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# JOSEPH FOURIER: A DIDACTIC PATH CROSSING LIFE AND WORKS

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ABSTRACT. In this work a didactic path crossing both the life and the works of Joseph Fourier is proposed with the purpose of furnishing an historical reference to some of the most important concepts inherited by him. Particular attention is addressed to the scholar's personality features and to the innovative methods and approaches employed in his research which encompassed not only Fourier analysis but also the greenhouse effect and climate science. The didactic proposal, addressed to university students, also includes a simple application of Fourier analysis as well as a comparison with the wavelet approach, this latter representing one of the newest and more interesting development of Fourier analysis.

## 1. J. Fourier: the character

**1.1. From the first studies to the "Académies".** Jean Baptiste Joseph Fourier, not Fourrier[1], was born in Auxerre on March 21, 1768 from a humble family native of Lorraine, his father was a simple tailor. He lost his parents very early, at the age of about seven to eight years.

His training began when he received the first elements of French and Latin from an organist named Pallais, music teacher at Auxerre cathedral and director of a secondary college (Duché 1871). Under the patronage of the bishop of Auxerre, Jean-Baptiste-Marie Champion de Cicé, he studied, first at the École Militaire of Auxerre, then at the Benedictine congregation of Saint-Maur. J. Fourier was an excellent student although he didn't try very hard. However, when he became passionate about mathematics, he began to study assiduously, so much so that he procured fragments of candles during the day, in order to continue, even at night, to solve problems of mathematics (Cousin 1840).

J. Fourier's education was completed in Paris at the Montaigu College. At sixteen, after finishing his philosophy course he returned to the cole Militaire where the Benedictines welcomed him as their favorite son. There he taught math lessons in conjunction with M. Bonnard, who had been his teacher and was proud to have him as a colleague (Duché 1871).

[1]The surname Fourier was in danger of being confused with Fourrier, a term that in French means harbinger, precursor.

Rejected by the army as not noble, he continued his studies as a Benedictine novice at the abbey of Saint-Benoît-sur-Loire (Cousin 1840). With the advent of the Revolution, in 1789, he abandoned the Benedictine novice habit and was hired as a teacher at the École Militaire in Auxerre, just 21 years old (Missirini 1843). It was an important year for Fourier, he submitted a first work of algebra to the Académie des Sciences and, thanks to his reputation as a professor, he also became a politician with great charisma. He taught in Auxerre until 1794, the year in which he was appointed a pupil of the École Normale (Cousin 1840; Duché 1871).

FIGURE 1. Portrait of J. Fourier.

J. Fourier was particularly interested in Gaspard Monge's descriptive geometry course who, having noticed his student talent, advised him to take an elementary course in mathematics which was highly appreciated by the students (Missirini 1843).

In the middle of the year 1795 the school was closed and thanks to J.L. Lagrange, P.S. Laplace and G. Monge he entered the École Polytechnique as assistant. Here he remained until the French expedition to Egypt, in May 1798, for which he was recruited by G. Monge on behalf of Napoleon Bonaparte. The young scientist set off for that exotic, mysterious, captivating and rich in history land which at that time was considered the ancient cradle of science, and especially of mathematics. In Egypt, the French founded the *"Institut d'Égypte"* strongly desired by Napoleon; it was he who put most of the study topics, for example, he proposed to found an observatory for astronomy and meteorology, a field in which J. Fourier will stand out. The first assembly took place on 24 August 1798 and Monge was elected as president, Bonaparte his deputy, and J. Fourier *secrétaire perpétuel* - perpetual secretary (Cousin 1840). J. Fourier, in Egypt, while also dealing with administrative aspects, worked on many scientific projects, including: a law and a thesis on the general resolution of algebraic equations, a machine to irrigate the earth fed by the wind and various researches of algebra and mechanics general (*Décade égyptienne* 1797; *Décade égyptienne* 1798).

When he returned in France, Napoleon he appointed him prefect of Isère on 2 January 1801. During his prefecture in Isére he completed two major works: the reclamation of the Bourgoin marshes and the Turin-Grenoble link road. It was not easy to make the works, J.

Fourier succeeded thanks to his scientific preparation and his exquisite character and gifted with great negotiating skills. The first allowed him to have the approval of the projects, the second the favor of the population; since the times of Louis XIV, the government had repeatedly tried, without results, to reclaim the marshes by supplying arable land, but because of the contrary claims of all the coastal communities and the conflict of opposing interests, it gave up. Fourier did it (Missirini 1843).

In 1814, during the First Restoration, thanks to its excellent reputation, it was also well received by the monarchy. But as soon as he returned from Elba, Napoleon, during the hundred days, appointed him Prefect of the Rhone (Cousin 1840).

After the Second Restoration, a companion of the expedition to Egypt, G.J.G. de Chabrol as prefect of the Seine and offered him a position as supreme director of the statistics office. By 1815, he had made the firm decision to deal only with science, presenting several texts to the *Académie des Sciences* which, on May 27, 1816, elected him as a member, but King Louis XVIII showed reservations towards J. Fourier because of his role during the Hundred Days. But the *Académie des Sciences* elected him unanimously on 11 May 1817. On 11 December he became a member of the Royal Society of London and on 14 December 1826 of the Académie française succeeding P.É. Lemonley. In addition, after Laplace's death, he was appointed chairman of the improvement board of the École Polytecnique.

**1.2. J. Fourier's strange habits.** J. Fourier was a universal scholar, he also explored the world of ancient traditions and Latin. The scientist had inherited, in spite of himself, from Egypt, a real disease, the habit and the need for extreme heat. Even in summer, he never went out without being very covered, he wore a frock coat over his coat and the servant who accompanied him always kept a large cloak ready. He suffered terribly all winter. He had used his physical talent to warm up well and once back in France he still regretted the Egyptian sun. In fact, he was staunchly convinced that the desert heat was the ideal condition for keeping healthy. In addition to covering himself with clothes and wrapping himself like a mummy, he lived in exaggeratedly heated rooms that his friends who had visited them compared to a union between the Sahara desert and hell (Missirini 1843).

As soon as he returned to Europe he began to suffer from severe rheumatism which was amplified at the slightest cold. He never left home for much of the winter, and his precautions only increased evil. His breathing was compromised, so much so that he was forced to sleep almost on his feet; in order not to stoop and suffocate he had to use a kind of box that held his body upright and left only his head and arms free. His illness led him to death on May 16, 1830 (Cousin 1840).

J. Fourier's obsession with extreme heat also influenced his scientific research, in fact among his studies stand out:

- The theory on the propagation of heat, widely explained in the work *"Théorie analytique de la chaleur"* whose definitive version was published in 1822, it is necessary to underline the importance of this work because it is the one in which the scientific masterpiece of the scholar emerges, that is the "Fourier Transform", a physico-mathematical concept of fundamental importance for various research sectors.

- The greenhouse effect studies which, alongside the studies on the propagation of heat, also led him to important climatological considerations, a science he pioneered.

## 2. From the "Théorie analytique de la chaleur" to the Fourier Transform

**2.1. Mathematics in Fourier's time.** During the first decades of the 1800s, the knowledge possessed by the mathematical world did not yet allow to fully express certain functions, for example those that had particular discontinuities or, in particular, those that showed a *"jump"*. As Ian Stewart wrote in "Taming the Infinite":

"*Before Fourier appeared on the scene, mathematicians were happy enough to know what a function was. It was a type of procedure, $f$, which took a number, $x$, and produced another number, $f(x)$. The numbers $x$ that allow a sensible function depend on the characteristics of $f$. If $f(x) = 1/x$, for example, then $x$ must be different from 0. If $f(x) = \sqrt{x}$, and we work with real numbers, then $x$ must be positive. Pressed, however, to give a definition, mathematicians tended to be somewhat vague. Their difficulties stemmed from being grappled with many different characteristics of the concept of function: not only what is a rule that associates a number $x$ with another number $f(x)$, but what properties does that rule possess, as continuity, differentiability , ability to be represented by a certain type of formula and so on.*"[2]

As mentioned before, a big problem was represented by describing the *"step function"*, the one for which:

$$f(x) = 0 \Rightarrow x \leq 0, \qquad f(x) = 1 \Rightarrow x > 0 \tag{1}$$

*"This function suddenly jumps from 0 to 1 when $x$ crosses 0. There was a widespread belief that the jump resulted in an obvious way from the double definition of the formula: from $f(x) = 0$ to $f(x) = 1$. At the same time it was thought that only in these cases there was a jump; that each uniquely defined formula avoided these jumps, and that a small variation in $x$ always determined a small variation in $f(x)$."*[3]

Similar considerations were expressed for complex field functions and logarithms; the former included two values for the square root, the logarithms, on the other hand, assumed an infinite number of values. It was difficult to set rules to obtain precise values from these functions and it was assumed that infinite different rules were valid simultaneously. J. Fourier, then, invented a method of his own to express certain functions: to use infinite sums of sines and cosines. This interpretation shocked the world of the greatest mathematicians, as Stewart writes:

*"it was Fourier who really irritated them, with his fantastic idea of writing each function as an infinite series of sinuses and cosines, developed on the occasion of his research on heat flow ."*[4]

This result is mainly due to the fact that the French expertly combined his great physical intuition with mathematics, abandoning the rigor of the latter for a few moments.
His description provides an excellent model for real cases, for example, the one considered by J. Fourier which envisaged a system consisting of a half-heated metal bar. Under these

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[2] from the book: Domare l'infinito. Storia della matematica dagli inizi alla teoria del caos, Stewart (2011).
[3] ibidem
[4] ibidem

conditions the temperature distribution could be well described very well with the Fourier development, even if there were some inaccuracies at the point of separation between the regions at different temperatures, this example will be explored later.

