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A NON-EQUILIBRIUM THERMODYNAMIC APPROACH TO STUDY THE BEHAVIOUR OF MAGNETIZABLE MEDIA

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ABSTRACT. In this didactic paper we formulate a linear theory for magnetizable media in the framework of classical irreversible thermodynamics (CIT) and using its general methods we investigate the behaviour of these media and the possibility of cross effects among magnetic phenomena, heat conduction, electric conduction and mechanical phenomena. In Galilean approximation all the processes occurring inside these materials are governed by two groups of laws: Maxwell equations and the balance equations. The entropy production is analyzed following the classical thermodynamic procedures and the constitutive equations in the linear anisotropic and isotropic case are derived. The obtained results have applications in several fields of applied sciences, like medicine, biology, physics, chemistry and others.

1. Introduction

Thermodynamics is one of metasciences having the tendency to unify the knowledge. In particular, classical irreversible thermodynamics (CIT) was founded by virtue of the studies by Onsager (1931a,b), Eckart (1940a,b, 1948), De Groot (1951), Meixner and Reik (1959), Prigogine (1961), De Groot and Mazur (1984), Kluitenberg (1984), Muschik (1989, 1993), Maugin (1999), Lebon *et al.* (2008), and Jou *et al.* (2010) and others. Thermodynamics is useful in particular problems belonging to the interdisciplinary sciences and its predictions are verified in the experiments and have rigorous foundations. To study the behaviour of a magnetizable medium we have to know the magnetic phenomena at microscopic and mesoscopic scale. The theory of magnetism is very charming (see Chikazumi 1966; Kittel 2004). Maugin (1976, 1977a,b) gives a phenomenological continuous description of the magnetization field in a deformable crystal below its magnetic phase-transition temperature T_{cr}, assuming that in each point of the considered medium there are interactions arising between each magnetic sublattice and the crystal lattice, called spin-lattice interactions, and also short-range magnetic interactions, called spin-interactions. In its continuum phenomenological theory for magnetizable bodies Maugin gives an explanation to the internal mechanisms in magnetizable bodies, defining particular tensors responsible of these interactions. In this paper a model for a magnetizable medium in the framework of CIT is given and for the sake of simplicity the medium is supposed magnetizable of

De Groot-Mazur type, in the sense that only the macroscopic variable magnetization **m**, describing its magnetic behaviour, is considered without internal variables, illustrating the mesoscopic behaviour of the medium.

In Kluitenberg (1973, 1977, 1981), Restuccia and Kluitenberg (1987, 1989), Restuccia (2010, 2014), and Restuccia *et al.* (2016) results were given for magnetizable media with internal variables arising from internal magnetic phenomena (see also Snoek 1938), responsible for magnetic relaxation. In CIT it is assumed the *local equilibrium hypothesis* (see Jou *et al.* 2010; Jou and Restuccia 2011) implying that all the variables defined in *reversible thermodynamics* (called equilibrium thermodynamics) have the same meaning, and that each point in the continuum system taken into account can be considered as a thermodynamic cell sufficiently large to allow it to be treated as macroscopic thermodynamic subsystem, but sufficiently small so that in each cell the reversible thermodynamics is valid. The relationships valid in equilibrium thermodynamics between state variables remain valid outside equilibrium provided that they are stated locally at each instant of time. Then, thermodynamic potentials and, the equations of state of the system keep their usual equilibrium form but at a local level, namely, for sufficiently small volume elements. The size d of these volumes should be bigger than the average distance traveled by the (charges, heat) carriers between two successive collisions, defined mean free path l ($d > l$). Furthermore, from the point of view of the temporal behavior, because small volume elements have a characteristic time, (called relaxation time τ) to reach internal equilibrium, it is supposed that the temporal rate of a physical perturbation is characterised by a time that is not smaller than τ in such a way that such elements will be able to reach internal equilibrium during the interval of the perturbation. The objective of this didactic paper is to formulate a linear theory for magnetizable media. Using the general methods of CIT we choose in the thermodynamic state vector as independent variables the specific internal energy u, the small strain tensor $\varepsilon_{\alpha\beta}$ and the specific magnetization **m** in order to investigate the possibility of cross effects among magnetic phenomena, heat conduction, electric conduction and mechanical phenomena. In Galilean approximation, taking into account Maxwell equations and the balance equations, the entropy production is analyzed and the phenomenological equations are formulated in the linear anisotropic case. It is seen that if the magnetizable medium is isotropic there are no contributions in the expression for the entropy production due to the cross effects among magnetic phenomena and the other irreversible phenomena. Also the constitutive equations are obtained in the anisotropic and isotropic case. The constitutive equations together with the Maxwell equations and the balance equations constitute a balanced system, which can be solved, when the initial and boundary conditions are known. As a didactic objective we give full details of the derivation of these equations with the aid of the methods of non-equilibrium thermodynamics. The obtained results have applications in several fields of applied sciences, like physics, chemestry, medicine, biology and others.

The paper is organized in the following way. In Section **2** we illustrate the didactic approaches used in the paper to formulate the model describing the behavior of magnetizable media in CIT. In Section **3** we present in Galilean approximation the Maxwell equations and the balance equations governing all the physical processes occurring inside the considered medium. In Sections **4** and **5** the thermodynamic state space is choosen and the intrinsic entropy production is analyzed describing all the irreversible processes inside the medium.

The phenomenological equations are derived for anisotropic magnetizable media following the classical procedures of CIT. The Onsager-Casimir relations are obtained, introducing the concepts of odd and even quantity, flux and affinity. The entropy production is worked out having the form of a bilinear form in the components of the physical quantities present in its physical contributions. In Section **6** the phenomenological equations and the entropy production are carried out in the isotropic case, where the magnetizable body has properties invariant respect to all rotations and inversion of the frame axes (see Jeffreys 1957; Famà and Restuccia 2020, where isotropic tensors are studied). In the isotropic case some detailed calculations are given regarding the expression for the intrinsic entropy production (see Appendix). In Sections **7** and **8** the linear equations of state for the magnetizable material systems are obtained in the linear anisotropic and isotropic case, considering as thermodynamic reference state a thermodynamic equilibrium state.

2. Didactic approaches

In this didactic paper we describe the behaviour of magnetizable media using some methodologies of teaching. More precisely, by a *classic method* we have started from the foundations of the study of the topic under consideration to introduce in a rigorous, sequential and detailed way the knowledge of the theories to apply to develop the model to be formulated. By a "*natural method*", that takes this name because remembers the child that when borns discovers a world having a language and an external reality to decipher, we have given some results correlated to the subject to study without giving demonstrations, referring to other works or monographs, where basic results were deduced. Also, we have applied a *functionality method*, in which the motivations and objectives of the studied topic are shown, emphasizing the important applications in the different fields of science, and therefore stimulating the learning.

Furthermore, basic ideas and concepts have been exposed.

3. The model

In this Section, in the framework of the classical thermodynamics of irreversible processes we formulate a model for anisotropic magnetizable media. The standard Cartesian tensor notation in a rectangular coordinate system is used and the equations governing the behavior of such media are considered in a current configuration $\mathscr{K}_t$. We assume that in Galilean approximation and for deformations and rotations of the medium which are small from a kinematical point of view, all the processes occurring in the considered bodies are governed by two groups of laws: Maxwell equations and balance equations.

Maxwell's equations for magnetizable media, in the rationalized Gauss system, keep the form (see De Groot and Mazur 1984; Maugin 1988)

$$rot\mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c}\mathbf{J}^{(el)}, \quad rot\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \tag{1}$$

$$div\mathbf{E} = \rho^{(el)}, \quad div\mathbf{B} = 0, \tag{2}$$

where $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{H}$, denote the electric field, the magnetic induction and the magnetic displacement, respectively, and $\mathbf{J}^{(el)}$ and $\rho^{(el)}$ are the density of the electric current per unit volume and the electric charge density per unit volume, which satisfy the following charge conservation law

$$\frac{\partial \rho^{(el)}}{\partial t} = -div\mathbf{J}^{(el)}. \tag{3}$$

Furthermore, we define the magnetization vector $\mathbf{M}$ and the magnetization per unit volume $\mathbf{m}$, called specific magnetization, respectively, by

$$\mathbf{M} = \mathbf{B} - \mathbf{H}, \quad \mathbf{m} = v\mathbf{M}, \tag{4}$$

where v is the specific volume, defined by $v = \rho^{-1}$, with ρ the density mass of the medium under consideration.

