DOI: 10.1478/AAPP.104S1A3

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 104, No. S1, A3 (2026)

TWO LEMMAS ON CONVEXITY WITH EXTERNALITIES IN NTU PARTITION FUNCTION GAMES

GIOVANNA BIMONTE [a*], LUIGI SENATORE [b] AND SALVATORE TRAMONTANO [c]

ABSTRACT. We study convexity in partition-function games with non-transferable utility under fixed externality schemes. We prove two lemmas: (i) a hereditary result showing that f-convexity is preserved by restriction to subgames that respect the externality scheme, and (ii) a reduction result establishing an equivalence between f-convexity and standard non-transferable utility (NTU) convexity for separable, idempotent and inclusion-preserving schemes. These lemmas enable the application of classical NTU results concerning convexity to partition-dependent environments, providing immediate corollaries such as the efficiency of the grand coalition and the non-emptiness of the f-core. The results clarify conceptually how restriction and a coalition-based partition rule collapse partition dependence. Furthermore, they provide a concise framework for applications involving positive or negative externalities.

1. Introduction

Convexity is a central structural property in cooperative games. In transferable-utility (TU) models, convexity delivers increasing marginal contributions, monotonicity of the core with respect to the coalition-expansion ordered relation, and core non-emptiness; in non-transferable-utility (NTU) models, analogous stability and efficiency results follow from standard convexity arguments (e.g., Scarf (1967)). With externalities — as in partition function form games (PFG) — the value of a coalition depends on the surrounding partition. Starting form Bimonte, Senatore, and Tramontano (2024), we model externalities via an externality scheme f that maps (S, π) to an environment for S, thereby separating the selection of external effects from the primitives of the coalition.

In Bimonte, Senatore, and Tramontano (2024), a f-based notions of convexity — both in ordinal and cardinal form — gets formalized and connected to efficiency of the grand coalition and non-emptiness of the f-core under mild regularity. Building on those results, this short note isolates two minimal lemmas that clarify and streamline the role of convexity with externalities: a hereditary lemma showing that f-convexity is preserved by restrictions to subgames that respect f; and a reduction lemma showing that, when f is separable, idempotent, and inclusion-preserving, f-convexity is equivalent to standard NTU convexity

of the f-induced characteristic game. These lemmas allow one to import classical NTU consequences of convexity directly into PFG environments.

Our approach aligns with both the coalition-structure perspective (Aumann and Dreze 1974) and selection-based treatments of externalities in partition-function games (Thrall and Lucas 1963; Myerson 1977; De Clippel and Serrano 2008). In this framework, separability establishes a standard environment for each coalition S, ensuring that the f-induced characteristic correspondence is well-defined (see McQuillin 2009; Do and Norde 2007). Idempotence and an inclusion-preserving condition then ensure coherence when transitioning between nested coalitions, a concept that our reduction theorem exploits.

The structure of the paper is as follows. Section **2** introduces the model and the essential setup and the definition of f-convexity. Section **3** states and proves the two lemmas and derives immediate corollaries on efficiency and core non-emptiness. Section **4** concludes.

2. The model

Let N be a finite set of players, $\Pi(N)$ the set of partitions of N. An NTU PFG is a pair (N,V) with a correspondence

$$V : 2^N \times \Pi(N) \longrightarrow \mathscr{P}(\mathbb{R}^N)$$

assigning to each coalition $S \subseteq N$ and partition $\pi \in \Pi(N)$ a subset $V(S,\pi)$ of $\mathbb{R}^N$ which represents the feasible utilities for the members of S, with the usual convention that utilities of players outside S are ignored.

For all $(S,\pi) \in 2^N \times \Pi(N)$, the feasible set $V(S,\pi)$ is nonempty, closed, and downward comprehensive in $\mathbb{R}^S$ (i.e., if $u \in V(S,\pi)$ and $0 \leq v \leq u$ componentwise, then $v \in V(S,\pi)$). We also adopt zero-normalization: $\mathbf{0} \in V(\varnothing,\pi)$; when needed, we identify $V(S,\pi)$ with its zero-completion in $\mathbb{R}^N$ in order to have subsets of the same space $\mathbb{R}^N$. We refer to these standing properties as Regularity.

Definition 1. An externality scheme is a function $f : 2^N \times \Pi(N) \to \Pi(N)$ such that

$$S \in f(S,\pi) \qquad \text{for every } S \subseteq N \text{ and every } \pi \in \Pi(N).$$

In words, for each coalition S and partition π, f assigns a partition of N that contains S as a block; we then set $V_f(S,\pi) := V\big(S, f(S,\pi)\big)$.

