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THE FABULOUS DESTINY OF RICHARD DEDEKIND

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Abstract. By using the preliminary results given in a previous divulgative note, we present here a concise and self–contained introduction to the construction of the real field as the unique, up to increasing isomorphism, Dedekind complete totally ordered field. Moreover, we also show the equivalence between the Dedekind completeness property on totally ordered fields and some meaningful well–known notions present in the literature, such as the Cauchy completeness on totally ordered Archimedean fields. This characterization result allows us to correctly encode the Dedekind completeness for totally ordered fields in the general abstract setting of metric spaces. We believe that the essential parts of the paper can be easily accessed by anyone with some experience in abstract mathematical thinking. The paper completes the lecture given by the second author during the International Workshop on *New Horizons in Teaching Science* in Messina on June 2018.

"We are justified in calling numbers
a free creation of the human mind"

Richard Dedekind

1. Introduction

Julius Wilhelm Richard Dedekind[1] studied in Göttingen, where he was a pupil of Johann Friedrich Carl Gauss. Later he taught in Göttingen, then in Zurich and finally at the Technische Hochschule of his hometown. He was among the first to exhibit the theories of Galois at university. Dedekind made fundamental contributions to Number Theory, in particular by rigorously specifying the concept of real numbers. Based on Cantor's Set Theory, he provided a non–intuitive clarification of the notion of natural number in purely set terms. Dedekind's contributions to Mathematics go far beyond that.

His other original research was directed towards the construction of the theory of the field of algebraic numbers; moreover, he was responsible for the introduction of the ideals, which opened further developments to the study of Modern Algebra. As

[1] Braunschweig, duchy of Braunschweig (now Germany) 1831–1916.

claimed by James S. Milne in 2017, the contribution of Dedekind laid the modern foundations of Algebraic Number Theory by finding the correct definition of the ring of integers in a number field, by proving that ideals factor uniquely into products of prime ideals in such ring, and by showing that, modulo principal ideals, they fall into finitely many classes.

In this didactic note we are mainly interested in the *Dedekind completeness condition* and in some of its consequences in Real Analysis. The Dedekind completeness condition, as well as the notion of continuity of the real line, the definition of the real field by means of cuts, the formulation of the Dedekind–Peano Axioms for natural numbers, are certainly topics which are familiar to any student who attends the first year of Mathematics and, in some cases, even the last years of high school. The same cannot be said, probably, for the important relationships that exist between these concepts and for the profound implications that these arguments have with the most important questions of Mathematics. The main purpose here is to discuss these links, which will ultimately lead us to discover, paraphrasing the title of the French movie *Le Fabuleux Destin d'Amélie Poulain* (2001), the fabulous destiny of R. Dedekind, who always hovers when it comes to Mathematics.

Section **3** is devoted to the fundamental notion of $\mathscr{D}$–completeness on totally ordered fields. The main result of this section characterizes the real field $\mathbb{R}$ as the unique $\mathscr{D}$–complete totally ordered field up to increasing isomorphism; see Theorem **3.3**. In Proposition **3.4** we also prove that the rational Archimedean field $\mathbb{Q}$ is not $\mathscr{D}$–complete.

As proved in Section **4** the $\mathscr{D}$–completeness is equivalent to the *least upper bound property*[2], as well as to the *greatest lower bound property.* We emphasize that the least upper bound property implies several classical and fundamental results of Mathematical Analysis valid for every continuous function on a closed and bounded interval. Among others, as in the interesting paper Corgnier and Valabrega 2013, we mention here: the intermediate value theorem; Weierstrass's theorem; the mean value theorem; Rolle's theorem; see, for instance, the classical textbooks of the Italian school of Mathematical Analysis: Cecconi and Stampacchia 1983; De Marco 1986; Giusti 1988; De Marco 1999; Lanconelli 2000 and Prodi 1970; Lanconelli 1998; Marcellini and Sbordone 1998; Fusco and Sbordone 2001; Pagani and Salsa 2015, 2016.