The Fourier series also made it possible to describe the *"step function"* previously mentioned, by means of a single development, considering it a portion of a square wave (Dhombres and Robert 1998; Stewart 2011).

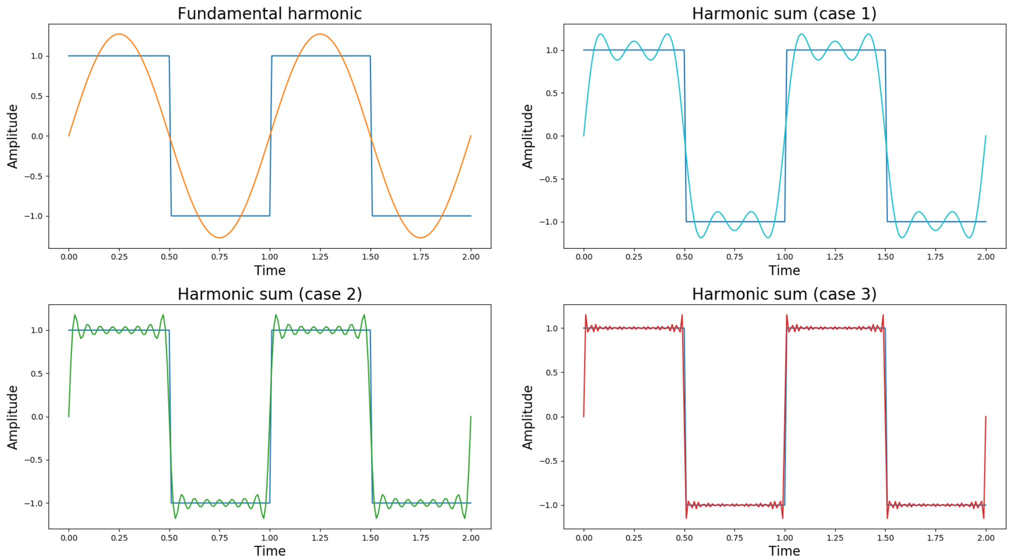



FIGURE 2. In the present figure the Fourier representation is superimposed on a square wave portion. In the first case only the fundamental harmonic is represented, in the second some secondary harmonics are added, in the third other harmonics are added, in the fourth still others. it is evident that, adding more and more terms of higher harmonics, the square wave is approximated more and more precisely.

**2.2. Heat conduction.** J. Fourier's first essay on heat conduction, entitled *"Mémoire sur la propagation de la chaleur dans les corps solides"*, was brought to the attention of the *Académie des Sciences* on December 21, 1807. The academicians, impressed, urged the scientist to present, for the grand prix of 1812, another work on the same subject.

J. Fourier won the prize with his new essay which represented an evolution from that of 1807, however provoking bitter, though not unfair, criticism from J.L. Lagrange, P.S. Laplace and A.M. Legendre. In fact, the aforementioned eminent mathematicians, while recognizing the novelty and importance of J. Fourier's work, argued that his mathematical reasoning was flawed and inaccurate.

The *"Théorie analytique de la chaleur"*, as we know it today, was published in 1822, in this work J. Fourier brought together all his previous works on heat; was defined by the esteemed Lord Kelvin, with the exception of some inaccuracies, a real *"great mathematical poem"* (Bell 2010).

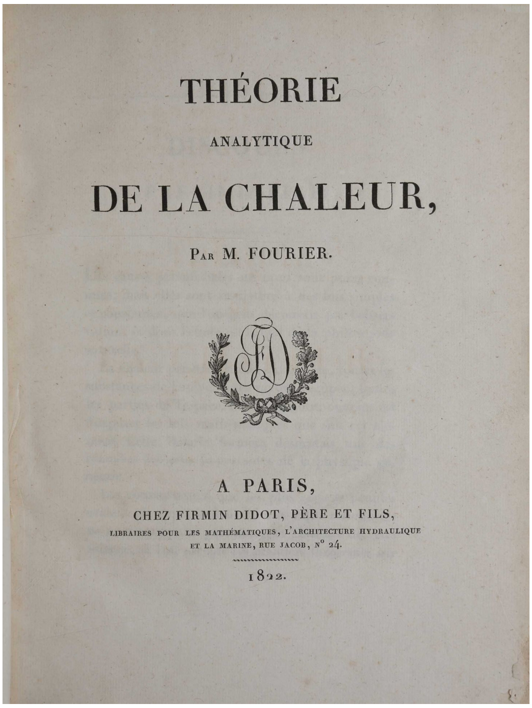

THÉORIE
ANALYTIQUE
DE LA CHALEUR,
PAR M. FOURIER.
A PARIS,
CHEZ FIRMIN DIDOT, PÈRE ET FILS,
LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE
ET LA MARINE, RUE JACOB, N° 24.
1822.

FIGURE 3. *"Théorie analytique de la chaleur"*, 1822 edition (Fourier 1822).

In J. Fourier's work, heat is considered an indestructible fluid and its propagation through the media is studied (Fourier 1822). By studying these phenomena he managed to derive the following relationship, called Fourier's law:

$$dQ = -k\frac{dT}{dn}dSdt \tag{2}$$

where $dS$ is an element of an isothermal surface, $dT\ /\ dn$ is the modulus of the temperature gradient, normal to $dS$, $dQ$ is the heat passing through $dS$ in the time interval $dt$ and $k$ is the thermal conductivity constant, typical of the material (Fourier 1822, 1878; Narasimhan 1999).

To study conduction he used partial differential equations, as P.S. Lagrange did for Newtonian systems. They proved to be a valid tool also taking into account that the conduction of heat was an interaction that took place by contact, one of the first to be studied, unlike the Newtonian one that occurred at even very large distances, such as those between two planets.

Although considerations were then made on other domains and certain conditions imposed, J. Fourier's starting system was as follows: a wall of thickness $s$ which separated two regions having different temperatures $T_1$ and $T_2$, both constant, with $T_1 > T_2$. Then, inside the wall, a $d\tau$ volume element, of surface $dS$ oriented normally to the $x$ axis and of thickness $dx$ concordant with $x$, consequently extends along the axis from the coordinate $x$ to $x + dx$.

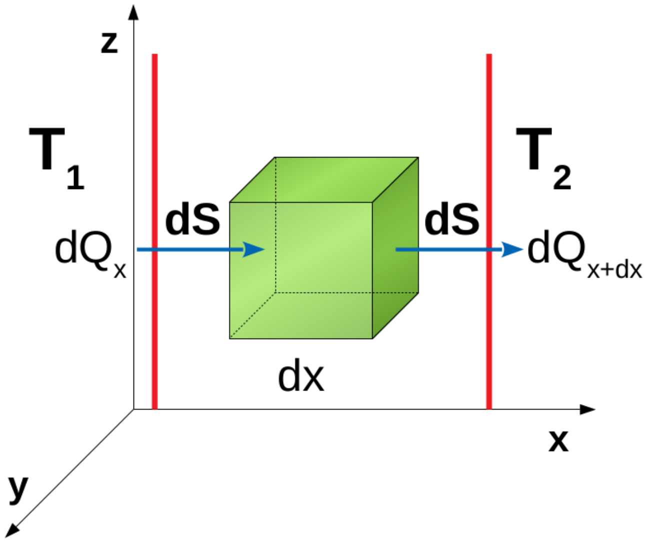



FIGURE 4. Heat propagation experiment scheme.

The heat will then move from the warmest to the coldest body until the thermal equilibrium is reached, so $T_1 = T_2$. For this reason, in the Fourier law a minus sign will appear to the second member to underline that the propagation is opposed to the temperature gradient. The mass $dm$ of the volume element $d\tau$ can be expressed:

$$dm = \rho d\tau \tag{3}$$

Considering the direction of heat diffusion, the $d\tau$ element receives a quantity of heat $dQ_1$ through the surface $dS_1$, of coordinate $x$, and transfers a quantity of heat $dQ_2$ through the surface $dS_2$, of coordinate $x+dx$. Considering that:

$$dS_1 = dS_2 = dS \qquad \text{and} \qquad d\tau = dSdx \tag{4}$$

You get:

$$dQ_1 = -k\left(\frac{\partial T}{\partial x}\right)_x dSdt \tag{5}$$

$$dQ_2 = -k\left(\frac{\partial T}{\partial x}\right)_{x+dx} dSdt \tag{6}$$

By developing $\left(\frac{\partial T}{\partial x}\right)_{x+dx}$ in Taylor Series truncated on the first order, it is possible to write:

$$dQ_2 = -k\left[\left(\frac{\partial T}{\partial x}\right)_x + \left(\frac{\partial^2 T}{\partial x^2}\right)_x dx\right] dSdt \tag{7}$$