Definition 2. Let $x \in \mathbb{R}^N$ be an allocation. We say x is *f-feasible* if

$$\exists\, \pi \in \Pi(N) \,:\, x \in V_f(N,\pi)$$

while x is *f-undominated* if

$$\nexists\, S \in 2^N \text{ and } y \in V_f(S,\pi) \text{ s.t. } y_i > x_i \;\forall\, i \in S.$$

The *f-core* is the set of f-feasible, f-undominated allocations.

We use the Minkowski sum $A \oplus B := \{a+b : a \in A,\, b \in B\}$ for any A, B subsets of the common Euclidean space $\mathbb{R}^N$.

Definition 3 (f-ordinal convexity)**.** The PFG (N,V) is *f-convex* if for all $S,T \subseteq N$ and all $\pi \in \Pi(N)$

$$V_f(S \cup T,\pi) \oplus V_f(S \cap T,\pi) \supseteq V_f(S,\pi) \oplus V_f(T,\pi). \tag{1}$$

In order to relate results proven for N to results on a substructure $S \subseteq N$, we require a canonical method of transitioning between the two levels. Zero-completion lets us embed payoffs of $T \subseteq S$ into $\mathbb{R}^N$, while restriction recovers the S–game by projecting back to $\mathbb{R}^S$, taking only the useful components.

This setup plays a key role in Lemma 6: we will extend a partition of S to a partition of N, apply f–convexity on N after zero–completion, and then project the inclusion back to S. Because externalities are selected by f as partitions of N, we additionally require that S *preserves* f, i.e. that the partitions chosen by f for coalitions contained in S can be realized using partitions that live entirely in S; this guarantees coherence of the f–selections across restriction.

Definition 4. For $S \subseteq N$, the restricted game (S, V^S) is pointwise defined by

$$V^S(T, \pi_S) := \left\{ u \in \mathbb{R}^T : \ (u, \mathbf{0}_{S \setminus T}) \in V(T, \pi_S) \right\}$$

for any choice of $T \subseteq S$, $\pi_S \in \Pi(S)$.

We say that S preserves f if there exists a map $f^S : 2^S \times \Pi(S) \to \Pi(S)$ such that, for every $T \subseteq S$ and every $\pi_S \in \Pi(S)$, there exists an extension $\tilde{\pi} \in \Pi(N)$ of π_S with

$$V\big(T, f^S(T, \pi_S)\big) \ = \ V\big(T, f(T, \tilde{\pi})\big).$$

The partitions selected by f for coalitions contained in S can be realized using partitions that live entirely in S, so (S, V^S) is compatible with the scheme f.

We call f-separable if, for each $S \subseteq N$, the map $\pi \mapsto f(S, \pi)$ is constant on $\Pi(N)$; equivalently, there exists a canonical partition $\pi^*(S)$ such that $f(S, \pi) = \pi^*(S)$ for all $\pi \in \Pi(N)$. Operationally, separability turns the partition selector into a coalition–indexed rule and makes $V\big(S, f(S, \pi)\big)$ independent of π.

Definition 5. The externality scheme $f : 2^N \times \Pi(N) \to \Pi(N)$ is separable if for every $S \subseteq N$ there exists a partition $\pi^*(S) \in \Pi(N)$ such that

$$f(S, \pi) = \pi^*(S) \qquad \text{for all } \pi \in \Pi(N).$$

Equivalently, for each fixed S, the map $\pi \mapsto f(S, \pi)$ is constant on $\Pi(N)$ that is

$$f(S, \pi) = f(S, \pi') \quad \forall \pi, \pi' \in \Pi(N).$$

By separability, $V_f(S, \pi)$ is constant in π. We write $W_f(S)$ for this set and note that

$$W_f(S) = V_f(S, \pi) \qquad \forall \pi \in \Pi(N).$$

In general, $W_f(S)$ depends only on S, not on π. Moreover, if f is idempotent as well, then the canonical selection is invariant under re-application of f so that

$$f(S, f(S, \pi)) = f(S, \pi)$$

and in particular $f\big(S, \pi^*(S)\big) = \pi^*(S)$.

Finally, inclusion–preserving asks for coherence across $S \subseteq T$, e.g., that the canonical choice for T restricts consistently to S.