Then, following Cohen and Ehrlich 1963, Theorem 5.1 we provide, in Theorem **4.1**, a list of statements equivalent to the $\mathscr{D}$–completeness on totally ordered fields. The topics and facts adduced are largely standard, though our choice of examples, problems, and manner of presentation may make some modest claim to freshness if not to novelty, but many of these lines of inquiry are pursued here in a greater detail if compared with other recent texts. For instance, an encyclopedic account of equivalent notions of $\mathscr{D}$–completeness was studied in Deveau and Teismann 2014. In the aforementioned paper a massive list of mostly familiar statements of Real Analysis has been considered, each of which turns out to be equivalent to $\mathscr{D}$–completeness in totally ordered fields; see also Hall 2010. To complete the picture

[2] In 1817 the importance of the least upper bound property was firstly recognized by B. Bolzano.

given in Section **4**, a concrete example of Cauchy complete non–Archimedean totally ordered field has been presented; see Corgnier and Valabrega 2017a, 2017b, 2015.

Finally, in Section **5** we exhibit some classical models for the real field $\mathbb{R}$. More precisely, we treat the following constructions of the reals:

- by Cauchy sequences (G. Cantor (1873));
- by Dedekind cuts (R. Dedekind (1872); E.A. Maier and D. Maier's (1973));
- by Decimals (S. Stevin (1585)).

2. The didactic approach

To provide an explicit, precise and systematic definition of the real numbers is a major step towards completing the arithmetization of Analysis. Further reflections in von Neumann, Gödel and Bernays Set Theory are based on a universal ordered field, namely the *surreal number field* which was defined in Knuth 1974 by following the *go endgame* by J.H. Conway. In the monumental Conway construction every other ordered field can be identified, by isomorphism, as a subfield of the surreals; see Knuth 2016 for a funny introduction on the subject.

No conscious attempt was made to grade the arguments according to their difficulty: they are arranged, in each section, in chronological order. Moreover, we also emphasize that for the sake of completeness and future didactic scopes, the arguments of the manuscript are summarize in a concise way.

The present paper goes well beyond a standard course on real numbers, and there is enough material for supplementary reading. As general references on the topics of this note we refer to the monographs Cohen and Ehrlich 1963; Burrill 1967 and the references therein.

Note: From now on we use notations and definitions given in Devillanova and Molica Bisci 2021.

3. $\mathscr{D}$–completeness: The real field $\mathbb{R}$

In this section $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ denotes a totally ordered field. However, we emphasize that the notion of $\mathscr{D}$–completeness given in this section **remains valid for a totally ordered set** $X = (X, \leq)$ as well as the validity of Theorem **3.1** given below; see De Marco 1986, Section B.1.2.

3.1. Dedekind completeness. Given X and Y two nonempty subsets of $\mathbb{K}$, we shall use the convention:

$$X \leq Y \Leftrightarrow (\forall x \in X, \forall y \in Y, x \leq y).$$

The field $\mathbb{K}$ is said to be **Dedekind complete**[3], briefly $\mathscr{D}$–complete, iff: for every X and Y nonempty subsets of $\mathbb{K}$ such that:

$$X \leq Y \Rightarrow \exists z \in \mathbb{K} : \forall x \in X, \forall y \in Y, x \leq z \leq y.^{4} \tag{1}$$

[3]We emphasize that a totally ordered field $\mathbb{K}$ is not necessarily Dedekind complete.

[4]We also write $X \leq z \leq Y$. With abuse of notation the simbol $X \leq z$ (resp. $z \leq Y$) means that $x \leq z$ (resp. $z \leq y$) for every $x \in X$ (resp. $y \in Y$). Similar notations are tacitely assumed for strict inequalities.

Let X be a nonempty subset of $\mathbb{K}$ and denote by:

$$\mathscr{U}_X \text{ the set of upper bounds for } X \text{ in } \mathbb{K},$$

as well as

$$\mathscr{L}_X \text{ the set of lower bounds for } X \text{ in } \mathbb{K}.$$

Moreover, when $\mathscr{U}_X \neq \emptyset$ and, respectively, $\mathscr{L}_X \neq \emptyset$ we set

$$\sup_{\mathbb{K}} X := \min_{\mathbb{K}} \mathscr{U}_X \qquad \text{and} \qquad \inf_{\mathbb{K}} X := \max_{\mathbb{K}} \mathscr{L}_X.$$

A nonempty set $X \subset \mathbb{K}$ such that $\mathscr{L}_X \neq \emptyset$ and $\mathscr{U}_X \neq \emptyset$ is bounded in $\mathbb{K}$. The class of bounded subsets of $\mathbb{K}$ is denoted by $\mathscr{B}_{\mathbb{K}}$.