Considerind the relation:

$$dQ = dmcdT = \rho d\tau cdT \tag{8}$$

The heat absorbed by the mass element $dm$ and volume $d\tau$, defined as $dQ$ can be expressed as the difference between $dQ_1$ and $dQ_2$:

$$dQ = dQ_1 - dQ_2 = k\frac{\partial^2 T}{\partial x^2} d\tau dt = \rho d\tau cdT \tag{9}$$

where $c$ is the specific heat of the $s$ thickness wall. Rearranging:

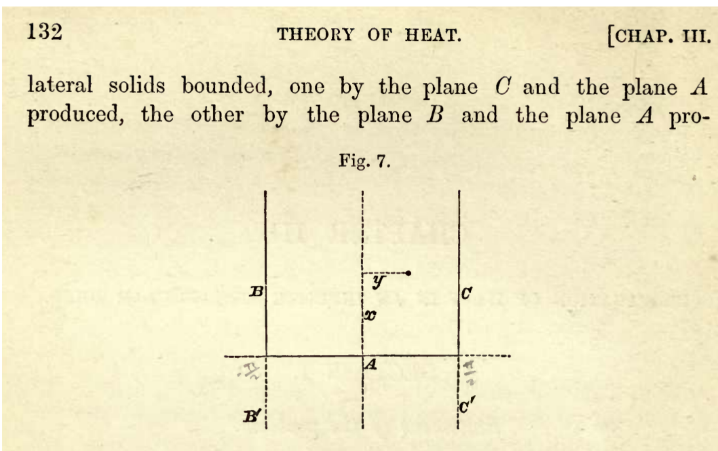
$$\frac{\partial^2 T}{\partial x^2} = \rho \frac{c}{k} \frac{\partial T}{\partial t} \tag{10}$$

This is the Fourier equation relating to the one-dimensional case $T = T(x,t)$ and expresses the variation in temperature, in the thickness considered, as a function of time $t$ and of the $x$ coordinate. It can also be generalized for the three-dimensional, that is case $T = T(x,y,z,t)$ (Fourier 1822, 1878; Wanderlingh 1976; Bailyn 1994; Halliday *et al.* 2001; Dehghani 2007; Bergman *et al.* 2011; Martellotta 2011; Petrillo 2012; Fourier 2013).

J. Fourier, at this point considered a very simple two-dimensional domain: an infinite rectangle with constant temperatures on its three edges, easily achievable for experimental comparison (See Figure 5). In this case, only the $x$ and $y$ coordinates are shown, in addition, the calculation is made for the stationary case, or rather when the temperature does not vary over time, therefore:

$$\frac{dT}{dt} = 0 \quad \rightarrow \quad \frac{\partial^2 T}{\partial t^2} = 0 \tag{11}$$

and it is possible to omit the time variable from equation.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lateral solids bounded, one by the plane $C$ and the plane $A$ produced, the other by the plane $B$ and the plane $A$ produced,

Fig. 7.



FIGURE 5. Diagram from *"The Analytical Theory of the Heat"* (Fourier 1878).

By obtaining an equation of this type (Fourier 1822, 1878) :

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{12}$$

It further simplified the geometry considering that the temperature varied only in the $x$ direction. In steady state conditions, the temperature will be a linear function of $x$ and the solution of a differential equation of this kind will have the form:

$$T = ax + b \tag{13}$$

By keeping the thickness $s$ and by imposing:

$$T_1 = b \rightarrow x = 0 \qquad \text{a}nd \qquad T_2 = as + b \rightarrow x = s \tag{14}$$

Substituting $b$ to $T_1$ in the expression of $T_2$ and rearranging it can be obtained:

$$a = \frac{T_2 - T_1}{s} \tag{15}$$

and:

$$T(x) = T_1 - \frac{T_1 - T_2}{s} x \tag{16}$$

Deriving we obtain the module of a gradient:

$$\left|\nabla T\right| = \left|\frac{dT}{dx}\right| = \frac{T_1 - T_2}{s} x \tag{17}$$

It is verified that, with increasing diffusivity $\eta$, the tendency of the material to respond to the perturbations imposed from the outside becomes faster. The shape explains that the temperature decreases in the wall from $T_1$ to $T_2$, with a constant gradient, therefore linearly. Considering a finished surface $S$, through it, in a time interval $t$, a quantity of heat will pass from the warmest to the coldest environment following the following equatio (Wanderlingh 1976; Bailyn 1994; Dehghani 2007; Martellotta 2011; Petrillo 2012).

$$Q = k \frac{T_1 - T_2}{s} St \tag{18}$$

The Fourier equation for the three-dimensional case is shown below, that is, in which the temperature depends on the three spatial coordinates $x$, $y$ and $z$ (Fourier 1878):

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \rho \frac{c}{k} \frac{\partial T}{\partial t} \tag{19}$$

We specify that J. Fourier uses a different convention, he writes $u$ and $D$ respectively instead of $T$ and $\rho$ used in this text.

To apply this equation J. Fourier will start from the simplest case, will calculate the distribution of temperatures when thermal equilibrium is reached, given the temperatures on the edge (Fourier 1822, 1878).

130. The function sought, which expresses the varied movement of heat in the interior of a solid of cubic form, must therefore be determined by the following conditions:

1st. It satisfies the general equation

$$\frac{dv}{dt} = \frac{K}{C.D}\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right);$$

2nd. It satisfies the three definite equations

FIGURE 6. Equation from *"The Analytical Theory of the Heat"* (Fourier 1878).

This relation, from the physical point of view, expresses the conservation of heat per unit of volume, considering an infinitesimal volume in the region crossed by the flow.

We assume that we have a known temperature at time $t = 0$. Furthermore, to describe the exchange of heat with the external environment, they are considered starting from the conditions of the thermal flow surrounding the domain.

In the case considered by J. Fourier, the stationary one, the temperature does not depend on time and is not expressed as a function of this. If, at the same time, $k$, the thermal conductivity constant is not temperature dependent, the Laplace equation is obtained. $k$, moreover, stands as a proportionality constant, which regulates the relationship between the quantity of heat that passes through a unit surface in units of time and the spatial temperature gradient perpendicular to the surface (Narasimhan 1999).

In the first version of the *"Théorie analytique de la chaleur"*, that of 1807, J. Fourier theorized the thermal conductivity mathematically rather than experimentally. J. Fourier developed this concept gradually, noting the differences due to the transition from discontinuous bodies to a continuous and uniform body (Fourier 1822, 1878; Grattan-Guinness 1972).

The theme of heat flow was still an innovative and infused concept. It will be in 1810 that, through a letter sent to a correspondent whose name is unknown today, J. Fourier will explain it completely for the first time (Herivel 1975).

Another parameter that can take on a very important meaning, both from an experimental and a theoretical point of view, is the specific heat, proposed by A.L. Lavoisier and P.S. Laplace in 1783. This relates the rate at which heat accumulates in an elementary volume with a corresponding change in temperature. Often it becomes operationally convenient to combine thermal conductivity and specific heat by defining a new parameter called thermal diffusivity, defined as follows:

$$\eta = \frac{k}{\rho c} \tag{20}$$

where $\rho$ is the density of the solid. It is shown that a material speeds up the responses to perturbations coming from outside, when diffusivity increases.

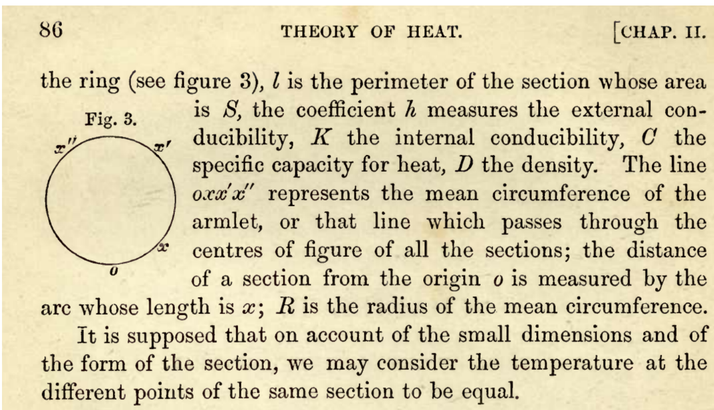
Shortly after the publication of the 1822 edition of J. Fourier's text, the scientific community, while expressing perplexity towards mathematical formalism, recognized the value of its contents, not only in relation to the propagation of heat, but in general as a basis for the development of other branches of science. In a few years the greatness was recognized even by great scholars, of the caliber of J.C.F. Sturm, C.L. Navier, S. Germain, P.G.L. Dirichlet and J. Liouville (Dhombres and Robert 1998).

A few years later many problems were extended to the phenomena of electrical conduction, and, still later, to those of molecular diffusion in liquids and solids. Dynamic gas theory has led to the analogy between gas diffusion and heat diffusion; J. Fourier's heat conduction model was also applied for the flow of blood through the capillary veins and for the flow of fluids in geological soils (Grattan-Guinness 1972; Herivel 1975; Dhombres and Robert 1998).

**2.3. Fourier series.** The interest for the scientist in the topic of heat and temperature led him to carry out a series of experiments followed by personal interpretations, for which J. Fourier started from a very interesting assumption. He interpreted the propagation of heat as if it were through waves. His research is documented in his work *"Théorie analytique de la chaleur"*, dating back to 1807, which was published, after several updates, only in 1822. This was defined by Sommerfield *"the Bible of the mathematical physics"* for its innovative content for the time, for the results, and especially for the fact that it predicted that a mathematical series of sines and cosines could be used to analyze heat conduction in solid bodies (Zappulla n.d.).