The *mass conservation law* has the form

$$\frac{\partial \rho}{\partial t} = -div\rho\mathbf{v}, \quad \text{or} \quad \rho\frac{\partial v}{\partial t} = div\mathbf{v}. \tag{5}$$

The *equation of motion* of a magnetizable medium in an electromagnetic field is given by (see De Groot and Mazur 1984)

$$\rho\frac{d\mathbf{v}}{dt} = div\boldsymbol{\tau} + \rho^{(el)}\mathbf{E} + \frac{1}{c}\mathbf{J}^{(el)} \times \mathbf{B} + (grad\mathbf{B})\cdot\mathbf{M} - \frac{1}{c}\frac{d}{dt}(\mathbf{M}\times\mathbf{E}) + \rho\mathbf{F}, \tag{6}$$

where d/dt is the material derivative, defined by $d/dt = \frac{\partial}{\partial t} + x_\gamma\frac{\partial}{\partial x_\gamma}$, where Einstein convention for repeated indices is used, $\mathbf{v}$ is the velocity field defined by $\mathbf{v} = \frac{d\mathbf{u}}{dt}$, with $\mathbf{u}$ the displacement field, $\boldsymbol{\tau}$ is the mechanical stress tensor and $\mathbf{F}$ is the volume force per unit of mass.

The *first law of thermodynamics* may be written in the form (see De Groot and Mazur 1984)

$$\rho\frac{du}{dt} = -div\mathbf{J}^{(q)} + \tau_{\alpha\beta}\frac{d\varepsilon_{\alpha\beta}}{dt} + \mathbf{J}^{(el)}\cdot\mathbf{E} + \rho\mathbf{B}\cdot\frac{d\mathbf{m}}{dt}, \tag{7}$$

where u is the specific internal energy (the internal energy per unit of mass) and $\mathbf{J}^{(q)}$ is the heat flux density, $\varepsilon_{\alpha\beta}$ is the small strain tensor defined by $\varepsilon_{\alpha\beta} = \frac{1}{2}(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha})$ $(\alpha, \beta = 1,2,3)$, and $\frac{d\varepsilon_{\alpha\beta}}{dt} = \frac{1}{2}(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha})$ $(\alpha, \beta = 1,2,3)$, being the deformations and the rotations of the medium supposed small from a kinematical of view. On the right hand side of Eq. (7) the first term is the heat supply, the second term is the work per unit time done by the mechanical stress, the third term is the Joule heat, $\rho\mathbf{B}\cdot\frac{d\mathbf{m}}{dt}$ is the work per unit time done by the magnetic induction to change the magnetization.

4. Entropy balance equation

Let us assume that the specific entropy s (i.e the entropy per unit of mass) depends on the specific internal energy u, the small strain tensor $\varepsilon_{\alpha\beta}$ and the specific magnetization $\mathbf{m}$. Thus, s is a constitutive function of the state space C

$$C = C\left(u, \varepsilon_{\alpha\beta}, \mathbf{m}\right),$$

$$s = s\left(u, \varepsilon_{\alpha\beta}, \mathbf{m}\right). \tag{8}$$

Following the general philosophy of classical irreversible thermodynamics, dissipative fluxes, gradients and time derivatives of the physical fields are not included in the state space and, and the local equilibrium hypotesis is assumed, i. e. out equilibrium each point of the considered continuum medium is considered as an elementary thermodynamic cell, where the reversible thermodynamics is applicable. According to the reversible thermodynamics, we define

$$T^{-1} = \frac{\partial}{\partial u} s\left(u, \varepsilon_{\alpha\beta}, \mathbf{m}\right), \tag{9}$$

$$\tau_{\alpha\beta}^{(eq)} = -\rho T \frac{\partial}{\partial \varepsilon_{\alpha\beta}} s\left(u, \varepsilon_{\alpha\beta}, \mathbf{m}\right), \tag{10}$$

$$\mathbf{B}^{(eq)} = -T \frac{\partial}{\partial \mathbf{m}} s\left(u, \varepsilon_{\alpha\beta}, \mathbf{m}\right), \tag{11}$$

where T is the equilibrium temperature (the absolute temperature), that is the inverse of the partial derivative of the entropy with respect to the specific internal energy , at strain tensor and specific magnetization constant. Analogous definitions are valid for the equilibrium stress tensor $\tau_{\alpha\beta}^{(eq)}$ and the equilibrium magnetic field $\mathbf{B}^{(eq)}$.

Considering very small deviations with respect to a local equilibrium state, we expand the entropy into Taylor's series with respect to this state and confining our considerations to the linear terms we obtain the differential of the expression of the entropy in a point of the thermodynamical phase space, corresponding to a local position $\mathbf{x}$ in a current configuration $\mathscr{K}_t$

$$s = s^{(eq)} + \frac{\partial s}{\partial u}\left(u - u^{(eq)}\right) + \frac{\partial s}{\partial \varepsilon_{\alpha\beta}}\left(\varepsilon_{\alpha\beta} - \varepsilon_{\alpha\beta}^{(eq)}\right) + \frac{\partial s}{\partial \mathbf{m}}\left(\mathbf{m} - \mathbf{m}^{(eq)}\right). \tag{12}$$

By virtue of the definitions (9)-(11), multiplying the obtained expression by ρT and using $\rho = \frac{1}{v}$, we obtain the following *Gibbs relation*

$$T ds = du - v\tau_{\alpha\beta}^{(eq)} d\varepsilon_{\alpha\beta} - \mathbf{B}^{(eq)} \cdot d\mathbf{m}, \tag{13}$$

giving the differential of the expression of the entropy. It follows that the time derivative of the entropy (8) in the point $\mathbf{x}$ in $\mathscr{K}_t$ has the form

$$T\frac{ds}{dt} = \frac{du}{dt} - v\tau_{\alpha\beta}^{(eq)} \frac{d\varepsilon_{\alpha\beta}}{dt} - \mathbf{B}^{(eq)} \cdot \frac{d\mathbf{m}}{dt}. \tag{14}$$

Thus, using equation (7) and the expression

$$-\frac{1}{T} div\mathbf{J}^{(q)} = -div\left(\frac{\mathbf{J}^{(q)}}{T}\right) - \frac{\mathbf{J}^{(q)}}{T^2} \cdot gradT, \tag{15}$$

from (14) we obtain

$$\rho\frac{ds}{dt} = -divJ^{(s)} + \sigma^{(s)}, \tag{16}$$

where $J^{(s)} = T^{-1}\mathbf{J}^{(q)}$ is the entropy flux and $\sigma^{(s)}$ is the intrinsic entropy production per unit volume and per unit time given by

$$\sigma^{(s)} = T^{-1}\left(\tau_{\alpha\beta}^{(vi)} \frac{d\varepsilon_{\alpha\beta}}{dt} + \mathbf{J}^{(q)} \cdot \mathbf{X}^{(q)} + \mathbf{J}^{(el)} \cdot \mathbf{E} + \rho \mathbf{B}^{(ir)} \cdot \frac{d\mathbf{m}}{dt}\right). \tag{17}$$

In the above equations the viscous stress tensor $\tau^{(vi)}_{\alpha\beta}$, the irreversible magnetic field $\mathbf{B}^{(ir)}$ and the field $\mathbf{X}^{(q)}$ are defined, respectively, by

$$\tau^{(vi)}_{\alpha\beta} = \tau_{\alpha\beta} - \tau^{(eq)}_{\alpha\beta}, \quad \mathbf{B}^{(ir)} = \mathbf{B} - \mathbf{B}^{(eq)}, \quad \mathbf{X}^{(q)} = -T^{-1} gradT. \tag{18}$$

In the expression (18) the intrinsic entropy production is defined non-negative quantity by the second law of thermodynamics: $\sigma^{(s)} \geq 0$. It is zero ($\sigma^{(s)} = 0$) when the material system is in thermodynamic equilibrium, *i.e.*, there is not irreversible processes inside the medium giving rise to $\sigma^{(s)}$. At thermodynamic equilibrium the sources (the body force, the energy source, the external entropy production) vanish, the velocity of the medium is constant or null, the gradients and the time derivatives of the fields are null. From (17), if we introduce the quantity

$$\sigma^{(i)} = \sigma^{(s)} T, \tag{19}$$

we have

$$\sigma^{(i)} = \tau^{(vi)}_{\alpha\beta} \frac{d\varepsilon_{\alpha\beta}}{dt} + \mathbf{J}^{(q)} \cdot \mathbf{X}^{(q)} + \mathbf{J}^{(el)} \cdot \mathbf{E} + \rho \mathbf{B}^{(ir)} \cdot \frac{d\mathbf{m}}{dt}, \tag{20}$$

From (20) it is seen that the quantity in the right hand side of (20) is additively composed of four contributions: the first is connected with the symmetric part of the velocity gradient field giving rise to viscous effects, the second term arises from heat conduction, the third term is due to the electric conduction, the term containing $\frac{d\mathbf{m}}{dt}$ is connected with entropy production due to an istantaneous change in the magnetization.