3. Results

We begin by fixing the link between the whole game and its substructures. Restriction and zero-completion yield a well-defined mapping from feasibility on N to feasibility on $S \subseteq N$, while preservation of f guarantees that the selections of f for coalitions in S are implementable within S. Under this compatibility, the convexity inclusion can be transported from N to S by extending partitions, applying the f–convex inequality on N, and projecting back to S. The first lemma formalizes this hereditary result.

Lemma 6 (Hereditary f-convexity)**.** *Let (N,V) be an NTU game in partition function form and let $f : 2^N \times \Pi(N) \to \Pi(N)$ be an externality scheme. If $S \subseteq N$ preserves f and (N,V) is f-convex, then the restricted game (S,V^S) is f^S-convex that is*

$$V^S_{f^S}(A \cup B, \pi_S) \;\oplus\; V^S_{f^S}(A \cap B, \pi_S) \;\supseteq\; V^S_{f^S}(A, \pi_S) \;\oplus\; V^S_{f^S}(B, \pi_S)$$

for all $A,B \subseteq S$ and all $\pi_S \in \Pi(S)$, where $V^S_{f^S}(T, \pi_S) := V^S\big(T,\, f^S(T, \pi_S)\big)$ for $T \subseteq S$.

Proof. Let $A,B \subseteq S$ and $\pi_S \in \Pi(S)$ and take $x \in V^S_{f^S}(A, \pi_S)$ and $y \in V^S_{f^S}(B, \pi_S)$. By the zero-normalisation assumption

$$(x, \mathbf{0}_{S\setminus A}) \in V\big(A, f^S(A, \pi_S)\big) \text{ and } (y, \mathbf{0}_{S\setminus B}) \in V\big(B, f^S(B, \pi_S)\big).$$

Since the S preserves f, for this π_S there exists a suitable extension $\tilde{\pi} \in \Pi(N)$ of π_S such that, for any $T \in \{A, B, A \cup B, A \cap B\}$

$$V\big(T, f^S(T, \pi_S)\big) \;=\; V\big(T, f(T, \tilde{\pi})\big).$$

Hence $(x, \mathbf{0}_{N\setminus A}) \in V\big(A, f(A, \tilde{\pi})\big)$ and $(y, \mathbf{0}_{N\setminus B}) \in V\big(B, f(B, \tilde{\pi})\big)$ as subsets of $\mathbb{R}^N$ by Regularity assumption.

The whole game (N,V) is f-convex so, applying (1) with the same partition $\tilde{\pi}$ and the coalitions A,B yields

$$V_f(A \cup B, \tilde{\pi}) \oplus V_f(A \cap B, \tilde{\pi}) \;\supseteq\; V_f(A, \tilde{\pi}) \oplus V_f(B, \tilde{\pi})$$

so that there exist $z \in V\big(A \cup B, f(A \cup B, \tilde{\pi})\big)$ and $w \in V\big(A \cap B, f(A \cap B, \tilde{\pi})\big)$ in $\mathbb{R}^N$ s.t.

$$(x, \mathbf{0}_{N\setminus A}) + (y, \mathbf{0}_{N\setminus B}) \;=\; z + w$$

by definition of Minkowski sum. Projecting both sides onto the coordinates of S and then onto $A \cup B$ and $A \cap B$ respectively

$$z^S \in V\big(A \cup B, f^S(A \cup B, \pi_S)\big), \qquad w^S \in V\big(A \cap B, f^S(A \cap B, \pi_S)\big),$$

due to the preservation equalities $V\big(T, f(T, \tilde{\pi})\big) = V\big(T, f^S(T, \pi_S)\big)$ for any choice of $T \in \{A \cup B, A \cap B\}$. By the definition of restriction, this is equivalent to

$$z^S \in V^S_{f^S}(A \cup B, \pi_S), \qquad w^S \in V^S_{f^S}(A \cap B, \pi_S).$$

Finally it follows that

$$(x, \mathbf{0}_{(A\cup B)\setminus A}) + (y, \mathbf{0}_{(A\cup B)\setminus B}) \;=\; z^S + w^S \quad \text{ in } \mathbb{R}^{A\cup B}.$$

Thus $x + y \in V^S_{f^S}(A \cup B, \pi_S) \oplus V^S_{f^S}(A \cap B, \pi_S)$, proving f^S-convexity of (S,V^S) as well. □

To extend standard NTU convexity results to the PFG setting, it is useful to collapse the dependence on the partition of N by associating to each coalition S a partition that is independent of π.