To arrive at his brilliant conclusions and the formulation of his series development, J. Fourier carried out a very interesting experiment which he also described in his work.

The test consisted in heating a half of a large metal ring in the fire, those used for anchors. As soon as this appeared incandescent, it was extracted from the fire and immediately immersed in sand, this had the task of insulating it thermally. The temperature was then measured in various positions of the ring. As you can imagine, half that had been in contact with the fire had a much higher temperature; on the contrary, the non-immersed area was less hot. Considering the temporal evolution of the system, the heat spread through the metal and the temperature was distributed more and more uniformly on the points of the circumference, tending to an equilibrium value.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the ring (see figure 3), $l$ is the perimeter of the section whose area is $S$, the coefficient $h$ measures the external conducibility, $K$ the internal conducibility, $C$ the specific capacity for heat, $D$ the density. The line $oxx'x''$ represents the mean circumference of the armlet, or that line which passes through the centres of figure of all the sections; the distance of a section from the origin $o$ is measured by the arc whose length is $x$; $R$ is the radius of the mean circumference.

It is supposed that on account of the small dimensions and of the form of the section, we may consider the temperature at the different points of the same section to be equal.



FIGURE 7. Diagram of the anchors ring, from *"The Analytical Theory of the Heat"* (Fourier 1878).

At this point the system can be simplified considerably by imagining cutting the ring at a point in the circumference and then deforming it to obtain a cylindrical object. Imagining that the latter has a negligible diameter, we refer to a situation that can be approximated with the one-dimensional case. This system was schematized as a body, extended along a single dimension of extremes $A$ and $B$, having constant thermal conductivity $k$.

J. Fourier then imagined a homogeneous conductive bar, insulated on the sides, provided with this assembly a certain amount of heat at the extreme $A$, capable of bringing it to the temperature $T_1$, while the extreme $B$ is heated up to the temperature $T_2$, with $T_1 > T_2$.

At both ends, he considered having thermostats that would allow $T_1$ and $T_2$ to be kept constant, thus preventing the possibility of achieving balance in the metal; the interesting thing he noticed was that, taking a point inside the bar, there were fluctuations in the temperature value; if we wanted to represent their values, we would obtain a curve. This represents an example in support of the scientist's belief about the propagation of heat by wave.

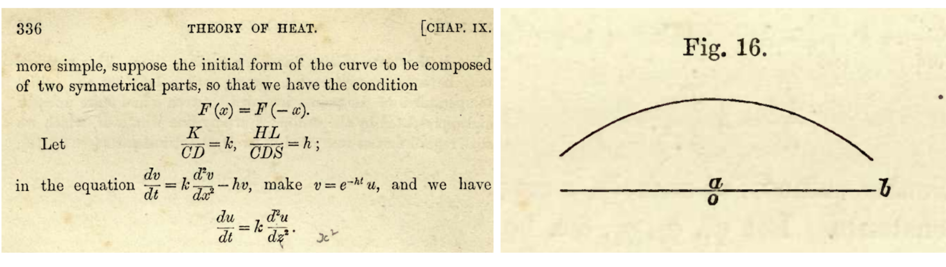
Going forward in the *"Théorie analytique de la chaleur"*, there is another particularly significant example that represents a fundamental intuition for the development of the trigonometric series. J. Fourier considered a system like the one in the Figure 8. A portion of an ideally infinite straight line was depicted which served as a model for a very long and negligible metal bar, in which the heat propagated along only one dimension. Considering two points marked with the letters $a$ and $b$ and imagining to provide heat only at the point $a$ bringing it to a certain temperature $T_1$, higher than $T_2$, at point $b$, this began to propagate along the bar until reaching the thermal equilibrium in the portion considered. Before reaching this condition, however, the measured temperature values showed a decreasing trend moving from $a$ to $b$ and represented described a portion of the curve. The temperature was therefore a function of the position $x$ and could be associated with it a generic function:

$$T = F(x) \tag{21}$$

But the point $a$, in this case, was not an extreme. On the contrary, it was a generic point placed along an infinite straight line. J. Fourier, on the basis of this, deduced that, taking the symmetric point of $b$, with respect to $a$, also the function of the measured temperature moving towards that point, should be symmetric:

$$F(x) = F(-x) \tag{22}$$

By representing them both an arch was obtained.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more simple, suppose the initial form of the curve to be composed of two symmetrical parts, so that we have the condition

$$F(x) = F(-x).$$

Let $\frac{K}{CD} = k, \quad \frac{HL}{CDS} = h;$

in the equation $\frac{dv}{dt} = k\frac{d^2v}{dx^2} - hv$, make $v = e^{-ht}u$, and we have

$$\frac{du}{dt} = k\frac{d^2u}{dx^2}.$$



FIGURE 8. Equation and diagram from *"The Analytical Theory of the Heat"* (Fourier 1878).

At this point the mathematician imagined to represent the curve described with a first sinusoid arc, of $\omega_f$ frequency, called fundamental harmonic. This sinusoid was then corrected by adding other sinuses and cosines of a greater period, called secondary harmonics, until it showed a trend that approximated well the curve section described by the temperature.

He then divided the temperature distribution into a sum of sinusoids, having different values of amplitude, period and phase, called harmonics. Moving towards the thermal equilibrium, the harmonics of the minor period, therefore of greater frequency, faded much faster than those of lower frequency, and the function was more and more regular. The Fourier series was born from the idea of writing a function as the sum of infinite sines and cosines. The development in Fourier series of a function $f(t)$ is defined as:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_f t) + b_n \sin(n\omega_f t) \tag{23}$$

In which the terms $a_n$ and $b_n$ are the so-called Fourier coefficients and can be expressed in the following way:

$$a_n = \frac{2}{T} \int f(t) \cos(n\omega_f t) dt \tag{24}$$

$$b_n = \frac{2}{T} \int f(t) \sin(n\omega_f t) dt \tag{25}$$

$\omega_f$ is called the fundamental pulsation and is obtained from:

$$\nu_f = \frac{1}{T} \qquad \omega_f = \frac{2\pi}{T} \qquad \omega_f = 2\pi\nu_f \tag{26}$$

where $T$ is the period of e $\nu_f$ the fundamental frequency. The Fourier series can be written in two forms, the first involves the use of complex numbers in exponential form:

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{i\omega_f t} \tag{27}$$

in which:

$$c_n = \frac{2}{T} \int f(t) e^{-i\omega_f t} \tag{28}$$

The second form, equivalent to the first:

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_f t + \Phi_n) \tag{29}$$

With:

$$A_n = \sqrt{a_n{}^2 + b_n{}^2} \qquad \tan\Phi_n = \frac{-b_n}{a_n} \tag{30}$$

For each index $n$, terms of amplitude $A_n$ and phase $\Phi_n$ are associated. In the case where $n = 1$ the returned values describe the fundamental harmonic. As $n$ increases, the amplitudes and phases of the upper harmonics with frequency $\nu_n$ are obtained (Fourier 1822, 1878; Dini 1880; Cercignani 1972; Katznelson 1976; Manca and Murri n.d.).

To study the distribution of temperatures, J. Fourier also considered other curvilinear trends associated with different cases.

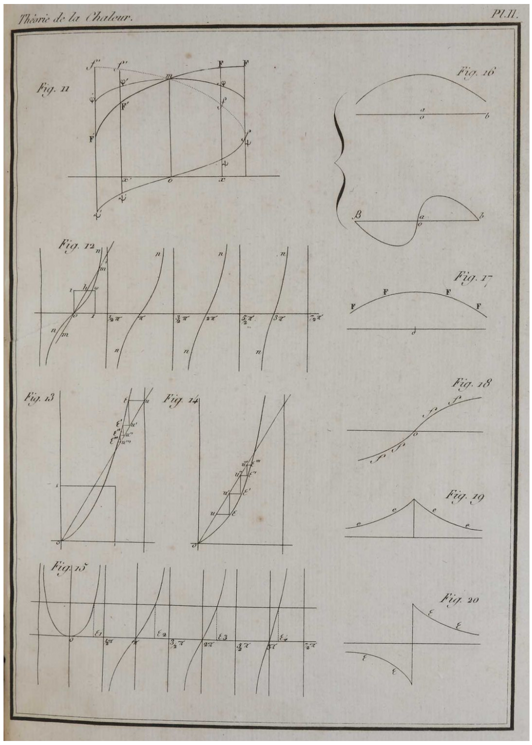



FIGURE 9. Designs of other domains, from *"Théorie analytique de la chaleur"* (Fourier 1822).

The results obtained by J. Fourier were not accepted immediately, his new method was totally in contrast with the cornerstones of the Mathematical Analysis of the early 1800s; as already mentioned, it was unthinkable to describe a discontinuous function, through a single function made up of sums of continuous functions. For example, J.L. Lagrange in 1807 did not allow the publication of the original work; Niels Henrik Abel in 1828 stated: *"the divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever"*. Fourier's analysis was totally at odds with the cornerstones of the Mathematical Analysis of the early 1800s; as already said, it was unthinkable to describe a discontinuous function, through a single function made up of sums of continuous functions. It must be said, however, that the proofs that appear in J. Fourier's writings are not always accurate; they represented a solution to the problem, seen through the eyes of a physicist who, for a moment, had moved away from mathematical rigor in order to build a good model. A few years later P.G.L. Dirichlet, C.F. Gauss and B. Riemann provided rigorous mathematical proofs, confirming the intuitions of J. Fourier, decreeing the birth of the Non-Differential Analysis (Dhombres and Robert 1998; Manca and Murri n.d.).