5. Phenomenological equations

From (20) it is seen that the intrinsic entropy production $\sigma^{(i)}$ is a bilinear form, having the form of a sum of terms, where each term is the product of the components of vectors, tensors, that represent fluxes, $\mathbf{J}_i$, and the components of vectors, tensors which represent thermodynamic forces or "affinities" $\mathbf{X}_i$, conjugate to the fluxes (see De Groot and Mazur 1984 and Kluitenberg 1984), i. e.

$$\sigma^{(i)} = \sum_{i}^{r} J_i \cdot X_i. \tag{21}$$

In our case $\sigma^{(i)}$ is sum of four terms. Following the linear theory by De Groot and Mazur (1984) (see Kluitenberg 1984) we can write the phenomenological equations in the form

$$J_i = \sum_{k=1}^{r} L_{ik} X_k \quad (i = 1, 2, ..., r). \tag{22}$$

The equations (22) are called phenomelogical equations and the quantities L_{ik} take the name of phenomenological coefficients and are constant. From expressions (22) we deduce that a thermodynamic flux does not depend only on the corresponding force, but also on other forces. All the macroscopic quantities, which occur in a phenomenological theory, come from statistical averages of functions of the coordinates and momenta of the microscopic particles constituting the system at mesoscopic level (see Kluitenberg 1984). The macroscopic quantities can depend on the speed of the microscopic particles, distinguishing them in even and odd functions. From the microscopic point of view a macroscopic quantity is called even (or odd), if it is a or even (odd) function of the speed of the microscopic particles. It is evident that the macroscopic quantities which depend on the velocity of the microscopic particles are odd functions of this velocity. The heat flux, the time derivative of the small strain tensor are examples of odd functions. While the temperature, the mass density, the kinetic energy are examples of even functions. Furthermore, the affinity conjugated to an even flux is an odd quantity, while the conjugate to an odd flux is an even quantity. Macroscopic quantities which do not depend on the velocity of these microscopic particles are considered even. From the macroscopic point of view we distinguish the macroscopic quantities in even and odd functions when they are even or odd under time reversal. Introducing the symbols "$^{(o)}$" and "$^{(e)}$" to indicate odd and even macroscopic quantities, respectively, the intrinsic entropy production (21) can be written in form

$$\sigma^{(i)} = \sum_{i=1}^{n} J_i^{(o)} X_i^{(e)} + \sum_{k=1}^{m} J_k^{(e)} X_k^{(o)}, \tag{23}$$

where the odd quantities $J_i^{(o)}$ and the even quantities $J_k^{(e)}$ are components of vectors, tensors, which represent the flows, while the even quantities $X_i^{(e)}$ and the odd quantities $X_k^{(o)}$ are components of vectors, tensors, which represent the affinities conjugated to flows. From (17) and (20), since T is an even function, $\sigma^{(i)}$ is an odd quantity. By virtue of (23) the phenomenological equations (22) take the following expression

$$J_i^{(o)} = \sum_{j=1}^{n} L_{ij}^{(o,e)} X_j^{(e)} + \sum_{k=1}^{m} L_{ik}^{(o,o)} X_k^{(o)} \qquad (i = 1,2,...,n) \tag{24}$$

$$J_l^{(e)} = \sum_{j=1}^{n} L_{lj}^{(e,e)} X_j^{(e)} + \sum_{k=1}^{m} L_{lk}^{(e,o)} X_k^{(o)} \qquad (l = 1,2,...,m) \tag{25}$$

Let us introduce the Onsager symmetry relations for the phenomenolgical coefficients (the minus sign which occurs in (28) is called the Casimir minus sign), coming from microscopic considerations (see De Groot and Mazur 1984, Kluitenberg 1984, Onsager 1931a and Onsager 1931b):

$$L_{ij}^{(o,e)} = L_{ji}^{(o,e)} \qquad (i, j = 1,2,...,n) \tag{26}$$

$$L_{kl}^{(e,o)} = L_{lk}^{(e,o)} \qquad (k,l = 1,2,...,m) \tag{27}$$

$$L_{ik}^{(o,o)} = -L_{ki}^{(e,e)} \qquad (i = 1,2,...,n; k = 1,2,...,m) \tag{28}$$

In a macroscopic theory they are postulated. They can be verified with the help of suitable experiments. On the other hand the concepts of odd and even variables are referred to microscopic notions.

Then, we choose the fluxes and thermodynamic forces (see Table 1).

TABLE 1. Fluxes and thermodynamic forces

Fluxes	$B_\alpha^{(ir)}$	$J_\alpha^{(el)}$	$J_\alpha^{(q)}$	$\tau_{\alpha\beta}^{(vi)}$
Affinities	$\rho \frac{dm_\beta}{dt}$	E_β	$-T^{-1} gradT$	$\frac{d\varepsilon_{\alpha\beta}}{dt}$

According the usual procedure of non equilibrium thermodynamics, by virtue of expression for the entropy production (20), we obtain the following phenomenological equations, in which the irreversible fluxes are linear functions of the thermodynamic forces

$$B_\alpha^{(ir)} = \rho L_{(M)\alpha\beta}^{(0,0)} \frac{dm_\beta}{dt} + L_{(M)\alpha\beta}^{(0,el)} E_\beta + L_{(M)\alpha\beta}^{(0,q)} X_\beta^{(q)} + L_{(M)\alpha\beta\gamma}^{(0,vi)} \frac{d\varepsilon_{\beta\gamma}}{dt}, \tag{29}$$

$$J_\alpha^{(el)} = \rho L_{(M)\alpha\beta}^{(el,0)} \frac{dm_\beta}{dt} + L_{\alpha\beta}^{(el,el)} E_\beta + L_{\alpha\beta}^{(el,q)} X_\beta^{(q)} + L_{\alpha\beta\gamma}^{(el,vi)} \frac{d\varepsilon_{\beta\gamma}}{dt}, \tag{30}$$

$$J_\alpha^{(q)} = \rho L_{(M)\alpha\beta}^{(q,0)} \frac{dm_\beta}{dt} + L_{\alpha\beta}^{(q,el)} E_\beta + L_{\alpha\beta}^{(q,q)} X_\beta^{(q)} + L_{\alpha\beta\gamma}^{(q,vi)} \frac{d\varepsilon_{\beta\gamma}}{dt}, \tag{31}$$

$$\tau_{\alpha\beta}^{(vi)} = \rho L_{(M)\alpha\beta\gamma}^{(vi,0)} \frac{dm_\gamma}{dt} + L_{\alpha\beta\gamma}^{(vi,el)} E_\gamma + L_{\alpha\beta\gamma}^{(vi,q)} X_\gamma^{(q)} + L_{\alpha\beta\gamma\zeta}^{(vi,vi)} \frac{d\varepsilon_{\gamma\zeta}}{dt}. \tag{32}$$

Phenomenological equation (29) is connected with irreversible changes in the magnetization. Eqs. (30), (31) and (32) represent the generalizations of Ohm's law, Fourier's law and Newton's law, respectively. $L_{(M)\alpha\beta}^{(0,0)}$ is a polar tensor of order two connected with magnetic phenomena, $L_{\alpha\beta}^{(q,q)}$ is the polar tensor of heat conductivity of order two, $L_{\alpha\beta}^{(el,el)}$ is the polar tensor of electric conductivity of order two, $L_{\alpha\beta\gamma\zeta}^{(vi,vi)}$ is the viscosity tensor of order four, $L_{(M)\alpha\beta}^{(0,el)}$, $L_{(M)\alpha\beta}^{(el,0)}$, $L_{(M)\alpha\beta}^{(0,q)}$, $L_{(M)\alpha\beta}^{(q,0)}$, are pseudotensors of order two connected with the interaction of the electric conduction and the heat flux with magnetic phenomena, respectively, $L_{\alpha\beta\gamma}^{(0,vi)}$, $L_{\alpha\beta\gamma}^{(vi,0)}$ are pseudotensors of order three connected with the interaction of viscous phenomena with magnetic phenomena, $L_{(M)\alpha\beta}^{(el,q)}$, $L_{(M)\alpha\beta}^{(q,el)}$, are polar tensors of order two and $L_{\alpha\beta\gamma}^{(el,vi)}$, $L_{\alpha\beta\gamma}^{(q,vi)}$, $L_{\alpha\beta\gamma}^{(vi,el)}$, $L_{\alpha\beta\gamma}^{(vi,q)}$ are polar tensors of order three connected with cross effects phenomena, having taken into account that $B_\alpha^{(ir)}$, $\frac{dm_\alpha}{dt}$ are psuedovectors (or axial vectors), $J_\alpha^{(el)}$, $J_\alpha^{(q)}$, $\tau_{\alpha\beta}^{(vi)}$, E_α, $X_\alpha^{(q)}$, $\frac{d\varepsilon_{\alpha\beta}}{dt}$ are polar tensors.

