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A GEOMETRIC MODEL FOR MAGNETIZABLE BODIES WITH INTERNAL VARIABLES.

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ABSTRACT. In a geometrical framework for thermo-elasticity of continua with internal variables we consider a model of magnetizable media previously discussed and investigated by Maugin. We assume as state variables the magnetization together with its space gradient, subjected to evolution equations depending on both internal and external magnetic fields. We calculate the entropy function and necessary conditions for its existence.

1. Introduction

In a previous paper [1] we applied a geometrization technique for thermodynamics of simple continua with internal variables (see [2, 3]) to a model of dielectric polarizable continua developed a few years ago by Maugin and his co-workers ([4, 5, 6, 7]). These models have a deep practical importance as they apply fruitfully in the investigation of electronic materials ([8]).
The aim of this paper is to investigate on similar bases the magnetic counterparts considered by Maugin in [4, 5], to construct a geometric model for the thermodynamics of these materials, providing a clearer meaning to the commonly used concepts of *processes* and *transformations*, to elucidate a clear ground suited to analyzing thermodynamic transformations out of equilibrium, and to derive the conditions for the existence of an entropy function.

2. Internal variables and entropy function

According to the theory developed in [2, 3] the entropy function in material bodies endowed with thermo-elastic properties can be investigated in the case of dissipative processes far from thermodynamical equilibrium by means of a theoretical model based on the method of internal variables, which are geometrized in a framework that generalizes the earlier notion of thermodynamical processes developed by Coleman and Owen [9]. As already discussed in [2, 3] the leading idea consists in assuming from the beginning that time resides on an equal footing with all other state variables, so that terms in dt will directly enter the *entropy form*. An *instantaneous state space* B_t is assumed to contain all state variables which fit the configuration of the element at time t and B_t is assumed to be

a manifold. The *total state space* is then given by the disjoint union

$$\mathcal{B} = \bigcup_t \{t\} \times B_t, \tag{1}$$

with a given natural structure of fiber bundle over the real line $\mathbb{R}$ where time flows [12, 13]. It will be called the *thermodynamic bundle*. If the instantaneous state space B_t does not vary in time the state space $\mathcal{B}$ reduces to a Cartesian product $\mathbb{R} \times B$. Moreover, following Noll [14] we consider an abstract space of *processes* which consists of a set Π of functions

$$P_t^i : [0, t] \to \mathcal{G}, \tag{2}$$

where $[0, t]$ is any time interval, the space $\mathcal{G}$ is a suitable target space suggested by the model, i is a label ranging in an unspecified index set for all allowed processes and $t \in \mathbb{R}$ is the so called *duration* of the process. For the given state space B we suppose that the set Π is such that the following hold:

- $\exists\, D \,:\, \Pi \to \mathcal{P}(B)$, where $\mathcal{P}(B)$ is the set of all subsets of B; D is the *domain function* and $D_t^i \equiv D(P_t^i)$ is called the domain of the i-th process (of duration t);
- $\exists\, R \,:\, \Pi \to \mathcal{P}(B)$; R is the *range function* and $R_t^i \equiv R(P_t^i)$ is called the range of the i-th process (of duration t);
- considering the restrictions

$$P_\tau^i = P_t^i \mid_{[0,\tau]} \quad (\tau \leq t) \tag{3}$$

new processes are obtained ("restricted processes") and they satisfy the following:[1]

$$\forall\, \tau < t \quad D(P_t^i) \subseteq D(P_\tau^i). \tag{4}$$

Then, a continuous function is defined

$$\rho : \mathbb{R} \times \Pi \to C^0(B_0, B_t), \tag{5}$$

so that for any instant of time t and for any process $P_t^i \in \Pi$ a continuous mapping called *transformation* (induced by the process) is generated.
For any given initial state $b \in D_t^i$ the transformed final state $\rho_t^i(b) \in R_t^i$ will be called, by an abuse of notation, the *value* of the process (at time t). We define now a function of time in the following way:

$$\lambda_b^i(\tau) = \begin{cases} b & \text{if } \tau = 0 \\ \rho_t^i b & \text{if } \tau \in]0, t] \end{cases} \quad \text{with } b \in D_t^i \tag{6}$$

so that we have

$$\lambda_b^i(t) = \rho_t^i(b) = \Phi^i(t, b), \tag{7}$$

with

$$\Phi^i(t, b) \,:\, \mathbb{R} \times B \to B. \tag{8}$$

The transformation for the system is a function

$$\sigma \,:\, \mathbb{R} \to \mathbb{R} \times B, \tag{9}$$

[1] This requirement expresses the intuitive physical idea that restricting the time interval allows a larger set of possible initial states.