The trigonometric series appears in chapter III of J. Fourier's masterpiece, in which we begin to define the movement of heat as follows:

*"Ce mouvements se décompose en une multitude de movements élémentaires,*

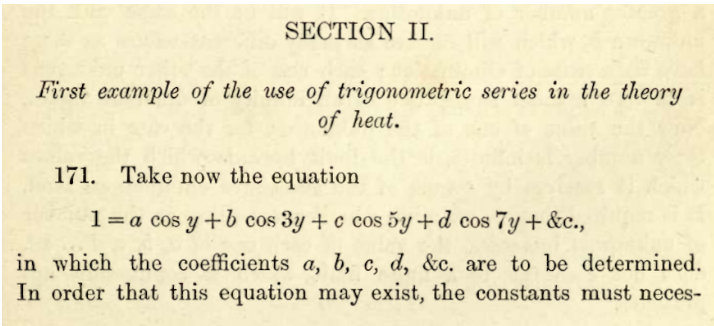
*dont chacun s'accomplit comme s'il était seul"* .

Translated:

*"This movement breaks down into a multitude of elementary movements, each of which is accomplished as if it existed alone" (Fourier 1822).*

Recall that the existence of the indestructible calorie flow was still hypothesized. The first trigonometric series that appears is the following:

$$1 = a\cos(y) + b\cos(3y) + c\cos 5(y) + d\cos(7y) + \ldots \tag{31}$$


SECTION II.

*First example of the use of trigonometric series in the theory of heat.*

**171.** Take now the equation

$$1 = a \cos y + b \cos 3y + c \cos 5y + d \cos 7y + \&c.,$$

in which the coefficients $a$, $b$, $c$, $d$, &c. are to be determined. In order that this equation may exist, the constants must neces-

FIGURE 10. Equation from *"The Analytical Theory of the Heat"*.

In which $a, b, c, d, \ldots$, are coefficients to be determined. Reference is then made to the validity of the series, in order for this equation to exist, the constants must necessarily satisfy the equations obtained by successive differentiations:

$$0 = a\sin(y) + 3b\sin(3y) + 5c\sin 5(y) + 7d\sin(7y) + \ldots \tag{32}$$

$$0 = a\cos(y) + 3^2 b\cos(3y) + 5^2 c\cos(5y) + 7^2 d\cos(7y) + \ldots \tag{33}$$

$$0 = a\sin(y) + 3^3 b\sin(3y) + 5^3 c\sin(5y) + 7^3 d\sin(7y) + \ldots \tag{34}$$

**2.4. Fourier Tansform.** The trigonometric series is presented in the manuscript *"Mémoire sur la propagation de la Chaleur dans les corps solides"*, dated 1807. We must wait for the updated version, which changed its name to *"Théorie analytique de la chaleur"*,, dated 1811, in order to read about Fourier transform, which appears for the first time in chapter IX, in relation to the transmission of heat in infinite-dimensional bodies. Heat transmission is also the most developed concept, in the variants in which this occurs between disjointed masses and in that which provides continuous bodies of finite size. Mathematically, there is a wide use of Fourier series development. In the 1811 version, J. Fourier addresses the problem only in the simplest case, the one-dimensional one.

The manuscript was further updated in 1822 and the demonstration of the convergence of the integral formula was present for the first time. The *"Théorie analytique de la chaleur"* of 1822 however already contains the demonstration of the convergence of the integral formula, absent in 1811, the terms "Transformed" and "Transformation" also appear, although they do not have, conceptually, the same meaning that is given to them today. In order to read the final version, it is necessary to wait for the years 1824 and 1826 (Fourier 1822, 1878, 2013; Manca and Murri n.d.).

Other great studious, especially mathematicians, also faced similar themes, mainly related to conductive and undulatory phenomena; some also had several analogies with those described by J. Fourier. For example, in 1815, A.L. Cauchy delivered a manuscript to the *Académie* on the propagation of waves on the surface of a liquid, which contained a demonstration of the convergence of its integral formula. A.L. Cauchy won the *Académie Prize* in 1815 and his text was published in 1827 (Cauchy 1882).

S.D. Poisson, however, won the award in 1816, with a work entitled *"Mémoire sur la distribution de la chaleur dans les corps solides"* which also contained the integral formula. The manuscript was printed immediately after and represents the first official publication of the integral formula (Poisson 1821).

A.L. Cauchy and S.D. Poisson, together with J. Fourier, described the state of a physical system that leads in any case to a second order partial differential equation, linear, with constant coefficients, whose solution is expressed in the form of an integral representation. The profound differences, however, were in the approach shown. A.L. Cauchy and S.D. Poisson aimed to demonstrate the convergence of the integral formula regardless of the starting physical problem. Above all, the former developed very extensive proofs that led to very rigorous mathematical conclusions. J. Fourier, on the other hand, gave priority to the solution of the physical problem, generalizing the method of serial development he devised to non-periodic functions. It seems that S.D. Poisson was inspired by both J. Fourier and A.L. Cauchy but there is no certainty; all three may have discovered the integral formula independently (Manca and Murri n.d.).

Regardless of the chronology of events, J. Fourier's contribution in the field of mathematical analysis was extremely important. The innovative transform concept has had a huge impact, especially in future decades, to date. At the time of J. Fourier it was not yet fully exploited and was never applied as an operator between functions (Manca and Murri n.d.). The Fourier transform is defined today:

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt \tag{35}$$

with $\omega$ which can take values from $-\infty$ to $+\infty$, and:

$$e^{-i\omega t} = \cos(\omega t) + i\sin(\omega t) \tag{36}$$

The Fourier Transform admits, under appropriate conditions, the existence of an inversion formula that allows to reconstruct the function from the transform. It is called inverse Fourier transform and is expressed in the form:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)e^{i\omega t}d\omega \tag{37}$$

$f(t)$, in this case, is represented as an integral sum of complex sinusoids, each of amplitude $\frac{1}{2\pi}\hat{f}(\omega)d\omega$.

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pour $u$ la valeur particulière $a$ cos. $q x e^{-kq^2t}$ ; $a$ et $q$ sont des constantes arbitraires. Soient $q_1, q_2, q_3, q_4 \ldots$ etc. une suite de valeurs quelconques, et $a_1, a_2, a_3, a_4 \ldots$ etc., une suite de valeurs correspondantes du coëfficient Q, on aura

$$u = a_1 \cos.(q_1 x) e^{-kq_1^2 t} + a_2 \cos.(q_2 x) e^{-kq_2^2 t} + a_3 \cos.(q_3 x) e^{-kq_3^2 t} + \text{etc.}$$

Supposons 1° que les valeurs $q_1, q_2, q_3, q_4 \ldots$ etc., croissent par degrés infiniment petits, comme les abscisses $q$ d'une certaine courbe; en sorte qu'elles deviennent égales à $dq$, $2dq$, $3dq$, $4dq, \ldots$ etc.; $dq$ étant la différentielle constante de l'abscisse; 2° que les valeurs $a_1, a_2, a_3, a_4 \ldots$ etc. sont proportionnelles aux ordonnées Q de la même courbe, et qu'elles deviennent égales à $Q_1\, dq$, $Q_2\, dq$, $Q_3\, dq \ldots$ etc. Q étant une certaine fonction de $q$. Il en résulte que la valeur de $u$ pourra être exprimée ainsi :

$$u = \int dq\, \text{Q}.\cos.\, q x\, e^{-kq^2 t}.$$

Q est une fonction arbitraire $fq$, et l'intégrale peut être prise de $q = 0$ à $q = \frac{1}{0}$. La difficulté se réduit à déterminer convenablement la fonction Q.

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Pour y parvenir, il faut supposer $t = 0$ dans l'expression de $u$ et l'égaler à $\text{F}x$. On a ainsi l'équation de condition

$$\text{F}x = \int dq\, \text{Q}.\cos.\, q x.$$

Si l'on mettait au lieu de Q une fonction quelconque de $q$,

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et que l'on achevât l'intégration depuis $q = 0$ jusqu'à $q = \frac{1}{0}$, on trouverait une fonction de $x$; il s'agit de résoudre la question inverse, c'est-à-dire, de connaître quelle est la fonction de $q$ qui, étant mise au lieu de Q, donnera pour résultat la fonction $\text{F}x$, problème singulier dont la solution exige un examen attentif.