By virtue of the symmetry of $\varepsilon_{\alpha\beta}$ from (10) it follows that also $\tau_{\alpha\beta}^{(eq)}$ is a symmetric tensor. Moreover, if we suppose that the mechanical stress tensor $\tau_{\alpha\beta}$ which occurs in the first law of thermodynamics and in the equation of motion is symmetric, it is seen from $(18)_1$ that also the viscous stress tensor $\tau_{\alpha\beta}^{(vi)}$ is a symmetric tensor.

Therefore, we obtain the following symmetry relations

$$L_{(M)\alpha\beta\gamma}^{(0,vi)} = L_{(M)\alpha\gamma\beta}^{(0,vi)}, \quad L_{(M)\alpha\beta\gamma}^{(vi,0)} = L_{(M)\beta\alpha\gamma}^{(vi,0)}, \tag{33}$$

$$L_{\alpha\beta\gamma}^{(vi,el)} = L_{\beta\alpha\gamma}^{(vi,el)}, \quad L_{\alpha\beta\gamma}^{(el,vi)} = L_{\alpha\gamma\beta}^{(el,vi)}, \quad L_{\alpha\beta\gamma}^{(q,vi)} = L_{\alpha\gamma\beta}^{(q,vi)}, \quad L_{\alpha\beta\gamma}^{(vi,q)} = L_{\beta\alpha\gamma}^{(vi,q)}, \tag{34}$$

$$L_{\alpha\beta\gamma\zeta}^{(vi,vi)} = L_{\beta\alpha\gamma\zeta}^{(vi,vi)} = L_{\alpha\beta\zeta\gamma}^{(vi,vi)} = L_{\beta\alpha\zeta\gamma}^{(vi,vi)}. \tag{35}$$

Since $\mathbf{B}^{(ir)}$, $\mathbf{j}^{(el)}$, $\mathbf{J}^{(q)}$ and $\frac{d\varepsilon_{\alpha\beta}}{dt}$ are odd functions under time reversal and $\rho\frac{d\mathbf{m}}{dt}$, $\mathbf{E}$, $\mathbf{X}^{(q)}$ and $\tau^{(vi)}_{\alpha\beta}$ are even functions under time reversal, we have

$$\left(B^{(ir)}_{\alpha}\right)^{(o)} = \left(L^{(0,0)}_{(M)\alpha\beta}\right)^{(o,e)}\left(\rho\frac{dm_{\beta}}{dt}\right)^{(e)} + \left(L^{(0,el)}_{(M)\alpha\beta}\right)^{(o,e)}(E_{\beta})^{(e)} + \left(L^{(0,q)}_{(M)\alpha\beta}\right)^{(o,e)}(X^{(q)}_{\beta})^{(e)} + \left(L^{(0,vi)}_{(M)\alpha\beta\gamma}\right)^{(o,o)}\left(\frac{d\varepsilon_{\beta\gamma}}{dt}\right)^{(o)}, \tag{36}$$

$$\left(J^{(el)}_{\alpha}\right)^{(o)} = \left(L^{(el,0)}_{(M)\alpha\beta}\right)^{(o,e)}\left(\rho\frac{dm_{\beta}}{dt}\right)^{(e)} + \left(L^{(el,el)}_{\alpha\beta}\right)^{(o,e)}(E_{\beta})^{(e)} + \left(L^{(el,q)}_{\alpha\beta}\right)^{(o,e)}(X^{(q)}_{\beta})^{(e)} + \left(L^{(el,vi)}_{\alpha\beta\gamma}\right)^{(o,o)}\left(\frac{d\varepsilon_{\beta\gamma}}{dt}\right)^{(o)}, \tag{37}$$

$$(J^{(q)}_{\alpha})^{(o)} = \left(L^{(q,0)}_{(M)\alpha\beta}\right)^{(o,e)}\left(\rho\frac{dm_{\beta}}{dt}\right)^{(e)} + \left(L^{(q,el)}_{\alpha\beta}\right)^{(o,e)}(E_{\beta})^{(e)} + \left(L^{(q,q)}_{\alpha\beta}\right)^{(o,e)}(X^{(q)}_{\beta})^{(e)} + \left(L^{(q,vi)}_{\alpha\beta\gamma}\right)^{(o,o)}\left(\frac{d\varepsilon_{\beta\gamma}}{dt}\right)^{(o)}, \tag{38}$$

$$\left(\tau^{(vi)}_{\alpha\beta}\right)^{(e)} = \left(L^{(vi,0)}_{\alpha\beta\gamma}\right)^{(e,e)}\left(\rho\frac{dm_{\gamma}}{dt}\right)^{(o)} + \left(L^{(vi,el)}_{\alpha\beta\gamma}\right)^{(e,e)}(E_{\gamma})^{(e)} + \left(L^{(vi,q)}_{\alpha\beta\gamma}\right)^{(e,e)}(X^{(q)}_{\gamma})^{(e)} + \left(L^{(vi,vi)}_{\alpha\beta\gamma\zeta}\right)^{(e,o)}\left(\frac{d\varepsilon_{\gamma\zeta}}{dt}\right)^{(o)}, \tag{39}$$

and the Onsager-Casimir reciprocity relations read

$$L^{(0,0)}_{(M)\alpha\beta} = L^{(0,0)}_{(M)\beta\alpha}, \tag{40}$$

$$L^{(el,el)}_{\alpha\beta} = L^{(el,el)}_{\beta\alpha}, \quad L^{(q,q)}_{\alpha\beta} = L^{(q,q)}_{\beta\alpha}, \tag{41}$$

$$L^{(vi,vi)}_{\alpha\beta\gamma\zeta} = L^{(vi,vi)}_{\gamma\zeta\alpha\beta}, \tag{42}$$

$$L^{(0,el)}_{(M)\alpha\beta} = L^{(el,0)}_{(M)\beta\alpha}, \quad L^{(0,q)}_{(M)\alpha\beta} = L^{(q,0)}_{(M)\beta\alpha}, \tag{43}$$

$$L^{(0,vi)}_{(M)\alpha\beta\gamma} = -L^{(vi,0)}_{(M)\beta\gamma\alpha}, \quad L^{(el,q)}_{\alpha\beta} = L^{(q,el)}_{\beta\alpha}, \quad L^{(el,vi)}_{\alpha\beta\gamma} = -L^{(vi,el)}_{\beta\gamma\alpha}, \tag{44}$$

$$L^{(q,vi)}_{\alpha\beta\gamma} = -L^{(vi,q)}_{\beta\gamma\alpha}. \tag{45}$$

Equations (33)-(44) reduce the number of independent components of the phenomenological tensors. Introducing the expressions for $\frac{d\mathbf{m}}{dt}$, $\mathbf{J}^{(el)}$, $\mathbf{J}^{(q)}$, and $\tau^{(vi)}_{\alpha\beta}$ in the entropy production (20), by virtue of Onsager-Casimir reciprocity relations, we obtain

$$\begin{aligned}\sigma^{(s)} = T^{-1}\Bigg(&\rho^2 L^{(0,0)}_{(M)\alpha\beta}\frac{dm_\alpha}{dt}\frac{dm_\beta}{dt} + L^{(el,el)}_{\alpha\beta}E_\alpha E_\beta + \\ &+ L^{(q,q)}_{\alpha\beta}X^{(q)}_\alpha X^{(q)}_\beta + L^{(vi,vi)}_{\alpha\beta\gamma\zeta}\frac{d\varepsilon_{\alpha\beta}}{dt}\frac{d\varepsilon_{\gamma\zeta}}{dt} + \\ &+ 2\rho L^{(0,el)}_{(M)\alpha\beta}\frac{dm_\alpha}{dt}E_\beta + 2L^{(el,q)}_{\alpha\beta}E_\alpha X^{(q)}_\beta + 2\rho L^{(0,q)}_{(M)\alpha\beta}\frac{dm_\alpha}{dt}X^{(q)}_\beta\Bigg) \geq 0.\end{aligned} \tag{46}$$

Relation (46) shows that the entropy production is a quadratic form in the components of the time derivatives of the specific magnetization vector, of the electric field, of the gradient of the temperature and in the components of the time derivative of the strain tensor. Furthermore, the entropy production is a non-negative definite quadratic form, *i.e.*,

$$\sigma^{(s)} \geq 0,$$

where the equality sign holds only if the above mentioned components vanish and the material system is in thermodynamic equilibrium. Several inequalities may be obtained for the components of the phenomenological tensors, resulting from the fact that all the elements of the main diagonal of the matrix associated to the quadratic form (46) must be non-negative, and all the principal minors of this matrix must also be non-negative (see Appendix 1 for more details, derived in the case when the magnetizable medium under consideration is isotropic), for instance

$$L^{(0,0)}_{(M)\alpha\alpha} \geq 0, \quad L^{(el,el)}_{\alpha\alpha} \geq 0, \quad L^{(q,q)}_{\alpha\alpha} \geq 0, \quad L^{(vi,vi)}_{\alpha\beta\alpha\beta} \geq 0.$$