such that for every local trivialization of the thermodynamic bundle one has

$$\sigma : t \to (t, \lambda_b^i(t)). \tag{10}$$

With these positions the transformation is interpreted as a curve in the union of all the state spaces such that it intersects the instantaneous state space just once, i.e. σ is a section of the thermodynamic bundle ([12],[13]).
In order to investigate thermodynamical process in non-equilibrium one extends the state space by introducing suitable *internal variables* (see, e.g., [10, 11, 15]). Following standard ideas (see [19, 20, 21]) we assume that the body is a "simple material" in the sense of Coleman and Noll whenever one refers only to its mechanical properties. They are described by a state space described by the variables $(\mathbf{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}, \nabla\boldsymbol{\alpha})$, where $\mathbf{F}$ is the *deformation gradient*. The total state space is then:

$$B = Lin(\mathcal{V}) \oplus \mathbb{R} \oplus \mathcal{V} \oplus W \oplus Lin(W) \tag{11}$$

where W is any vector space accounting for (yet unspecified) internal variables α and $Lin(W)$ accounts for their space gradients $\nabla\alpha$. A process is a piecewise continuous function of time

$$P_t = [\mathbf{L}, -\nabla \cdot \mathbf{q}, \mathbf{\Lambda}, \mathbf{\Sigma}, \mathbf{\Gamma}] \tag{12}$$

where $\mathbf{L}$ is the *velocity gradient*, $\mathbf{q}$ is the *heat flux* per unit of mass, $\mathbf{\Lambda}$ accounts for the time evolution of the gradient of temperature $\boldsymbol{\beta} = \nabla\theta$ and $\mathbf{\Sigma}, \mathbf{\Gamma}$ for the time evolution of the internal variable α and of its gradient $\nabla\alpha$, respectively. The theory is completed (as in [2]) by: (i) assuming dynamical equations for $\mathbf{F}, e, \boldsymbol{\beta}, \boldsymbol{\alpha}$ and $\nabla\boldsymbol{\alpha}$; (ii) imposing a phenomenological definition of the extra flux of entropy; (iii) suitably defining the *entropy action* s for the theory by the standard prescription

$$s = -\int_0^t \frac{1}{m} \nabla \cdot \mathbf{J}_s dt,$$

where $m \neq 0$ is the mass density and $\mathbf{J}_s$ is the *total flux of entropy*; (iv) calculating out of processes and of the action s the *total entropy* S of the theory; (v) calculating out of S and of the second principle of thermodynamics the Clausius-Duhem inequality together with the relevant thermodynamical restrictions on the state variables. This method was fruitfully applied in [1] to deformable dielectrics. We shall hereafter apply it to magnetizable media corresponding to the model which we shall shortly recall in Section 3.

3. Models of deformable magnetizable media

Working as in [4, 5, 22] in a suitable Galilean quasi-static approximation, we assume that the medium is formed by n molecular species, each one of them giving rise to a *field of magnetization* per unit of mass denoted by $\boldsymbol{\mu}_\alpha$ $(\alpha = 1, ..., n)$, which in turn induces a *spin density* (per unit mass):

$$\mathbf{s}_\alpha = \gamma_\alpha^{-1} \boldsymbol{\mu}_\alpha \tag{13}$$

where $\gamma_\alpha \neq 0$ is the α-th *gyromagnetic ratio*. The *total magnetization* $\boldsymbol{\mu}$ and the *total spin density* $\mathbf{s}$ are defined by:

$$\boldsymbol{\mu} = \sum_\alpha \boldsymbol{\mu}_\alpha, \ \mathbf{s} = \sum_\alpha \mathbf{s}_\alpha = \sum_\alpha \gamma_\alpha^{-1} \boldsymbol{\mu}_\alpha. \tag{14}$$