En développant le signe de l'intégrale, on écrira comme il suit l'équation dont il faut déduire la valeur de Q :

$$\text{F}x = dq\,\text{Q}_1 \cos.\, q_1 x + dq\,\text{Q}_2 \cos.\, q_2 x + dq\,\text{Q}_3 \cos.\, q_3 x + dq\,\text{Q}_4 \cos.\, q_4 x + dq\,\text{Q}_5 \cos.\, q_5 x + \text{etc.}$$

Pour faire disparaître tous les termes du second membre, excepté un seul, on multipliera de part et d'autre par $dx$ cos. $rx$, et l'on intégrera ensuite par rapport à $x$ depuis $x = 0$ jusqu'à $x = n\pi$, $n$ étant un nombre infini; $r$ représente une grandeur quelconque égale à l'une des suivantes : $q_1, q_2, q_3, q_4 \ldots$ etc., ou ce qui est la même chose $dq$, $2dq$, $3dq$, $4dq \ldots$ etc. Soit $q_i$ une valeur quelconque de la variable $q$, et $q_j$ une autre valeur qui est celle que l'on a prise pour $r$; on aura $r = q_j\, dq$ et $q = q_i\, dq$. On considérera ensuite le nombre infini $n$ comme exprimant combien l'unité de longueur contient de fois l'élément $dq$, en sorte que l'on aura $n = \frac{1}{dq}$. En procédant à l'intégration, on reconnaîtra que la valeur de l'intégrale $\int dx \cos.\, qx \cos.\, rx$ est nulle, toutes les fois que $r$ et $q$ sont des grandeurs différentes; mais cette même valeur de l'intégrale est $\frac{1}{2} n\pi$, lorsque $q = r$. Il suit de là que l'intégration élimine dans le second

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membre tous les termes, excepté un seul : savoir, celui qui contient $q_j$ ou $r$. La fonction qui affecte ce même terme est $\text{Q}_j$, on aura donc

$$\int dx\, \text{F}x.\cos.\, qx = dq.\text{Q}_j \tfrac{1}{2} n\pi,$$

et mettant pour $n\, dq$ sa valeur 1, on a

$$\frac{\pi \text{Q}_j}{2} = \int dx\, \text{F}x \cos.\, qx :$$

on trouve donc en général $\frac{\pi \text{Q}}{2} = \int_0^{\frac{1}{0}} dx\, \text{F}x \cos.\, qx$. Ainsi, pour déterminer la fonction Q qui satisfait à la condition proposée, il faut multiplier la fonction donnée $\text{F}x$ par $dx$ cos. $qx$, et intégrer de $x$ nulle à $x$ infinie, en multipliant le résultat par $\frac{2}{\pi}$; c'est-à-dire, que de l'équation

$$\text{F}x = \int dq\, fq \cos.\, qx,$$

on déduit celle-ci, $fq = \frac{2}{\pi} \int dx\, \text{F}x \cos.\, qx$, la fonction $\text{F}x$ représentant les températures initiales d'un prisme infini dont une partie intermédiaire seulement est échauffée. En substituant la valeur de $fq$ dans l'expression de $\text{F}x$, on obtient l'équation générale

$$\frac{\pi}{2} \text{F}x = \int dq \cos.\, qx \int dx\, \text{F}x \cos.\, qx. \qquad (\varepsilon)$$

347.

Si l'on substitue dans l'expression de $v$ la valeur que l'on a trouvée pour la fonction Q, on a l'intégrale suivante, qui contient la solution complète de la question proposée

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valle X est infini : alors les limites $a$ et $b$ sont évidemment des constantes entièrement arbitraires.

419.

Le théorême exprimé par l'équation (B) offre aussi diverses applications analytiques, que nous ne pourrions exposer sans nous écarter de l'objet de cet ouvrage; mais nous énoncerons le principe dont ces applications dérivent.

On voit que, dans le second membre de l'équation

$$fx = \frac{1}{2\pi} \int_a^b d\alpha\, f\alpha \int_{-\infty}^{+\infty} dp \cos.\, (px - p\alpha), \qquad (\text{B})$$

la fonction $fx$ est tellement transformée, que le signe de fonction $f$ n'affecte plus la variable $x$, mais une variable auxiliaire $\alpha$. La variable $x$ est seulement affectée du signe cosinus. Il suit de là que, pour différencier la fonction $fx$ par rapport à $x$, autant de fois que l'on voudra, il suffira de différencier le second membre par rapport à $x$ sous le signe cosinus. On aura donc, en désignant par $i$ un nombre entier quelconque,

$$\frac{d^{2i}}{dx^{2i}} fx = \pm \int d\alpha\, f\alpha \int dp\, p^{2i} \cos.\, (px - p\alpha).$$

On écrit le signe supérieur lorsque $i$ est pair, et le signe inférieur lorsque $i$ est impair. On aura en suivant cette même règle relative au choix du signe :

$$\frac{d^{(2i+1)}}{dx^{2i+1}} fx = \mp \frac{1}{2\pi} \int d\alpha\, f\alpha \int dp\, p^{2i+1} \sin.\, (px - p\alpha).$$

On peut aussi intégrer plusieurs fois de suite, par rapport à $x$, le second membre de l'équation (B); il suffit d'é-

FIGURE 11. First examples of Fourier Transform in the *"Théorie analytique de la chaleur"*, 1822 edition (Fourier 1822).

$\hat{f}(\omega)$ expresses the harmonic richness of the signal relative to the omega frequencies considered. It is shown that the anti-transform of the transform coincides with the starting function $f$ (Bracewell 1999; Lizorkin 2002; James 2011; Manca and Murri n.d.).

The transformed today turns out to be an operator, as well as widely used, with great scientific charm, for this reason it is still the object of continuous studies (Manca and Murri n.d.). Many, especially scientists, have exposed themselves about the Transform and about J. Fourier himself, among them the British physicist Sir James Jeans (1877-1946) who said: *"Fourier's theorem tells us that every curve, regardless of its nature or the way it was originally obtained, can be reproduced exactly, by superimposing a sufficient number of simple harmonic curves. In a nutshell, each curve can be constructed accumulating waves"* (Pickover 2012).

From the operational point of view it represents one of the greatest successes of mathematics and science in general, it is among the most widespread thanks to its close association with the signals. Especially in the case of electrical signals, since they are present in various fields such as: telecommunications, engineering, music, biology, medicine, information technology and research in general (Castorina *et al.* 2018; Manca and Murri n.d.). In signal theory, in fact, the formulas of J. Fourier's Transform allow the passage from the time domain to the frequency domain and vice versa. A signal described instant by instant by a wave can be broken down into a series of harmonics and transformed into another function. The following paragraph will discuss a simple but explanatory example.

**2.5. Simple FT applications.** To give an example, among many others, of the possible applications of the Fourier Transform, we will deal with the simplest possible case. It will be assumed to have a generic sinusoidal signal, of $A_1$ amplitude and $\omega_s$ frequency, which is disturbed by a background noise, also periodic, having its own $A_2$ amplitude and $\omega_n$ frequency. This model, suitably adapted, can well represent real situations, such as when talking on the phone you perceive a constant and annoying background hum, or the disturbances that are felt in audio reproduction devices in case of problems related to the earthing of the electrical system.
The application of the Transform, followed by the inverse Transform, allows to subtract the contribution of noise.

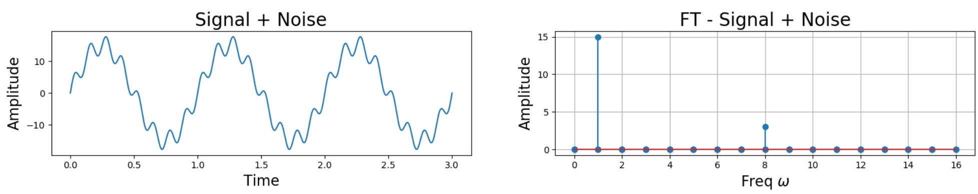



FIGURE 12. Starting signal with noise, and its FT.

The starting signal is shown on the left of Figure 12, it is clear from the graph that the known function sinus relative to the signal shows significant effects due to noise. We will now proceed to break down the signal into its two components through the Transform, passing from the time domain to the frequency domain. This, calculated through an FFT algorithm, is graphed to the right of the signal. The spectrum of the FT shows, as was legitimate, two very narrow peaks related to the $\omega_s$ and $\omega_n$ frequencies. Now, the two contributions relating to signal and noise are quite distinct and it is possible to subtract that relating to noise. Then you can proceed by applying the inverse Transform which reconstructs the signal, net of noise, again in the time domain.

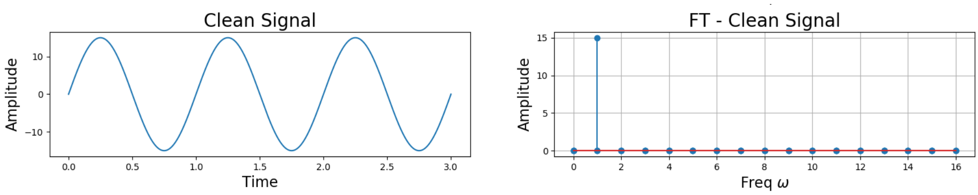



FIGURE 13. Signal without noise and its FT.

The noise-free signal is shown in the Figure 13, flanked by its FT. By observing the latter, it is possible to experimentally verify, in a certain sense, the validity of the Fourier inversion theorem, since the spectrum is identical to that of the FT of the starting signal, obtained immediately after the noise subtraction, before apply the inverse Transform which allows to reconstruct the "Clean Signal".

It should be noted that it was voluntarily chosen to examine a signal of amplitude $A_1 > A_2$ and $\omega_s < \omega_n$ in order to be able to easily distinguish the two contributions in the spectra. Through the application of FT we have therefore obtained a noise-free signal, the concept obviously can be generalized to more complex cases.