6. Phenomenological equations and entropy production for magnetizable media isotropic with respect to all the rotations and inversions of the frame of axes

The existence of spatial symmetry properties in a material system simplifies the form of the phenomenological equations (29)-(32) so that the Cartesian components of the fluxes do not depend on all the Cartesian components of the thermodynamic forces (*Principle of Curie symmetry*). For example, in isotropic magnetizable media (whose properties are invariant with respect to all orthogonal rotations and inversions of the frame axes), it can be shown that fluxes and thermodynamic forces of different tensorial character (polar or pseudo) do not couple (see De Groot and Mazur 1984, Jeffreys 1957 and Famà and Restuccia 2020). We introduce the deviator $\tilde{A}_{\alpha\beta}$ and the scalar part A of an arbitrary tensor field of order two $A_{\alpha\beta}$ defined by

$$\tilde{A}_{\alpha\beta} = A_{\alpha\beta} - \frac{1}{3}A_{\gamma\gamma}\delta_{\alpha\beta} \qquad \text{and} \qquad A = \frac{1}{3}A_{\gamma\gamma}. \tag{47}$$

Hence,

$$A_{\alpha\beta} = \tilde{A}_{\alpha\beta} + A\delta_{\alpha\beta} \qquad \text{and} \qquad \tilde{A}_{\gamma\gamma} = 0. \tag{48}$$

Moreover, if $A_{\alpha\beta}$ is symmetric also $\tilde{A}_{\alpha\beta}$ is symmetric.

In the isotropic case, taking into account that the phenomenological tensor $L^{(vi,vi)}_{\alpha\beta\gamma\zeta}$, obeys the symmetry relations (35) and the Onsager relations (42), it satisfies the following relation

$$L^{(vi,vi)}_{\alpha\beta\gamma\zeta} = L^{(vi,vi)}_{\beta\alpha\gamma\zeta} = L^{(vi,vi)}_{\alpha\beta\zeta\gamma} = L^{(vi,vi)}_{\beta\alpha\zeta\gamma} = L^{(vi,vi)}_{\gamma\zeta\beta\alpha} = L^{(vi,vi)}_{\gamma\zeta\alpha\beta} = L^{(vi,vi)}_{\zeta\gamma\alpha\beta} = L^{(vi,vi)}_{\zeta\gamma\beta\alpha}, \tag{49}$$

and has the form (see Famà and Restuccia 2020, where particular studies on isotropic tensors have been done, in the case of porous media filled by fluids)

$$L_{\alpha\beta\gamma\delta} = L_1 \left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\beta\gamma}\delta_{\alpha\delta}\right) + L_2\, \delta_{\alpha\beta}\,\delta_{\gamma\delta}. \tag{50}$$

Assuming $L_1 = \frac{1}{2}\eta_s$ and $L_2 = \frac{1}{3}(\eta_v - \eta_s)$, one gets

$$L^{(vi,vi)}_{\alpha\beta\gamma\zeta} = \frac{1}{2}\eta_s(\delta_{\alpha\gamma}\delta_{\beta\zeta} + \delta_{\beta\gamma}\delta_{\alpha\zeta}) + \frac{1}{3}(\eta_v - \eta_s)\delta_{\alpha\beta}\delta_{\gamma\zeta}, \tag{51}$$

where η_s and η_v are the shear and bulk viscosity, respectively.
The polar tensors of order three vanish, *i.e.*,

$$L_{\alpha\beta\gamma} = 0; \tag{52}$$

the pseudo tensors of order three keep the form

$$L_{\alpha\beta\gamma} = L\varepsilon_{\alpha\beta\gamma}, \tag{53}$$

but, because of the symmetry properties of $L_{\alpha\beta\gamma}$, they are null

$$L_{\alpha\beta\gamma} = 0; \tag{54}$$

the polar tensors of order two $L_{\alpha\beta}$ take the form

$$L_{\alpha\beta} = L\delta_{\alpha\beta}, \tag{55}$$

which reduces to

$$L_{\alpha\beta} = 0, \tag{56}$$

if $L_{\alpha\beta}$ has zero trace;
the pseudo tensors of order two vanish, *i.e.*,

$$L_{\alpha\beta} = 0; \tag{57}$$

the polar and pseudo vectors L_α vanish, *i.e.*,

$$L_\alpha = 0. \tag{58}$$

Taking into consideration that $\mathbf{B}^{(ir)}$ and $\frac{d\mathbf{m}}{dt}$ are pseudovectors and $\mathbf{J}^{(el)}$, $\mathbf{J}^{(q)}$ and $\tau^{(vi)}_{\alpha\beta}$ polar tensors, the phenomenological equations (29)-(32) keep the form

$$\mathbf{B}^{(ir)} = \rho L^{(0,0)}_{(M)} \frac{d\mathbf{m}}{dt}, \tag{59}$$

$$\mathbf{J}^{(el)} = L^{(el,el)}\mathbf{E} + L^{(el,q)}\mathbf{X}^{(q)}, \tag{60}$$

$$\mathbf{J}^{(q)} = L^{(q,el)}\mathbf{E} - k\, gradT, \tag{61}$$

where

$$L^{(q,q)}_{\alpha\beta} = L^{(q,q)}\delta_{\alpha\beta} = -k\,T\,\delta_{\alpha\beta}$$

with k heat conductivity coefficient,

$$\tau^{(vi)}_{\alpha\beta} = \frac{1}{2}\eta_s(\delta_{\alpha\gamma}\delta_{\beta\zeta}+\delta_{\beta\gamma}\delta_{\alpha\zeta})\frac{d\varepsilon_{\gamma\zeta}}{dt}+\frac{1}{3}(\eta_v-\eta_s)\delta_{\alpha\beta}\delta_{\gamma\zeta}\frac{d\varepsilon_{\gamma\zeta}}{dt} =$$

$$\eta_s(\frac{d\varepsilon_{\alpha\beta}}{dt}-\frac{1}{3}\frac{d\varepsilon_{\gamma\gamma}}{dt}\delta_{\alpha\beta})+\eta_v\frac{1}{3}\frac{d\varepsilon_{\gamma\gamma}}{dt}\delta_{\alpha\beta} =$$

$$\eta_s\frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt}+\eta_v\frac{1}{3}\frac{d\varepsilon_{\gamma\gamma}}{dt}\delta_{\alpha\beta} = \eta_s\frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt}+\eta_v\frac{d\varepsilon}{dt}\delta_{\alpha\beta}, \tag{62}$$

being

$$\frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt} = \frac{d\varepsilon_{\alpha\beta}}{dt}-\frac{1}{3}\frac{d\varepsilon_{\gamma\gamma}}{dt}\delta_{\alpha\beta}, \quad \text{and} \quad \frac{d\varepsilon}{dt} = \frac{1}{3}\frac{d\varepsilon_{\gamma\gamma}}{dt}, \tag{63}$$

the deviator and the scalar part of the tensor field of order two $\frac{d\varepsilon_{\alpha\beta}}{dt}$, respectively, and $\frac{d\varepsilon_{\alpha\beta}}{dt}$ symmetric. Furthermore, because $\frac{d\varepsilon_{\alpha\beta}}{dt}$ is symmetric also its deviatoric part is symmetric. From the definitions (63) we have

$$\frac{d\varepsilon}{dt} = \frac{1}{3}div\mathbf{v} \quad \text{and} \quad \frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt} = \frac{1}{2}\left(\frac{\partial v_\alpha}{\partial x_\beta}+\frac{\partial v_\beta}{\partial x_\alpha}-\frac{2}{3}\delta_{\alpha\beta}div\mathbf{v}\right). \tag{64}$$

Then, the entropy production (46) in the case in which we assume that the isotropic media have the mass density ρ constant takes the form

$$\sigma^{(i)} = L^{(0,0)}_{(M)}\left(\frac{d\mathbf{M}}{dt}\right)^2+L^{(el,el)}(\mathbf{E})^2+2L^{(el,q)}\mathbf{E}\cdot\mathbf{X}^{(q)}+L^{(q,q)}\left(\mathbf{X}^{(q)}\right)^2+$$

$$\eta_s\left(\frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt}\right)^2+3\eta_v\left(\frac{d\varepsilon}{dt}\right)^2 \geq 0, \tag{65}$$

where we have defined $\mathbf{M} = \rho\mathbf{m}$.

From (65) it is seen that the entropy production is a non-negative bilinear form in the components of time derivative of the magnetization, in the electric field, in the temperature gradient, and in the deviator and the scalar part of the small strain tensor (see in Appendix its matrix representation $\sigma^{(i)} = X_i\mathscr{L}_{ij}X_j$, with X_i, X_j and $\mathscr{L}_{ij}$ suitable matrices).