The vector $\boldsymbol{\mu}$ is assumed (as in [4]-[6]) to satisfy the following equation:

$$\dot{\boldsymbol{\mu}} = \omega \times \boldsymbol{\mu} = -\gamma \boldsymbol{B}^{eff} \times \boldsymbol{\mu}, \tag{15}$$

where $\times$ denotes vectorial product in $\mathbb{R}^3$ and the *effective magnetic field* $\boldsymbol{B}^{eff}$ is assumed to be:

$$\boldsymbol{B}^{eff} = \mathbf{B} +^L \mathbf{B} + m^{-1}(\nabla \cdot^L \mathbb{B}), \tag{16}$$

where: $\mathbf{B}$ is the external magnetic field; ${}^L\mathbf{B}$ is a vector which accounts for the intermagnetic-sublattice interaction; ${}^L\mathbb{B}$ is a rank two tensor which accounts for the *spin-interactions* (i.e. the short range intra- and inter-magnetic interactions). Standard arguments in electrodynamics state that the following dynamical equation holds (in absence of external forces):

$$div\mathbf{t} + \mathbf{f}^{em} = m\dot{\mathbf{v}}, \tag{17}$$

in which $\mathbf{f}^{em}$ is the *electromotive force*; $\mathbf{v}$ is the velocity of body particles; $\mathbf{t}$ is the non-symmetric *Cauchy stress tensor*, given in components by:

$$t^{ij} = T^{ij} + \hat{t}^{[ij]}, \tag{18}$$

where $[\ ]$ denotes skew-symmetrization and $\mathbf{T} = ||T^{ij}||$ is the standard symmetric *elastic stress tensor* of Cauchy; $\hat{\mathbf{t}} = \boldsymbol{\mu} \times^L \mathbf{B} = ||\hat{t}_{ij}||$ is the *interaction stress tensor* defined by:

$$\hat{t}^{ij} = m\boldsymbol{\mu}^j({}^L B)^i. \tag{19}$$

This model is based on the assumption that elasto-mechanical phenomena should couple, in concrete materials, both to magnetization $\boldsymbol{\mu}$ and to its gradient $\nabla\boldsymbol{\mu}$ (we remark that, as in [4], this corresponds to the case of ferromagnetism). The thermodynamical properties of such continua lead to extra terms in the relevant Clausius-Duhem inequality. Following the model proposed in [3], we assume that the body $\mathcal{C}$ (with regular boundary $\partial\mathcal{C}$) is regularly embedded into Euclidean space $\mathbb{R}^3$ by a regular family of instantaneous time-dependent configurations $\mathcal{C}_t$. The *rate of deformation* $\mathbf{L}$ is given by:

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} \tag{20}$$

being $\mathbf{F}$ invertible. In the thermodynamical framework in which the outcome of dissipative structures is involved a different relation between the heat flux $\mathbf{q}$ and the entropy flux $\mathbf{J}_s$ has to be postulated. This is due to the presence of an additional term $\mathbf{k}$, called in the literature *extra entropy flux* [16], which is assumed to include contributions from the presence of internal variables. More precisely one has:

$$\mathbf{J}_s = \frac{1}{\theta}\mathbf{q} + \mathbf{k}. \tag{21}$$

The technique developed by the French school (see [6]) for investigating the relevant dynamical relations consists in a clever mix-up between the principle of virtual powers and the two fundamental principles of thermodynamics. Denoting by $p_{(i)}$ the *virtual power* of internal forces:

$$p_{(i)} = \mathbf{T} \cdot \mathbf{D} - m^L\mathbf{B} \cdot \dot{\boldsymbol{\mu}} +^L \mathbb{B} \cdot \nabla\dot{\boldsymbol{\mu}}, \tag{22}$$

in components:

$$p_{(i)} = T^{ij}D_{ij} - m^L B^i \boldsymbol{\mu}_{/i} +^L \mathbb{B}^{ij}\dot{\boldsymbol{\mu}}_{i/j}, \tag{23}$$