**2.6. Evolutions of the Fourier Transform: The Wavelet Transform.** The FT allows you to break down a signal and reconstruct it, equal to itself, without loss of information. However, this is valid only for stationary signals, to put it better: analysis by FT, although localized in frequency, does not provide any information on time evolution.

To overcome the limits of FT, the concept of fenestrated Transform was introduced; this analysis consists in the introduction of a "window" function of a determined amplitude which moves along the time axis by performing an "localized over time" FT. In 1946 the Hungarian Dennis Gabor, using the Gauss function as a window function, experimented with the STFT, Short Time Fourier Transform. However, inaccuracies were found due to the choice of function. The STFT returned an equal resolution over time for ever higher frequencies, and the width of the window function did not vary during the analysis (Gabor 1946; Caccamo and Magazù 2018b).

In the 70s, however, the Frenchman Jean Morlet, during geophysical experiments, proposed a different method to apply the STFT while keeping the window function fixed by changing the width of the same window by means of compression and expansion procedures (Morlet *et al.* 1982; Caccamo and Magazù 2018b).

So it was that in 1984 J. Morlet, together with Alex Grossmann, introduced the "Wavelet" analysis. The application of this allows you to overcome the time limits of J. Fourier's analysis, thus resulting significantly higher in terms of performance (Caccamo and Magazù 2018b).

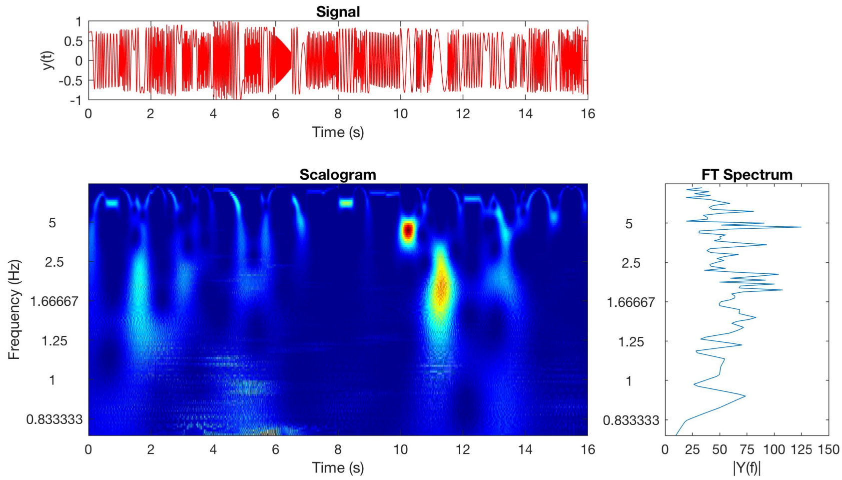


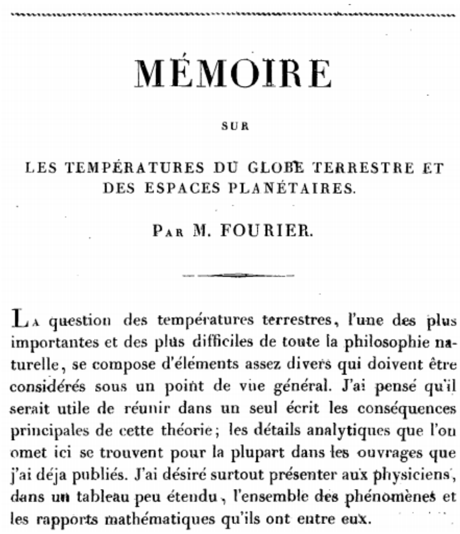

FIGURE 14. FT and Wavelet analysis.

The image in Figure 14 represents the spectrum of an acoustic signal that evolves over time, to which the FT and Wavelet analyzes are applied. We can deduce what was said before, the FT provides valuable information about the frequency but there is no temporal information; this limit is overcome with the Wavelet approach (Caccamo and Magazù 2016; Caccamo and Magazù 2018a; Caccamo *et al.* 2018; Caccamo and Magazù 2018b).

## 3. Fourier, towards the Greenhouse Effect

The French scientist is remembered primarily for the series, the Transform and the law for the conduction of heat, but his contribution to the sciences was certainly not limited to this. This section will discuss some very important consequences and considerations deriving above all from the extension of the concepts described above to larger systems.

J. Fourier will move from the anchor ring of the *"Théorie analytique de la chaleur"*, described in section 1, to the planet Earth and the interplanetary spaces of the work *"Mémoire sur les températures du globe terrestre et des espaces planétaires"*. The results, also in this case, had an incredible impact, above all because they were the basis for future research, including topics studied still, today after 200 years, and allow to crown J. Fourier as an innovator and absolute pioneer. One of the first questions he dealt with was the determination, through thermodynamic considerations, of the age of the Earth.



MÉMOIRE

SUR

LES TEMPÉRATURES DU GLOBE TERRESTRE ET DES ESPACES PLANÉTAIRES.

PAR M. FOURIER.

LA question des températures terrestres, l'une des plus importantes et des plus difficiles de toute la philosophie naturelle, se compose d'éléments assez divers qui doivent être considérés sous un point de vue général. J'ai pensé qu'il serait utile de réunir dans un seul écrit les conséquences principales de cette théorie; les détails analytiques que l'on omet ici se trouvent pour la plupart dans les ouvrages que j'ai déja publiés. J'ai désiré surtout présenter aux physiciens, dans un tableau peu étendu, l'ensemble des phénomènes et les rapports mathématiques qu'ils ont entre eux.

FIGURE 15. *"Mémoire sur les températures du globe terrestre et des espaces planétaires"* (Fourier 1827).

**3.1. The age of the Earth.** With the approach of 1800, the scientific community, driven by the great strides made over the centuries, refused to take the date of birth of the planet imposed by the Church as correct a priori. The Christian world was firm on the belief that the planet originated in 4004 b.C. and was consequently about 5800 years old.

The first to start the studies was G.L. Leclerc de Buffon proposing one on the theory based on the study of fossils and erosion of rocks (Krivine 2017).

In 1812 J. Fourier, based on the mathematical methods treated in the *"Mémoire sur la propagation de la Chaleur dans les corps solides"* and in the *"Théorie analytique de la chaleur"*, tries to answer the question by considering the Earth as a body that, initially very hot, it was gradually cooling. He had obtained a formula like:

$$\Gamma = \frac{T_0 - T_c}{\sqrt{\pi k t}} \tag{38}$$

Where $\Gamma$ indicates the temperature gradient on the surface at time $t$ which represents the time elapsed from the initial conditions, $k$ is the physical constant of conductivity, with $T_0$ and $T_c$, instead, the temperatures are indicated.

He then thought he could describe the age of the planet as the time needed for the Earth to pass from the initial temperature (to creation) to the temperature that he himself measured. By making the time explicit, you can easily get:

$$t = \frac{(T_0 - T_c)^2}{\pi k \Gamma^2} \tag{39}$$

The physical constant of conductivity $k$ is assigned the value of that of iron; with $T_0$ and $T_c$ the temperatures are still indicated, $t$ represents the cooling time (Fourier 1822, 1827, 2013). It is believed that the value assigned to $k$ depends on the findings that J. Fourier was able to make on the experiments conducted by G.L.L. Buffon on his ironworks near Montbard, which attributed the planet to an age of 74 047 years (Deparis 2001).

The results obtained from various experimental and theoretical tests, even by adopting extreme parameters, were very far from the approximately 5800 years set by the Church and universally accepted. About forty years later, Lord Kelvin again applied J. Fourier's calculations by optimizing them and realizing a numerical application. Kelvin studied this question for decades, so much so that many erroneously attribute the formula to it. In reality it had the merit to use it with more precise values than those used by J. Fourier and G.L.L. Buffon, resulting in an age of about 100 million years. J. Fourier probably shared his thoughts with F. Arago. The latter, in the eulogy which he pronounced after the death of J. Fourier, confessed:

*"Among Fourier's formulas, there is one, intended to give the value of the secular cooling of the globe and the number of centuries since the origin of this cooling. The highly controversial question of the antiquity of our land, including its incandescence period, therefore boils down to a thermometric determination. Unfortunately, this point of theory is subject to serious difficulties. Furthermore, the thermometric determination, due to its excessive smallness, would have been reserved for centuries to come"* (Arago 1833).

New reflections led him to publish the manuscript *"Mémoire sur les températures du globe terrestre et des espaces planétaires"* in 1824.

**3.2. "Mémoire sur les températures du globe terrestre et des espaces planétaires".** From the first lines of the essay the ambitious intentions of the French scientist are understood, his goal was to create a solid basis for future studies on the temperature of the Earth. The goal has undoubtedly been achieved, given that still today the work is considered today as the basis for understanding the thermal balance of planetary atmospheres and for tackling *"the problem of the temperature of the Earth in its cosmological context"* (Grinevald 1992).

One of the fundamental concepts studied is that of the exchange of energy between the earth and its environment. J. Fourier identified three main sources to which the Earth owes the heat received:

- The sun's rays, which immediately connected to the diversity of climates in different places on the planet;
- The radiation of the stars surrounding the solar system;
- Geothermal heat, defined as the primitive heat that the Earth has preserved inside (Fourier 1824, 1827, 1890).