The following inequalities result from the fact that all the elements of the main diagonal of the symbolic matrix $\{\mathscr{L}_{ij}\}$ associated to the bilinear form (65) must be non-negative (see Appendix)

$$L^{(0,0)}_{(M)} \geq 0, \quad L^{(el,el)} \geq 0, \quad L^{(q,q)} \geq 0, \quad \eta_s \geq 0, \quad \eta_v \geq 0.$$

Other relations can be obtained from the non-negativity of the major minors.

7. Reference state and thermodynamic equilibrium

Let us consider a reference state of the medium (Kluitenberg 1984) , with an arbitrary (but fixed) uniform temperature $T_{(0)}$ in which the mechanical stress tensor $\tau_{\alpha\beta}$ and the magnetic field $\mathbf{B}$ vanish in the material system. Such a state with fixed temperature, mechanical stress and magnetic field may be obtained by suitable experimental arrangements. Also, we require that the reference state (indicated by the symbol "$_{(0)}$") is a state of thermodynamic equilibrium. Furthermore, being $\tau^{(eq)}_{\alpha\beta}$ and $\mathbf{B}^{(eq)}$ functions of the temperature $T_{(0)}$, the strain tensor $\varepsilon_{\alpha\beta}$ and the magnetization m_α, we require that in this state the value $\varepsilon_{(0)\alpha\beta}$ for the strain tensor and the value $m_{(0)\alpha}$ for the magnetization vector are such that

$$\tau^{(eq)}_{\alpha\beta}\left(T_{(0)},\varepsilon_{(0)\alpha\beta},m_{(0)\alpha}\right)=0, \tag{66}$$

$$\mathbf{B}^{(eq)}\left(T_{(0)},\varepsilon_{(0)\alpha\beta},m_{(0)\alpha}\right)=0. \tag{67}$$

Since the tensor $\tau^{(eq)}_{\alpha\beta}$ is symmetric, Eqs. (66)-(67) form a set of 9 equations for the values of the 6 independent components of the symmetric strain tensor $\varepsilon_{(0)\alpha\beta}$ and the values of the 3 components of the vector $m_{(0)\alpha}$. All strains will be measured with respect to this state and we choose the tensor $\varepsilon_{\alpha\beta}$ and axial vector $m_{(0)\alpha}$ so that they vanish in the reference state. Thus, $\varepsilon_{(0)\alpha\beta}=m_{(0)\alpha}=0$. Hence,

$$\tau^{(eq)}_{\alpha\beta}=0,\quad \mathbf{B}^{(eq)}=0,$$

$$\text{for}\quad T=T_{(0)}\quad\text{and}\quad \varepsilon_{(0)\alpha\beta}=m_{(0)\alpha}=0. \tag{68}$$

From (66) the considered magnetizable medium is in a state of thermodynamic equilibrium if the entropy production vanishes. It follows from the positive definite character of this quantity that it vanishes if

$$\frac{d\mathbf{m}}{dt}=0,\quad \mathbf{E}=0,\quad gradT=0,\quad \frac{d\varepsilon}{dt}=0,\quad \frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt}=0, \tag{69}$$

It follows from (69) that the reference state is a state of thermodynamic equilibrium, provided that ε , $\tilde{\varepsilon}_{\alpha\beta}$ and the vector $\mathbf{m}$ (determined by (66)-(67)) are kept constant. Moreover, from (69) the electric field must be kept vanishing in this state of thermodynamic equilibrium. We note that in the reference state the medium has the uniform temperature $T_{(0)}$ and hence grad T vanishes in this state. Moreover, by virtue of conditions (69) and the phenomenological equation (32) the viscous stress tensor $\tau^{(vi)}_{\alpha\beta}$ vanishes in the thermodynamic equilibrium and from $(18)_1$ it follows that

$$\tau_{\alpha\beta}=\tau^{(eq)}_{\alpha\beta}. \tag{70}$$

8. Linear equations of state and constitutive relations for anisotropic and isotropic media

In this Section we derive in the anisotropic case and in linear approximation the constitutive theory for the system under consideration. We recall that the total mass density ρ has been assumed to be constant. Thus, we define the specific free energy f by $f=u-Ts$ and with the aid of Gibbs relation (13) we obtain for the differential of f

$$df=-sdT+v\tau^{(eq)}_{\alpha\beta}d\varepsilon_{\alpha\beta}+\mathbf{B}^{(eq)}\cdot d\mathbf{m}. \tag{71}$$

Because of (71) one gets

$$s=-\frac{\partial}{\partial T}f\left(T,\varepsilon_{\alpha\beta},\mathbf{m}\right), \tag{72}$$

$$\mathbf{B}^{(eq)}=\frac{\partial}{\partial\mathbf{m}}f\left(T,\varepsilon_{\alpha\beta},\mathbf{m}\right). \tag{73}$$

$$\tau^{(eq)}_{\alpha\beta} = \rho \frac{\partial}{\partial \varepsilon_{\alpha\beta}} f\left(T, \varepsilon_{\alpha\beta}, \mathbf{m}\right). \tag{74}$$

Applying the potential method (De Groot and Mazur 1984 and Kluitenberg 1984), we expand the free energy up the second-order approximation around the thermodynamic equilibrium state defined in Section 7, and indicated by "$/_0$. Introducing the deviations of the indipendent variables from this reference state in the following way

$$\hat{T} = T - T_0, \ \mid \frac{\hat{T}}{T_0} \mid \ll 1, \ \hat{\varepsilon}_{\alpha\beta} = \varepsilon_{\alpha\beta} - \left(\varepsilon_{\alpha\beta}\right)_0, \ \mid \frac{\varepsilon_{\alpha\beta}}{\left(\varepsilon_{\alpha\beta}\right)_0} \mid \ll 1,$$

$$\hat{m}_\alpha = m_\alpha - (m_\alpha)_0, \ \mid \frac{m_\alpha}{(m_\alpha)_0} \mid \ll 1, \tag{75}$$

we have also

$$(s)_0 = s_0, \quad (u)_0 = u_0, \quad \left(\tau_{\alpha\beta}\right)_0 = 0, \quad (\varepsilon_{\alpha\beta})_0 = 0, \quad (B^{(eq)}_\alpha)_0 = 0, \quad (m_\alpha)_0 = 0. \tag{76}$$

Thus, we have the following expansion for the free energy

$$f = f_0 + \left(\frac{\partial f}{\partial T}\right)_0 (T - T_0) + \left(\frac{\partial f}{\partial m_\alpha}\right)_0 (m_\alpha - (m_\alpha)_0) + \left(\frac{\partial f}{\partial \varepsilon_{\alpha\beta}}\right)_0 (\varepsilon_{\alpha\beta} - (\varepsilon_{\alpha\beta})_0) +$$

$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial T^2}\right)_0 (T - T_0)^2 + \left(\frac{\partial^2 f}{\partial T \partial m_\alpha}\right)_0 (T - T_0)\,(m_\alpha - (m_\alpha)_0 +$$

$$\left(\frac{\partial^2 f}{\partial T \partial \varepsilon_{\alpha\beta}}\right)_0 (T - T_0)\,(\varepsilon_{\alpha\beta} - (\varepsilon_{\alpha\beta})_0) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial m_\alpha \partial m_\beta}\right)_0 (m_\alpha - (m_\alpha)_0)\,(m_\beta - (m_\beta)_0) +$$

$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial \varepsilon_{\alpha\beta} \partial \varepsilon_{\gamma\nu}}\right)_0 (\varepsilon_{\alpha\beta} - (\varepsilon_{\alpha\beta})_0)(\varepsilon_{\gamma\nu} - (\varepsilon_{\gamma\nu})_0) + O(h^3), \tag{77}$$

being

$$\left(\frac{\partial f}{\partial T}\right)_0 = -s_0, \quad \left(\frac{\partial f}{\partial \varepsilon_{\alpha\beta}}\right)_0 = \left(\tau_{\alpha\beta}\right)_0 = 0, \quad \left(\frac{\partial f}{\partial m_\alpha}\right)_0 = \left(B^{eq}_\alpha\right)_0 = 0.$$