(where $/i$ denotes covariant derivation), one ends up with the following equation in absence of heat source by radiation:

$$m\dot{e} = p_{(i)} - \nabla \cdot \mathbf{q}. \tag{24}$$

Here and in the sequel $\mathbf{A}\cdot\mathbf{B}$ denotes the full contraction of vectors and tensors, i.e. $\mathbf{A}\cdot\mathbf{B} = tr(\mathbf{A}^T\mathbf{B})$ where T denotes transposition. We make now a Legendre transformation on the energy e to replace it by the free energy

$$\Psi = e - S\theta, \tag{25}$$

so that the second principle of thermodynamics leads to the Clausius-Duhem inequality under the form:

$$-m(\dot{\Psi} + S\dot{\theta}) + p_{(i)} - \frac{1}{\theta}\mathbf{q}\cdot\nabla\theta \geq 0; \tag{26}$$

θ is the thermodynamic temperature (here supposed to be such that $0 < \theta \leq \theta_c$, since the range of temperature considered is much below the Curie ferromagnetic phase-transition temperature θ_c).

4. Entropy function in ferromagnets

We are now ready to apply the general scheme of our previous papers [2, 3], recalled in Section 2, to the general model of Section 3.
We specify that the space W of (11) is in this case $\mathcal{V} \simeq \mathbb{R}^3$, where the time derivative of the magnetization vector $\boldsymbol{\mu}$ lives; in other words we assume as state variables $(\mathbf{F}, e, \nabla\theta, \boldsymbol{\mu}, \nabla\boldsymbol{\mu})$. According to equation (12) we replace the process P_t by the following specific function

$$P_t(\tau) = [\mathbf{L}(\tau), h(\tau), \mathbf{\Lambda}(\tau), \mathbf{\Sigma}(\tau), \mathbf{\Gamma}(\tau)] \tag{27}$$

where $h(\tau) = -\nabla\cdot\mathbf{q}$, $\mathbf{\Lambda}(\tau) = \dot{\beta}$, $\mathbf{\Sigma}(\tau) = -\gamma\boldsymbol{B}^{eff}$, with $\boldsymbol{B}^{eff}$ defined by (16) and $\mathbf{\Gamma}(\tau) = \dot{\nabla\boldsymbol{\mu}}$. Following the general method of processes, we assume that the state variables obey the following dynamical system:

$$\begin{cases} \dot{\mathbf{F}} = \mathbf{L}\mathbf{F} \\ m\dot{e} = p_{(i)} + h(\tau) \\ \dot{\boldsymbol{\beta}} = \mathbf{\Lambda}(\tau) = \nabla\cdot\mathbf{J}_{\nabla\theta} + \sigma_{\nabla\theta} \\ \dot{\boldsymbol{\mu}} = \mathbf{\Sigma}(\tau)\times\boldsymbol{\mu} \\ (\dot{\nabla\boldsymbol{\mu}}) = \mathbf{\Gamma}(\tau) = \nabla\cdot\mathbf{J}_{\nabla\boldsymbol{\mu}} + \sigma_{\nabla\boldsymbol{\mu}} \end{cases} \tag{28}$$

where $\mathbf{J}_{\nabla\theta}$ and $\sigma_{\nabla\theta}$ are the *current* and *source* term associated to $\nabla\theta$, respectively; $\mathbf{J}_{\nabla\boldsymbol{\mu}}$ and $\sigma_{\nabla\boldsymbol{\mu}}$ are the current and the source terms associated to the gradient of magnetization $\nabla\boldsymbol{\mu}$. This system determines a linear morphism $\mathbf{G}$ defined on the fiber bundle of processes in the following way:

$$\mathbf{G} : (\mathbf{F}, e, \nabla\theta, \boldsymbol{\mu}, \nabla\boldsymbol{\mu}, \mathbf{L}, h, \mathbf{\Lambda}, \mathbf{\Sigma}) \to (\mathbf{F}, e, \nabla\theta, \boldsymbol{\mu}, \nabla\boldsymbol{\mu}, \dot{\mathbf{F}}, \dot{e}, \dot{\nabla\theta}, \dot{\boldsymbol{\mu}}, \dot{\nabla\boldsymbol{\mu}}) \tag{29}$$