Lemma 7 (Reduction to the f-induced game)**.** *Let (N,V) be an NTU partition function form game and let $f : 2^N \times \Pi(N) \to \Pi(N)$ be separable, idempotent, and inclusion-preserving externality scheme. For each $S \subseteq N$, define*

$$W_f(S) := V\big(S, f(S,\pi)\big) \qquad \textit{for any} \pi \in \Pi(N),$$

which is well defined by separability. Then the following are equivalent:

(i) *(N,V) is f-convex;*
(ii) *W_f is (ordinally) convex in the usual NTU sense.*

Proof. Let (N,V) be an NTU game and $f : 2^N \times \Pi(N) \to \Pi(N)$ be an externality scheme that is separable, idempotent, and inclusion–preserving.

(i)⇒(ii) Let us assume that (N,V) is f–convex. Let $S,T \subseteq N$ and take arbitrary allocations $x \in W_f(S)$ and $y \in W_f(T)$. By definition of W_f there exists $\pi \in \Pi(N)$ with $x \in V\big(S, f(S,\pi)\big)$ and $y \in V\big(T, f(T,\pi)\big)$. Applying f–convexity at the same π gives

$$V_f(S\cup T,\pi) \,\oplus\, V_f(S\cap T,\pi) \,\supseteq\, V_f(S,\pi) \,\oplus\, V_f(T,\pi).$$

so there exist $z \in V\big(S\cup T, f(S\cup T,\pi)\big)$ and $w \in V\big(S\cap T, f(S\cap T,\pi)\big)$ such that $x+y = z+w$. By the definition of W_f we have $z \in W_f(S\cup T)$ and $w \in W_f(S\cap T)$, so $x+y \in W_f(S\cup T) \oplus W_f(S\cap T)$, i.e. W_f is (ordinally) convex.

(ii)⇒(i) Viceversa, let us assume that the game W_f is (ordinally) convex. Let $S,T \subseteq N$, fix a partition $\pi \in \Pi(N)$ and take arbitrary allocations $x \in V_f(S,\pi) = V\big(S, f(S,\pi)\big)$ and $y \in V_f(T,\pi) = V\big(T, f(T,\pi)\big)$. By separability, $f(S,\pi)$ and $f(T,\pi)$ do not depend on the partition π, hence $x \in W_f(S)$ and $y \in W_f(T)$. By convexity of W_f there exist $z \in W_f(S\cup T)$ and $w \in W_f(S\cap T)$ such that $x+y = z+w$. Using again the separability hypothesis

$$W_f(S\cup T) = V\big(S\cup T, f(S\cup T,\pi)\big) = V_f(S\cup T,\pi)$$
$$W_f(S\cap T) = V\big(S\cap T, f(S\cap T,\pi)\big) = V_f(S\cap T,\pi).$$

Therefore the allocation $x+y \in V_f(S\cup T,\pi) \oplus V_f(S\cap T,\pi)$ by definition of Minkowski sum and, by arbitrariness of x,y we have

$$V_f(S\cup T,\pi) \,\oplus\, V_f(S\cap T,\pi) \,\supseteq\, V_f(S,\pi) \,\oplus\, V_f(T,\pi) \qquad \forall\, \pi \in \Pi(N)$$

i.e. the whole game (N,V) is f–convex.

□

The reduction in Lemma 7 allows us to transfer classical NTU results from the induced characteristic correspondence W_f back to the original PFG with externalities.

Corollary 8 (Efficiency of the grand coalition)**.** *Under the same hypotheses of Lemma 7, if W_f is (ordinally) convex, then any f-core allocation is Pareto efficient for N (with respect to the feasible set $W_f(N)$, hence to $V_f(N,\pi)$ for any π).*

Proof. By Lemma 7, we may work in the characteristic NTU game W_f. Let x be in the f-core. By contradiction, let us suppose that x were not Pareto efficient for N, then there would exist $y \in W_f(N)$ such that $y_i \geq x_i$ for all $i \in N$ and $y_j > x_j$ for some j. According to the standard (weak) NTU core notion, the coalition N would block x, and this is contradicting the fact that x is in the core. Therefore x is Pareto efficient.

If one adopts the strong blocking variant used in Section **2** (strict improvement for all members of the blocking coalition), the same argument yields strong efficiency for N (no $y \in W_f(N)$ with $y_i > x_i$ for all i). With the weak core notion—standard in NTU theory — the conclusion is the usual Pareto efficiency. □

Remark 9. Our f-core adopts the standard NTU blocking notion: a coalition S blocks x if it can realize a $y \in W_f(S)$ that satisfies $y_i > x_i$ for all $i \in S$ that is a sort of strict improvement. If one uses the weak variant, the conclusion of Corollary 8 is still Pareto efficiency for N; with the strong variant, one obtains strong efficiency, i.e. no $y \in W_f(N)$ with $y_i > x_i$ for all i.