As usually in thermodynamics, in (77) we have called the second partial derivatives of free energy with respect to the considered independent variables using the name of the phenomenological coefficients, measurable by experiments and supposed constant, coming from their physical interpretation. Some of these coefficients have well-known physical meanings. Thus, one has

$$\frac{\partial^2 f}{\partial T^2} = -\frac{c_v}{T_0}, \quad \frac{\partial^2 f}{\partial T \partial \varepsilon_{\alpha\beta}} = \rho^{-1} K_{\alpha\beta}, \quad \frac{\partial^2 f}{\partial T \partial m_\alpha} = a^{(0)}_{(M)\alpha},$$

$$\frac{\partial^2 f}{\partial \varepsilon_{\alpha\beta} \partial m_\gamma} = a^{(vi,0)}_{(M)\alpha\beta\gamma}, \quad \frac{\partial^2 f}{\partial \varepsilon_{\alpha\beta} \partial \varepsilon_{\gamma\delta}} = \rho^{-1} c_{\alpha\beta\gamma\delta}, \quad \frac{\partial^2 f}{\partial m_\alpha \partial m_\beta} = \rho \, a^{(0,0)}_{(M)\alpha\beta}. \tag{77}$$

where c is the specific heat, $c_{\alpha\beta\gamma\zeta}$ are the elastic coefficients, $K_{\alpha\beta}$ are the thermoelastic constants, $a^{(0)}_{(M)\alpha}$, $a^{(0,0)}_{(M)\alpha\beta}$, $a^{(vi,0)}_{(M)\alpha\beta\gamma}$ are the other phenomenological coefficients expressing simple and coupled interactions effects among the acting fields. Furthermore, we have taken into account the physical dimensions of the physical quantities present (the introduction of the minus sign comes from physical reasons) and because some phenomenological coefficients satisfy some symmetry relations, coming from the invariance properties with respect to the priority of derivation of free energy with respect to the independent variables, this symmetry reduces the number of independent components of these phenomenological tensors. In particular, we have

$$c_{\alpha\beta\gamma\zeta}=c_{\beta\alpha\gamma\zeta}=c_{\alpha\beta\zeta\gamma}=c_{\beta\alpha\zeta\gamma}=c_{\gamma\zeta\alpha\beta}=c_{\gamma\zeta\beta\alpha}=c_{\zeta\gamma\alpha\beta}=c_{\zeta\gamma\beta\alpha},$$

$$a^{(0,0)}_{(M)\alpha\beta}=a^{(0,0)}_{(M)\beta\alpha}\quad a^{(vi,0)}_{(M)\alpha\beta\gamma}=a^{(vi,0)}_{(M)\beta\alpha\gamma}=a^{(0,vi)}_{(M)\gamma\alpha\beta}=a^{(0,vi,)}_{(M)\gamma\beta\alpha}. \tag{78}$$

Thus, one gets

$$f=f_0-s_0T+\frac{1}{2}\left(-\frac{c_v}{T_0}(T-T_0)^2-\frac{2}{\rho}K_{\alpha\beta}(T-T_0)\varepsilon_{\alpha\beta}+2a^{(0)}_{(M)\alpha}(T-T_0)m_\alpha\right)+$$
$$\frac{1}{2}\rho\, a^{(0,0)}_{(M)\alpha\beta}m_\alpha m_\beta+\frac{1}{2\rho}c_{\alpha\beta\gamma\delta}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}+a^{(vi,0)}_{(M)\alpha\beta\gamma}\varepsilon_{\alpha\beta}\,m_\gamma, \tag{79}$$

where ρ is supposed constant. Rewriting the expression (79) for the free energy we have

$$f=f_0-s_0(T-T_0)+\frac{1}{2}\left(-\frac{c_v}{T_0}(T-T_0)^2-\frac{2}{\rho}K_{\eta\xi}(T-T_0)\varepsilon_{\eta\xi}+2a^{(0)}_{(M)\eta}(T-T_0)m_\eta\right)+$$
$$\frac{1}{2}\rho a^{(0,0)}_{(M)\eta\xi}m_\eta m_\xi+\frac{1}{2\rho}c_{\eta\xi\gamma\delta}\,\varepsilon_{\eta\xi}\varepsilon_{\gamma\delta}+a^{(vi,0)}_{(M)\eta\xi\gamma}\varepsilon_{\eta\xi}\,m_\gamma. \tag{80}$$

By virtue of the state laws (72)-(73) one gets

$$s=s_0+\frac{c_v}{T_0}(T-T_0)+\frac{1}{\rho}K_{\alpha\beta}\,\varepsilon_{\alpha\beta}-a^{(0)}_{(M)\alpha}m_\alpha, \tag{81}$$

$$B^{(eq)}_\alpha=\frac{\partial f}{\partial m_\alpha}=a^{(0)}_{(M)\alpha}(T-T_0)+a^{(vi,0)}_{(M)\eta\xi\gamma}\varepsilon_{\eta\xi}\,\delta_{\alpha\gamma}+\rho a^{(0,0)}_{(M)\alpha\beta}m_\beta=$$
$$a^{(0)}_{(M)\alpha}(T-T_0)+a^{(0,vi)}_{(M)\alpha\beta\gamma}\varepsilon_{\beta\gamma}+a^{(0,0)}_{(M)\alpha\beta}M_\beta \tag{82}$$

$$\tau^{(eq)}_{\alpha\beta}=\rho\frac{\partial f}{\partial\varepsilon_{\alpha\beta}}=\frac{1}{2}\left(c_{\eta\xi\gamma\delta}\varepsilon_{\gamma\delta}\delta_{\alpha\eta}\,\delta_{\beta\xi}+c_{\eta\xi\gamma\delta}\varepsilon_{\eta\xi}\,\delta_{\alpha\gamma}\delta_{\beta\delta}\right)-K_{\eta\xi}(T-T_0)\delta_{\alpha\eta}\,\delta_{\beta\xi}+$$
$$a^{(vi,0)}_{(M)\eta\xi\gamma}\delta_{\alpha\eta}\,\delta_{\beta\xi}m_\gamma.$$

$$\tau^{(eq)}_{\alpha\beta}=\frac{1}{2}\left(c_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}+c_{\eta\xi\alpha\beta}\varepsilon_{\eta\xi}\right)-K_{\alpha\beta}(T-T_0)+a^{(vi,0)}_{(M)\alpha\beta\gamma}m_\gamma=$$
$$c_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}-K_{\alpha\beta}(T-T_0)+a^{(vi,0)}_{(M)\alpha\beta\gamma}m_\gamma. \tag{83}$$

Then, from the phenomenolgical equations (29) and (32) the constitutive laws for the anisotropic case read

$$\tau_{\alpha\beta}=\tau^{(eq)}_{\alpha\beta}+L^{(vi,0)}_{(M)\alpha\beta\gamma}\frac{dM_\gamma}{dt}+L^{(vi,el)}_{\alpha\beta\gamma}E_\gamma+L^{(vi,q)}_{\alpha\beta\gamma}X^{(q)}_\gamma+$$
$$+L^{(vi,vi)}_{\alpha\beta\gamma\delta}\frac{d\varepsilon_{\gamma\delta}}{dt}, \tag{84}$$

$$B_\alpha = B_\alpha^{(eq)} + L^{(0,0)}_{(M)\alpha\beta}\frac{dM_\beta}{dt} + L^{(0,el)}_{(M)\alpha\beta}E_\beta + L^{(0,q)}_{(M)\alpha\beta}X_\beta^{(q)} + \\ + L^{(0,vi)}_{(M)\alpha\beta\gamma}\frac{d\varepsilon_{\beta\gamma}}{dt}, \tag{85}$$

where $\tau_{\alpha\beta}^{(eq)}$ and $B_\alpha^{(eq)}$ are given by (82) and (83). The constitutive equations for $J_\alpha^{(el)}$ and q_α are the same as the phenomenological equations (30) and (31), respectively.

In the isotropic case, taking into account the relations (52) and (58) and that the fourth order tensor $c_{\alpha\beta\gamma\zeta}$ obeys the symmetry relations (78) and has the form (50) with $L_1 = \mu$ and $L_2 = \lambda$, *i.e.*,

$$c_{\alpha\beta\gamma\delta} = \mu\,(\delta_{\alpha\gamma}\,\delta_{\beta\delta} + \delta_{\beta\gamma}\,\delta_{\alpha\delta}) + \lambda\,\delta_{\alpha\beta}\,\delta_{\gamma\delta},$$

where λ and μ are the constant phenomenological coefficients called the Lamé constants, the state laws (81)-(83) keep the form

$$s = s_0 + \frac{cv}{T_0}(T - T_0) + \frac{1}{\rho}K\,\varepsilon_{\gamma\gamma},$$

$$\tau_{\alpha\beta}^{(eq)} = -K\,\delta_{\alpha\beta}\,(T - T_0) + \mu(\delta_{\alpha\gamma}\delta_{\beta\zeta} + \delta_{\beta\gamma}\delta_{\alpha\zeta})\varepsilon_{\gamma\zeta} + \lambda\,\delta_{\alpha\beta}\,\delta_{\gamma\zeta}\,\varepsilon_{\gamma\zeta} = \\ -K\,\delta_{\alpha\beta}\,(T - T_0) + \lambda\,\varepsilon_{\gamma\gamma}\,\delta_{\alpha\beta} + 2\,\mu\,\varepsilon_{\alpha\beta}\,, \tag{86}$$

$$B_\alpha^{(eq)} = \rho a^{(0,0)}_{(M)}\,\delta_{\alpha\beta}\,m_\beta = a^{(0,0)}_{(M)}\,M_\alpha\,. \tag{87}$$

Finally, from (86), (87), (59) and (62) we obtain the constitutive laws for $\tau_{\alpha\beta}$ and B_α in the isotropic case

$$\mathbf{B} = a^{(0,0)}_{(M)}\mathbf{M} + L^{(0,0)}_{(M)}\frac{d\mathbf{M}}{dt}, \tag{88}$$

$$\tau_{\alpha\beta}^{(vi)} = -K\delta_{\alpha\beta}(T - T_0) + \lambda\,\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu\varepsilon_{\alpha\beta} + \eta_s\frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt} + \eta_v\frac{d\varepsilon}{dt}\delta_{\alpha\beta}, \tag{89}$$

The contitutive equations for for $J_\alpha^{(el)}$ and q_α are the same as the phenomenological equations (60) and (61), respectively.