which in matrix form is expressed by:

$$\begin{aligned} &(\mathbf{F}, e, \nabla\theta, \boldsymbol{\mu}, \nabla\boldsymbol{\mu}, \dot{\mathbf{F}}, \dot{e}, \dot{\nabla\theta}, \dot{\boldsymbol{\mu}}, \dot{\nabla\boldsymbol{\mu}})^T = \\ &= \begin{pmatrix} \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{pmatrix} (\mathbf{F}, e, \nabla\theta, \boldsymbol{\mu}, \nabla\boldsymbol{\mu}, \mathbf{L}, h, \mathbf{\Lambda}, \mathbf{\Sigma})^T \end{aligned} \tag{30}$$

with

$$\mathbf{A} = \begin{pmatrix} \mathbf{F} & 0 & 0 & 0 & 0 \\ \frac{\mathbf{T}}{m} & \frac{1}{m} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \times\boldsymbol{\mu} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{31}$$

Using the standard procedures of [2, 9] the entropy function is defined by:

$$s(t) = -\int_0^t \frac{1}{m}\nabla\cdot\mathbf{J}_s d\tau, \tag{32}$$

so that, using (21), we have:

$$s = \int_0^t -\frac{1}{m\theta}\nabla\cdot\mathbf{q}d\tau + \int_0^t \frac{1}{m\theta^2}\mathbf{q}\cdot\nabla\theta d\tau - \int_0^t \frac{1}{m}\nabla\cdot\mathbf{k}d\tau. \tag{33}$$

Inserting the relevant equations $(28)_1$, $(28)_2$ and (22) one obtains the following expression for $\nabla\cdot\mathbf{q}$:

$$\nabla\cdot\mathbf{q} = \mathbf{T}\cdot(\dot{\mathbf{F}}\mathbf{F}^{-1}) - m\,{}^L\mathbf{B}\cdot\dot{\boldsymbol{\mu}} + {}^L\mathbb{B}\cdot\nabla\dot{\boldsymbol{\mu}} - m\dot{e}, \tag{34}$$

so that, using the algebraic relation $\mathbf{T}\cdot(\dot{\mathbf{F}}\mathbf{F}^{-1}) = (\mathbf{T}\mathbf{F}^{-T})\cdot\dot{\mathbf{F}}$, where for simplicity we set $\mathbf{F}^{-T} = (\mathbf{F}^{-1})^T$, we end up with:

$$s = \int_\sigma -\frac{1}{\theta}\Big[\frac{1}{m}\mathbf{T}\mathbf{F}^{-T}\cdot d\mathbf{F} - de + \Big(-{}^L\mathbf{B}\cdot\dot{\boldsymbol{\mu}} + \frac{1}{m}{}^L\mathbb{B}\cdot\nabla\dot{\boldsymbol{\mu}} - \frac{1}{m\theta}\mathbf{q}\cdot\nabla\theta + \frac{\theta}{m}\nabla\cdot\mathbf{k}\Big)dt\Big], \tag{35}$$

where the explicit expression for $\mathbf{k}$ will be calculated in a forthcoming paper. As in [2] the closure conditions for the 1-form Ω which defines entropy (35) as the integral $\int_\sigma \Omega$ on a path σ in the thermodynamical extended space $\mathbb{R}\times B$ (i.e. $\Omega = \omega_\mu dq^\mu + \omega_0 dt$) may be easily found by imposing $d\Omega = 0$ (see [12, 13]). From (35) one first get the following:

$$\begin{aligned} s &= \int_\sigma -\frac{1}{\theta}\Big[\frac{1}{m}\mathbf{T}\mathbf{F}^{-T}\cdot d\mathbf{F} - de - {}^L\mathbf{B}\cdot d\boldsymbol{\mu} + \frac{1}{m}{}^L\mathbb{B}\cdot d\nabla\boldsymbol{\mu} + \\ &+ \Big(-\frac{1}{m\theta}\mathbf{q}\cdot\nabla\theta + \frac{\theta}{m}\nabla\cdot\mathbf{k}\Big)dt\Big], \end{aligned} \tag{36}$$

