7). In this work it has been given a computational procedure to solve the evolutionary variational inequality, which characterizes the dynamic Walrasian price equilibrium. With this procedure we have not carried out any discretization of the problem, because we have a global approximation of the equilibrium in the interval $[0, T]$.

The abstract model of dynamic equilibrium, that we have considered in this paper, is a generalization of a first useful abstract model studied in Ref. 12 and Ref. 13. The motivation for this time-dependent approach is that we can't but consider that each phenomenon of our economic and physical world is not stable with respect to the time.

This problem is of interest to operations researchers and also to economists and, more recently, to computer scientists.

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