9. Conclusions

In this paper a model for magnetizable medium was formulated in CIT, in the local equilibrium hypothesis, where it has been assumed that at any point of the considered continuous medium the reversible thermodynamics is valid and thus all variables defined and the known thermodynamic relations among these variables in this equilibrium thermodynamics remain significant, like temperature, entropy, internal energy, thermodynamic potentials, equations of state. For simplicity we have treated the magnetizable medium as a De Groot-Mazur medium, where only the macroscopic variable magnetization describes the megnetizable medium taken into account. Also, in Restuccia and Kluitenberg 1987, Restuccia and Kluitenberg 1989, Restuccia 2010, Restuccia 2014 and Restuccia *et al.* 2016 particular studies were done by the author on magnetizable media, using n arbitrary internal variables describing the magnetic relaxation of these media, arising from n arbitrary microscopic phenomena, following the classical procedures of CIT. In this paper in the anisotropic and isotropic case the phenomenological equations for magnetizable media were derived, when the independent variables are the specific internal energy u, the small strain tensor $\varepsilon_{\alpha\beta}$ and the specific magnetization $\mathbf{m}$. The set of Maxwell equations, balance equations and constitutive relations, illustrating the magnetic, electric, thermal and mechanical behaviour of the considered medium, permits to close the system of equations describing the medium under consideration. The procedures of CIT have been explained and discussed and also the entropy production has been commented as a bilinear form giving informations on the signum of the phenomenological coefficients present in the derived phenomenological equations. Also, starting from a particular reference state, that coincides with an equilibrium thermodynamic state, the expressions at equilibrium for the magnetic induction and for the stress tensor have been calculated, in order to obtain the constitutive equations, that represent laws connected with irreversible changes in the magnetization and generalizations of Ohm's law, Fourie's law and Newton's law.

Appendix

In this Appendix we represent the matrix $\{\mathscr{L}_{ij}\}$. Entropy production in the isotropic case (65) can be also rewritten in the form (see also Van *et al.* 2021 where anaogous studies were done in the case of a rigid body with an internal variable)

$$\sigma^{(i)} = L^{(0,0)}_{(M)}\delta_{\alpha\beta}\frac{dM_\alpha}{dt}\frac{dM_\beta}{dt} + L^{(el,el)}\delta_{\alpha\beta}E_\alpha E_\beta + \quad + L^{(q,q)}\delta_{\alpha\beta}X^{(q)}_\alpha X^{(q)}_\beta + \eta_s\frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt}\frac{d\tilde{\varepsilon}_{\gamma\delta}}{dt}\delta_{\alpha\gamma}\delta_{\beta\delta} + 3\eta_v\left(\frac{d\varepsilon}{dt}\right)^2, \tag{90}$$

where we have disregarded the contribution $L^{(el,q)}\delta_{\alpha\beta}E_\alpha X^{(q)}_\beta$. In symbolic matrix notation the entropy production (90) keeps the form

$$X_i\mathscr{L}_{ij}X_j \geq 0, \tag{91}$$

where

$$\begin{aligned}\{X_i\} &= \left\{\frac{dM_\alpha}{dt}\ ;\ E_\alpha\ ;\ X^{(q)}_\alpha\ ;\ \frac{d\tilde{\varepsilon}_{\alpha\beta}}{dt}\ ;\ \frac{d\varepsilon}{dt}\right\} = \\ &= \{\frac{dM_1}{dt}\ ;\ \frac{dM_2}{dt}\ ;\ \frac{dM_3}{dt}\ ;\ E_1\ ;\ E_2\ ;\ E_3\ ;\ X^{(q)}_1\ ;\ X^{(q)}_2\ ;\ X^{(q)}_3\ ; \\ &\quad \frac{d\tilde{\varepsilon}_{11}}{dt}\ ;\ \frac{d\tilde{\varepsilon}_{12}}{dt}\ ;\ \frac{d\tilde{\varepsilon}_{13}}{dt}\ ;\ \frac{d\tilde{\varepsilon}_{22}}{dt}\ ;\ \frac{d\tilde{\varepsilon}_{23}}{dt}\ ;\ \frac{d\varepsilon}{dt}\}, \\ &(i = 1,\ldots,15),\end{aligned} \tag{92}$$

$$\{X_j\} = \begin{Bmatrix} \frac{dM_\beta}{dt} \\ E_\beta \\ X^{(q)}_\beta \\ \frac{d\tilde{\varepsilon}_{\gamma\varsigma}}{dt} \\ \frac{d\varepsilon}{dt} \end{Bmatrix}, \quad (j = 1,\ldots,15). \tag{93}$$

Then, for $\mathscr{L}_{ij}$ we introduce the following notation

$$\{\mathscr{L}_{ij}\} = \left(\begin{array}{c|c|c|c|c} \overset{3\times3}{\mathscr{L}^{(0,0)}_{(M)\alpha\beta}} & \overset{3\times3}{0} & \overset{3\times3}{0} & \overset{3\times5}{0} & \overset{3\times1}{0} \\ \hline \overset{3\times3}{0} & \overset{3\times3}{\mathscr{L}^{(el,el)}_{\alpha\beta}} & \overset{3\times3}{0} & \overset{3\times5}{0} & \overset{3\times1}{0} \\ \hline \overset{3\times3}{0} & \overset{3\times3}{0} & \overset{3\times3}{\mathscr{L}^{(q,q)}_{\alpha\beta}} & \overset{3\times5}{0} & \overset{3\times1}{0} \\ \hline \overset{5\times3}{0} & \overset{5\times3}{0} & \overset{5\times3}{0} & \overset{5\times5}{\mathscr{L}^{(vi,vi)}_{\alpha\beta\gamma\varsigma}} & \overset{5\times1}{0} \\ \hline \overset{1\times3}{0} & \overset{1\times3}{0} & \overset{1\times3}{0} & \overset{1\times5}{0} & \overset{1\times1}{\eta_v} \end{array}\right) \quad (i,j=1,\dots,15), \tag{94}$$

in which $\overset{n\times m}{0}$ is the symbolic null matrix of dimension $n\times m$ and the sub-matrices, appearing inside, have the following form

$$\mathscr{L}^{(0,0)}_{(M)\alpha\beta} = \mathscr{L}^{(0,0)}_{(M)}\delta_{\alpha\beta} = \begin{pmatrix} \mathscr{L}^{(0,0)}_{(M)} & 0 & 0 \\ 0 & \mathscr{L}^{(0,0)}_{(M)} & 0 \\ 0 & 0 & \mathscr{L}^{(0,0)}_{(M)} \end{pmatrix}, \tag{95}$$

$$\mathscr{L}^{(el,el)}_{\alpha\beta} = \mathscr{L}^{(el,el)}\delta_{\alpha\beta} = \begin{pmatrix} \mathscr{L}^{(el,el)} & 0 & 0 \\ 0 & \mathscr{L}^{(el,el)} & 0 \\ 0 & 0 & \mathscr{L}^{(el,el)} \end{pmatrix}, \tag{96}$$

$$\mathscr{L}^{(q,q)}_{\alpha\beta} = \mathscr{L}^{(q,q)}\delta_{\alpha\beta} = \begin{pmatrix} \mathscr{L}^{(q,q)} & 0 & 0 \\ 0 & \mathscr{L}^{(q,q)} & 0 \\ 0 & 0 & \mathscr{L}^{(q,q)} \end{pmatrix}, \tag{97}$$

$$\mathscr{L}^{(vi,vi)}_{\alpha\beta\gamma\delta} = \begin{pmatrix} \eta_s & 0 & 0 & 0 & 0 \\ 0 & \eta_s & 0 & 0 & 0 \\ 0 & 0 & \eta_s & 0 & 0 \\ 0 & 0 & 0 & \eta_s & 0 \\ 0 & 0 & 0 & 0 & \eta_s \end{pmatrix}. \tag{98}$$

In (98) it has been taken into consideration that only 5 components of the deviator are independent.

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