which better exploits the integrand as a 1-form with coefficients for both differentials $d\boldsymbol{\mu}$ and $d(\nabla\boldsymbol{\mu})$ of the state variables, as claimed. In both expressions (35) and (36) one can recognize the explicit dependence on time which makes the total state space, i.e. a state space varying in time, a more appropriate model for this kind of irreversible phenomena. In the following we will derive the closure relations which will give the necessary conditions for the existence of the above upper-potential. By applying the closure conditions for the 1-form (see for example [2],[3]) we find the following necessary conditions for the existence of the entropy function during the analyzed process:

$$\frac{\partial}{\partial\mathbf{F}}\Big(\frac{1}{\theta}{}^L\mathbf{B}\Big) = \frac{\partial}{\partial\boldsymbol{\mu}}\Big(-\frac{1}{m\theta}\mathbf{T}\mathbf{F}^{-T}\Big), \tag{37}$$

$$\frac{\partial}{\partial\mathbf{F}}\Big(\frac{1}{\theta}\Big) = \frac{\partial}{\partial e}\Big(-\frac{1}{m\theta}\mathbf{T}\mathbf{F}^{-T}\Big), \tag{38}$$

$$\frac{\partial}{\partial \mathbf{F}}\Big(\frac{1}{\theta m}^{L}\mathbb{B}\Big)=\frac{\partial}{\partial \nabla\boldsymbol{\mu}}\Big(\frac{1}{m\theta}\mathbf{T}\mathbf{F}^{-T}\Big), \tag{39}$$

$$\frac{\partial}{\partial \mathbf{F}}\Big(\frac{1}{m\theta^2}\mathbf{q}\cdot\nabla\theta-\frac{1}{m}\nabla\cdot\mathbf{k}\Big)=\frac{\partial}{\partial t}\Big(-\frac{1}{m\theta}\mathbf{T}\mathbf{F}^{-T}\Big), \tag{40}$$

$$\frac{\partial}{\partial e}\Big(\frac{1}{\theta}^{L}\mathbf{B}\Big)=\frac{\partial}{\partial \boldsymbol{\mu}}\Big(\frac{1}{\theta}\Big), \tag{41}$$

$$\frac{\partial}{\partial e}\Big(-\frac{1}{\theta m}^{L}\mathbb{B}\Big)=\frac{\partial}{\partial \nabla\boldsymbol{\mu}}\Big(\frac{1}{\theta}\Big), \tag{42}$$

$$\frac{\partial}{\partial t}\Big(\frac{1}{\theta}\Big)=\frac{\partial}{\partial e}\Big(\frac{1}{m\theta^2}\mathbf{q}\cdot\nabla\theta-\frac{1}{m}\nabla\cdot\mathbf{k}\Big), \tag{43}$$

$$\frac{\partial}{\partial \boldsymbol{\mu}}\Big(-\frac{1}{m\theta}^{L}\mathbb{B}\Big)=\frac{\partial}{\partial \nabla\boldsymbol{\mu}}\Big(\frac{1}{\theta}^{L}\mathbf{B}\Big), \tag{44}$$

$$\frac{\partial}{\partial \boldsymbol{\mu}}\Big(\frac{1}{m\theta^2}\mathbf{q}\cdot\nabla\theta-\frac{1}{m}\nabla\cdot\mathbf{k}\Big)=\frac{\partial}{\partial t}\Big(\frac{1}{\theta}^{L}\mathbf{B}\Big)\quad , \tag{45}$$

$$\frac{\partial}{\partial \nabla\boldsymbol{\mu}}\Big(\frac{1}{m\theta^2}\mathbf{q}\cdot\nabla\theta-\frac{1}{m}\nabla\cdot\mathbf{k}\Big)=\frac{\partial}{\partial t}\Big(-\frac{1}{m\theta}^{L}\mathbb{B}\Big). \tag{46}$$

These relations express a sort of "irrotationality" of the entropy 1-form during the analyzed transformation and ensure the (local) existence of a potential function S for entropy (see [9]).

