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Finite Volume methods for nonconservative hyperbolic systems: Application to shallow-flows

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The motivating question of this work was the design of numerical schemes for P.D.E. systems that can be written under the form

$$\partial_t w + \partial_x F(w) = \mathcal{B}(w) \cdot \partial_x w + S(w)\partial_x \sigma, \tag{0.1}$$

where the unknown $w(x,t)$ takes values on an open convex set D of $\mathbb{R}^N$; F is a regular function from D to $\mathbb{R}^N$; $\mathcal{B}$ is a regular matrix function from D to $\mathcal{M}_{N\times N}(\mathbb{R})$; S, a function from D to $\mathbb{R}^N$; and $\sigma(x)$, a known function from $\mathbb{R}$ to $\mathbb{R}$.

System (0.1) includes as particular cases: systems of conservation laws ($\mathcal{B} = 0$, $S = 0$), systems of conservation laws with source term or balance laws ($\mathcal{B} = 0$), as well as some coupled system of balance laws.

More precisely, the discretization of the shallow water systems that govern the flow of one shallow layer or two superposed shallow layers of immiscible homogeneous fluids was focused (see http://www.damflow.org). The corresponding systems can be written, respectively, as a balance law or a coupled system of two balance laws. Systems with similar characteristics also appear in other flow models such as two-phase flows.

It is well known that standard methods that solve correctly systems of conservation laws can fail in solving systems of balance laws, specially when approaching equilibria or near to equilibria solutions. Moreover, they can produce unstable methods when they are applied to coupled systems of conservation or balance laws. In the context of the numerical analysis of systems and coupled systems of balance laws, many authors have studied the design of well-balanced schemes, that is, schemes that preserve some equilibria: see [1] for a review.

Among the main techniques used in the derivation of well-balanced numerical schemes, one of them consists of choosing first a standard conservative scheme for the discretization of the flux terms and then discretizing the source and the coupling terms in order to obtain a consistent scheme which solves correctly a predetermined family of equilibria. If this first procedure is followed, the calculation of the correct discretization of the source and the coupling terms depends both on the specific problem and the conservative numerical scheme chosen, and it may become rather cumbersome.

Another technique consists of considering (0.1) as a particular case of one-dimensional quasi-linear hyperbolic system

$$\frac{\partial W}{\partial t} + \mathcal{A}(W)\frac{\partial W}{\partial x} = 0, \quad x \in \mathbb{R}, t > 0, \tag{0.2}$$

by adding to the system the trivial equation

$$\frac{\partial \sigma}{\partial t} = 0.$$

Once the system is rewritten under this form, piecewise constant approximations of the solutions are considered, then are updated by means of approximate Riemann solvers at the intercells.

If this second procedure is followed, the main difficulty both from the mathematical and the numerical points of view comes from the presence of nonconservative products, which makes difficult even the definition of weak solutions. Many papers have been devoted to the definition and stability of nonconservative products, and its application to the definition of weak solutions of nonconservative hyperbolic systems. In this work we assume the definition of nonconservative products as Borel measures given by Dal Maso, LeFloch, and Murat in [3]. This definition, which depends on the choice of a family of paths in the phases space, allows one to give a rigorous definition of weak solutions of (0.2). A family of paths in $\Omega \subset \mathbb{R}^N$ is a locally Lipschitz map

$$\Phi\colon [0,1] \times \Omega \times \Omega \mapsto \Omega,$$

such that:

- $\Phi(0; W_L, W_R) = W_L$ and $\Phi(1; W_L, W_R) = W_R$, for any $W_L, W_R \in \Omega$;
- for every arbitrary bounded set $\mathcal{O} \subset \Omega$, there exists a constant k such that

  $$\left| \frac{\partial \Phi}{\partial s}(s; W_L, W_R) \right| \le k |W_R - W_L|,$$

  for any $W_L, W_R \in \mathcal{O}$ and almost every $s \in [0,1]$;
- for every bounded set $\mathcal{O} \subset \Omega$, there exists a constant K such that

  $$\left| \frac{\partial \Phi}{\partial s}(s; W_L^1, W_R^1) - \frac{\partial \Phi}{\partial s}(s; W_L^2, W_R^2) \right| \le K(|W_L^1 - W_L^2| + |W_R^1 - W_R^2|),$$

  for any $W_L^1, W_R^1, W_L^2, W_R^2 \in \mathcal{O}$ and almost every $s \in [0,1]$.

Together with the definition of weak solutions, a notion of *entropy* has to be chosen as the usual Lax's concept or one related to an entropy pair. The classical theory of simple waves of hyperbolic systems of conservation laws and the results concerning the solutions of Riemann problems can then be extended to systems (0.2).

The choice of the family of paths may be, in general, a difficult task. The goal of this work is, once the choice is done, to provide a theoretical framework for the numerical approximation of the corresponding weak solutions of a strictly hyperbolic system (0.2) whose characteristic fields are either genuinely nonlinear or linearly degenerate.

The main concept of this framework is that of path-conservative numerical schemes: given a family of paths Φ, a numerical scheme is said to be Φ-conservative if it can be written under the form:

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\left(D_{i-1/2}^{+} + D_{i+1/2}^{-}\right), \tag{0.3}$$

where

$$D_{i+1/2}^{\pm} = D^{\pm}(W_{i-q}^n, \ldots, W_{i+p}^n),$$

D^- and D^+ being two continuous functions from Ω^{p+q+1} to Ω satisfying:

$$D^{\pm}(W, \ldots, W) = 0, \quad \forall W \in \Omega, \tag{0.4}$$

and

$$\begin{aligned} & D^-(W_{-q}, \ldots, W_p) + D^+(W_{-q}, \ldots, W_p) \\ & \quad = \int_0^1 \mathcal{A}(\Phi(s; W_0, W_1)) \frac{\partial \Phi}{\partial s}(s; W_0, W_1)\, ds, \end{aligned} \tag{0.5}$$

for every $W_i \in \Omega$, $i = -q, \ldots, p$.

This definition generalizes the usual concept of conservative numerical scheme for system of conservation laws: if (0.2) is a system of conservation laws, i.e. $\mathcal{A}$ is the Jacobian of a flux function F, then, every numerical scheme which is path-conservative is consistent and conservative in the usual sense. And, conversely, a consistent conservative numerical scheme is path-conservative for every family of paths Φ.

The generalizations of the classical methods of Roe introduced in [6] or Godunov methods are particular cases of path-conservative methods based on approximate Riemann solvers fitting this general definition.

Another key-concept is that of well-balanced numerical scheme: the general definition introduced in [4] will be recalled.

This theoretical framework makes it possible also to generalize to systems of the form (0.2) the construction of high order methods based on reconstruction techniques (see [2]).

Some applications of this general theory in the context of one and two-layer Shallow Water systems will be shown.

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