From these he tried to determine a balance of the heat flow, on which the temperatures were dependent. He distinguished in particular two contributions, linked to the sources mentioned above. The one due to the flow of geothermal heat that comes from the center of the Earth; the other to the radiation incident on the surface, characterized by the day/night cycle and the alternation of the seasons. Furthermore, recognizing the linear nature of the diffusion of heat in a solid, he studied the two contributions separately, being able, later, to add up their effects.

The geothermal flow, therefore according to which the heat propagated from the center of the Earth towards the surface, was the most treated part in the text of J. Fourier, probably because it was possible to base both on direct observations and on his mathematical formalism which well described the heat propagation in solid bodies. He observed that the temperature increased by about 1 °C when the depth increased by 30 or 40 meters, deducing that the heat flow due to this gradient has only a very small impact on the surface temperature. The oceans, on the other hand, had an inverse behavior to that of the earth's soil, the temperature decreased with increasing depth (Dufresne 2006).

To demonstrate that this observation was not in contradiction with the existence of geothermal flows, he explained the principle of the functioning of the thermohaline circulation as follows:

*"When the temperature of the upper layers of the liquid becomes lower than that of the lower parts, although the temperature of the melting ice is only a few degrees higher, the density of these upper layers will increase; they will go down more and more and will occupy the bottom of the basins which will cool down due to their contact; at the same time the warmer and lighter waters will rise to replace the upper waters and infinitely variable movements will be established in the liquid masses, the general effect of which will be to transport heat to the higher regions" (Fourier 1824, 1827, 1890).*

Due to convection, the deep ocean contains water whose temperature is the one for which the density is maximum. Finally, he concluded that heat exchanges with the interior of the Earth played a negligible role on the equilibrium temperature of the Earth's surface.

For the exchange of heat due to radiation, J. Fourier observed that the seasonal cycle of heat flow and temperature diminished moving away from the surface. Furthermore, he assumed that the Earth's atmosphere had to, due to an unclear phenomenon, retain part of the heat that the planet had received from the outside (Dufresne 2006; Caccamo *et al.* 2019).

**3.3. The basis for the Greenhouse Effect.** J. Fourier considered the atmospheric processes similarly to those observed in the experiments performed some time earlier by Horace Benedict de Saussure, whose experimental setup was very similar to a greenhouse (Saussure and H. 1779; Dufresne 2006).

The setup used consisted of a thermally insulated box, having a black background and topped with a triple glass. This device was a real precursor of the solar thermal collector, used for example to obtain hot water. H.B.De Saussure observed that the temperature inside the box, exposed to the sun, reached very high values compared to the outside.

The concept of *"dark heat"* was already known, discovered by William Herschel about twenty five years earlier, which referred to infrared radiation at the time. It had been discovered but not yet well characterized. In spite of this, J. Fourier, with a very brief and incomplete treatment on infrared radial exchanges, gave his personal and valid interpretation to the result of H.B. de Saussure. He considered solar radiation to be different from dark heat in that it was visible and, as the French said, *"bright"*. This, however, once entered the box, would have heated the air and the internal walls, losing its *"brightness"* and retaining only typical properties of the dark heat, which remained stuck in the container as it was unable to pass through it. The temperature therefore tended to increase, until the heat was compensated by that dissipated.

J. Fourier also drew a parallel between window panes and Earth's atmosphere: both are transparent to visible radiation and opaque to infrared radiation. The atmosphere absorbs this infrared radiation by limiting the amount of energy transferred to space (Fourier 1824, 1827, 1890; Dufresne 2006). These are the words of J. Fourier:

*"Thus the temperature is increased by the interposition of the atmosphere, because the heat finds fewer obstacles to penetrate the air, being in the state of light, compared to those it finds in the air when it is converted into dark heat"* (Fourier 1824, 1827, 1890).

So J. Fourier, analyzing it and describing its essential principles, laid the foundations for what will later be called **Greenhouse Effect** (Bard 2004; Pierrehumbert 2004) John Tyndall, some time later, will explain the important role of the greenhouse effect in the alternation day/night and that of the seasons (Tyndall n.d.).

Today, the greenhouse effect is one of the most studied phenomena, especially in the field of climatology. In particular, an attempt is made to investigate the behavior of water vapor, which is the main greenhouse gas, the vertical distribution of which represents one of the main ones in the process of estimating past and future climate changes (Dufresne 2006).

**3.4. Atmospheric and oceanic motions.** J. Fourier demonstrated his awareness that the atmosphere and the ocean could carry heat in their movement although he believed that liquids in general conducted heat hardly. On the other hand, he considered them capable, in combination with the centrifugal force, of mixing.

The scientist, however, underestimated the contribution of these heat transports and considered their effects negligible, too weak to compete with the greenhouse effect. He also believed that they had no influence on the average temperature difference between the equator and the poles.

J. Fourier was most likely misled because he did not have the appropriate means to quantify the transport of heat by the atmosphere and the ocean; the first reliable results in this regard were obtained only in the 70s of the last century, thanks to the use of satellites. The heat transported in total is now measured with good precision but there are still strong uncertainties about the percentages carried individually by the atmosphere or by the waters.

Another correct intuition relates to the temperature of the interplanetary space. Assuming that the polar regions do not receive solar radiation in the winter months and that the geothermal contributions of ocean and atmosphere transport motions are negligible, J. Fourier suggested a temperature for the planetary space similar to that of the polar regions in winter, about 220K. However, the value differs from the estimated 165-200K today, precisely because of the incorrect interpretation of the role of the transport mechanism due to the oceans and atmosphere.

He then explained that the value he theorized, which today can be defined as too high, was due to the billions of celestial bodies that radiated from every direction. But even this was overestimated because the interstellar space emits a radiation corresponding to that of a 3K black body. He further justified his results by describing apocalyptic scenarios which, according to him, would have occurred following a lower spatial temperature value (Fourier 1824, 1827, 1890; Dufresne 2006).

He also argued that a zero temperature of the interplanetary space would make the Earth too sensitive to a change in the Earth-Sun distance, to a change in eccentricity; texts that appeared later supported J. Fourier's hypotheses, arguing that the glacial-interglacial periods could be connected to variations in the relative positions of the Earth and the Sun (Fourier 1827; Bard 2004).

**3.5. Science refers to J. Fourier … and not only for the FT.** J. Fourier, again, as a great innovator, can be called the initiator and father of Ecophysics, since many basic contents for this scientific sector rest their foundations on the theories and arguments of the illustrious French scientist. Often, we arrive at the Fourier Equation, even considering much more recent and current concepts. For example, if you want to calculate Gibbs' *availability* $W$, defined:

$$dW = (P - P_0)dV - TdS\left(1 - \frac{T_0}{T}\right) = dW_f + dW_t \tag{40}$$

In irreversible natural processes the $dW$ is dissipated. In special cases such as shock wave detonation, availability can be extracted. Non-equilibrium thermodynamics is the extension of the equation:

$$dE = dQ - PdV \tag{41}$$

to a system described by measurable fields:

$$T = T(x,y,z,t) \qquad P = P(x,y,z,t)$$

$$\rho = \rho(x,y,z,t) \qquad \vec{v} = \vec{v}(x,y,z,t)$$

The differential form becomes a set of partial differential equations that express the conservation law for energy density and impulse density, that is, they must have the form of continuity equations. In the simple case of a rigid system where the fields $\rho$, $\vec{v}$, $P$ do not exist, the conservation law

$$dQ = dU \tag{42}$$

becomes an equation of continuity

$$\frac{\partial e(x,y,z,t)}{\partial t} = -div\vec{q}(x,y,z,t) \tag{43}$$

Where $e$ is the scalar field internal energy density and is the heat conduction field. If we are in case there are no fluid motions, that is when

$$e = \rho c T \tag{44}$$

Where $c$ is the specific heat at constant volume and

$$\vec{q} = -k\nabla T \qquad where \quad \nabla \quad \text{is a gradient} \tag{45}$$

The continuity equation becomes the Fourier equation:

$$\frac{\Delta T}{\Delta t} = \eta \Delta t \tag{46}$$

where $\Delta$ is the Laplacian operator and $\eta = k/\rho c$.

The Fourier equation is particularly advantageous because a linear differential equation and contains only one unknown field (Sertorio 2009).

## 4. Conclusions

In the present work, which proposes a didactic path addressed to university students, the J. Fourier's works and approaches are analyzed and discussed in the framework of historical and biographic information.
In particular, at the outset a description of J. Fourier's personality is proposed in order to highlight two curious aspects of this scholar. The first one concerns the fact that this "extravagant" scientist, who folded and adapted mathematical laws to physical systems, opposed himself to the great mathematicians of his time, who often looked at him with greta skepticism. The second one is linked to a singular conviction, gained during the Egyptian period, which brought him to wear very heavy coats and clothes even in summer; this conviction probably influenced the choice of the main topics dealt, i.e. the series and the transform analysis for the heat transport process and the greenhouse effect for the atmosphere thermal energy balance. It is also put into evidence how the work carried out by J. Fourier, besides having an enormous impact on physics, was also the starting point for the development of new analysis developments as, for example, the wavelet approach.

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