References

[1] M. Francaviglia, L. Restuccia, and P. Rogolino, "Entropy production in polarizable bodies with internal variables", *J. Non-Equilib. Thermodyn.*, **29**, 221-235 (2004).

[2] M. Dolfin, M. Francaviglia, and P. Rogolino, "A geometric perspective on irreversible thermodynamics with internal variables", *J. Non-Equilib. Thermodyn.*, **23**, 250-263 (1998).

[3] M. Dolfin, M. Francaviglia, and P. Rogolino, "A geometric model on the thermodynamics of simple materials", *Periodica Polytechnica Serie Mech. Eng.*, **43**, (1999).

[4] G.A. Maugin, "A continuum theory of deformable ferrimagnetic bodies. I. Field equations", *J. Math. Phys.*, **17**, 1727-1738, (1976).

[5] G.A. Maugin, "A continuum theory of deformable ferrimagnetic bodies. II. Thermodynamics, constitutive theory", *J. Math. Phys.*, **17**, 1739-1751, (1976).

[6] G.A. Maugin, "The method of virtual power in Continuum Mechanics: Application to coupled fields", *Acta Mechanica*, **35**, 1-70, (1980).

[7] G.A. Maugin, *Continuum Mechanics of electromagnetic solids*, North-Holland Series in Applied Mathematics and Mechanics (North-Holland, Amsterdam 1988).

[8] C. Kittel, *Introduction to solid state physics*, 3rd Ed. (John Wiley and Sons, New York, 1966).

[9] B. D. Coleman and D. R. Owen , "A mathematical foundation for thermodynamics", *Arch. Rat. Mech. Anal.*, **54**, 1-104 (1974).

[10] G.A. Maugin and W. Muschik, "Thermodynamics with internal variables. Part I. General Concepts", *J. Non-Equilib. Thermodyn.*, **19**, 217-249 (1994).

[11] G.A. Maugin and W. Muschik, "Thermodynamics with internal variables. Part II. Applications", *J. Non-Equilib. Thermodyn.*, **19**, 250-289 (1994).

[12] M. Francaviglia, *Elements of differential and Riemannian geometry*, (Bibliopolis, Napoli, 1988).

[13] J. Marsden and R. Abraham, *Foundations of mechanics*, (Addison-Wesley, Reading, 1978).
[14] W. Noll, "A new mathematical theory of simple materials", *Arch. Rat. Mech. Anal.*, **48**, 1-50 1972.
[15] J. Verhas, *Thermodynamics and Rheology*, (Kluwer, Dordrecht, 1997).
[16] G.A. Maugin, "Internal Variables and dissipative structures", *J. Non-Equilib. Thermodyn.*, **15**, 173-192 (1990).
[17] W. Muschik, *Non-equilibrium thermodynamics with applications to solids*, CISM Courses and lectures, **336** (Springer-Verlag, Berlin, 1994).
[18] F. Bampi and A. Morro , *Non-equilibrium thermodynamics: A hidden variable approach*, 211-232, in "Recent Developments in Non-equilibrium thermodynamics", J. Casas-Vazquez, D. Jou, and G. Lebon Ed.s (Springer-Verlag, Berlin, 1984).
[19] C. A. Trusdell, *Termodinamica Razionale*, (Accademia Nazionale dei Lincei, Roma, 1976).
[20] C. Trusdell and C. C. Wang, *Introduction to Rational Elasticity*, (Noordhoff, Leyden, 1973).
[21] C. Truesdell and W. Noll, *The non-linear field theories of mechanics*, Handbuch der Physik, Vol. **III/3**, (Springer, Berlin, 1965).
[22] G.A. Maugin and J. Pouget, "Acoustic equations for one-domain ferroelectric", *J. Acoust. Soc. Am.*, **68**, 575-587 (1980).
[23] B. D. Coleman and M. E. Gurtin, "Thermodynamics with internal state variables", *J. Chem. Phys.*, **47**, 597-613 (1967).

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