Corollary 10 (Non-emptiness of the f-core)**.** *Under regularity assumption and the same hypothesis in the Lemma 7, if W_f is f-convex then the $f-$core is non-empty.*

Proof. By Lemma 7, we may work with the characteristic NTU game W_f. Regularity assumption ensures that $W_f(S)$ is nonempty, closed, and downward comprehensive for every $S \subseteq N$. Since W_f is convex, the core of W_f is nonempty by the classical NTU core non-emptiness theorem of Scarf (1967). Any allocation in Core(W_f) is undominated in V_f by construction, hence belongs to the f-core. □

4. Conclusion

This paper outlines two fundamental results concerning convexity with externalities in NTU partition function form games. Firstly, hereditary f-convexity demonstrates that convexity behaves consistently under restriction, provided that the substructure preserves the externality scheme. This justifies local analysis of relevant coalition subsystems, while maintaining the global convexity property. Secondly, the reduction to the f-induced characteristic correspondence W_f establishes an explicit bridge to the standard NTU setting. Under the conditions of separability, idempotence and inclusion-preservation of f, f-convexity is equivalent to classical NTU convexity of W_f. These two properties make it possible to derive well-known consequences, such as the efficiency of the grand coalition and the non-emptiness of the core, by appealing directly to standard NTU results.

Taken together, the results show that convexity—beyond mere superadditivity—stabilises the f-core in partition-dependent environments. For applications, this yields a concise framework: restriction localises arguments, and a coalition-based partition rule eliminates partition dependence, simplifying analyses with positive or negative externalities.

The assumptions that define the class of schemes to which the reduction applies are clear, yet not straightforward. They make explicit the design of f, specifying both the selection rule for environments and its coherence across nested coalitions.

A natural direction for further research is to relax the separability or inclusion-preserving conditions — for example, by allowing local or π-measurable schemes — while ensuring that the conclusions can be transferred across environments. Furthermore, it could be

considered moving from ordinal to cardinal convexity to sharpen structural results, such as the selection properties of cores and their links to value-like solutions.

Acknowledgments

The authors are grateful to the Scientific Committee of the #ApplMath2025 conference for the invitation and for the careful organization of the scientific program. We also thank the conference participants for helpful comments and discussions.

References

Aumann, R. J. and Dreze, J. H. (1974). "Cooperative games with coalition structures". *International Journal of game theory* **3**(4), 217–237.

Bimonte, G., Senatore, L., and Tramontano, S. (2024). "Efficiency and the core in NTU games in partition function form". *Annals of Operations Research*, 1–12.

De Clippel, G. and Serrano, R. (2008). "Marginal contributions and externalities in the value". *Econometrica* **76**(6), 1413–1436.

Do, K. H. P. and Norde, H. (2007). "The Shapley value for partition function form games". *International Game Theory Review* **9**(02), 353–360.

McQuillin, B. (2009). "The extended and generalized Shapley value: Simultaneous consideration of coalitional externalities and coalitional structure". *Journal of Economic Theory* **144**(2), 696–721.

Myerson, R. B. (1977). "Graphs and cooperation in games". *Mathematics of operations research* **2**(3), 225–229.

Scarf, H. E. (1967). "The core of an N person game". *Econometrica: Journal of the Econometric Society*, 50–69.

Thrall, R. M. and Lucas, W. F. (1963). "N-person games in partition function form". *Naval Research Logistics Quarterly* **10**(1), 281–298.

---

[a] University of Salerno,
Department of Economics and Statistics,
Via Giovanni Paolo II, 132, 84084 Fisciano, Italy

[b] University of Salerno,
Department of Economics and Statistics,
Via Giovanni Paolo II, 132, 84084 Fisciano, Italy

[c] University of Salerno,
Department of Mathematics,
Via Giovanni Paolo II, 132, 84084 Fisciano, Italy

[*] To whom correspondence should be addressed | email: gbimonte@unisa.it

Paper contributed to the#ApplMath2025 International Workshop
held in Messina, Italy (9-11 June 2025) under the patronage of the *Accademia Peloritana dei Pericolanti*.

Manuscript received 24 July 2025; published online 18 November 2026

© 2026 by the author(s); licensee *Accademia Peloritana dei Pericolanti* (Messina, Italy). This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution 4.0 International License (https://creativecommons.org/licenses/by/4.0/).