Theorem 3.1. *Let $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ be a totally ordered field. The following assertions are equivalent:*

(a) $\mathbb{K}$ *is $\mathscr{D}$–complete*;

(b) *for every nonempty $X \subseteq \mathbb{K}$, if $\mathscr{U}_X \neq \emptyset$ then* $\sup_{\mathbb{K}} X$ *exists*;

(c) *for every nonempty $X \subseteq \mathbb{K}$, if $\mathscr{L}_X \neq \emptyset$ then* $\inf_{\mathbb{K}} X$ *exists.*

Proof. (a) $\Rightarrow$ (b) Let $X \subset \mathbb{K}$, $X \neq \emptyset$ and $\mathscr{U}_X \neq \emptyset$. Clearly $X \leq \mathscr{U}_X$ and, by (a), there exists $z \in \mathbb{K}$ such that $X \leq z \leq \mathscr{U}_X$. Note that $z = \sup_{\mathbb{K}} X$, *i.e.*, $z = \min_{\mathbb{K}} \mathscr{U}_X$, indeed, $z \leq \mathscr{U}_X$ and $z \in \mathscr{U}_X$ since $z \geq X$.

(b) $\Rightarrow$ (a) Let $X, Y \subseteq \mathbb{K}$, $X, Y \neq \emptyset$, and $X \leq Y$. Let us prove that there exists $z \in \mathbb{K}$ such that $X \leq z \leq Y$. Now $X \neq \emptyset$ and $\mathscr{U}_X \neq \emptyset$ since $Y \subset \mathscr{U}_X$ and $Y \neq \emptyset$. By (b) there exists $z = \sup_{\mathbb{K}} X$. Again, since $Y \subset \mathscr{U}_X$, it follows that $z = \min_{\mathbb{K}} Y \leq Y$. On the other hand, owing to $z \in \mathscr{U}_X$, one has $X \leq z$.

(a) $\Rightarrow$ (c) Let $X \subset \mathbb{R}$, $X \neq \emptyset$ and $\mathscr{L}_X \neq \emptyset$. Clearly $\mathscr{L}_X \leq X$ and, by (a), there exists $z \in \mathbb{K}$ such that $\mathscr{L}_X \leq z \leq X$. Note that $z = \inf_{\mathbb{K}} X$, *i.e.*, $z = \max_{\mathbb{K}} \mathscr{L}_X$, indeed $\mathscr{L}_X \leq z$ with $z \in \mathscr{L}_X$ since $z \leq X$.

(c) $\Rightarrow$ (a) Let $X, Y \subseteq \mathbb{K}$, $X, Y \neq \emptyset$, and $X \leq Y$. Let us prove that there exists $z \in \mathbb{K}$ such that $X \leq z \leq Y$. Now $Y \neq \emptyset$ and $\mathscr{L}_Y \neq \emptyset$ since $X \subset \mathscr{L}_Y$ and $X \neq \emptyset$. By (c) there exists $z = \inf_{\mathbb{K}} Y$. Again, since $X \subset \mathscr{L}_Y$, it follows that $X \leq \inf_{\mathbb{K}} Y = z$. On the other hand, owing to $z \in \mathscr{L}_Y$, one has $z \leq Y$. □

The next property will be used in the sequel.

Proposition 3.2. *Let $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ be a totally ordered Archimedean field and let $\varphi : \mathbb{K} \to \mathbb{K}$ be an increasing function, i.e., such that*

$$\forall x, y \in \mathbb{K}, \ \ x < y \ \textit{in} \ \mathbb{K} \Rightarrow \varphi(x) < \varphi(y) \ \textit{in} \ \mathbb{K}.$$

Assume that $\varphi|_{\mathbb{Q}} = id_{\mathbb{Q}}$.[5] *Then $\varphi = id_{\mathbb{K}}$.*

Proof. Let $x \in \mathbb{K}$ and $\varepsilon > 0$ in $\mathbb{K}$. By Devillanova and Molica Bisci 2021, Proposition 5.5, the field $\mathbb{Q}$ is dense in $\mathbb{K}$. Hence, there are $p, q \in \mathbb{Q}$ such that $x - \varepsilon < p < x < q < x + \varepsilon$. Since φ is increasing and $\varphi|_{\mathbb{Q}} = id_{\mathbb{Q}}$, one has $p = \varphi(p) < \varphi(x) < \varphi(q) = q$ so that $x - \varepsilon < \varphi(x) < x + \varepsilon$. The last inequality and the density of $\mathbb{Q}$ in $\mathbb{K}$ ensure that $\varphi(x) = x$. The proof is complete. □

[5]Note that every totally ordered field $\mathbb{K}$ contains a copy of the rational field $\mathbb{Q}$.

3.2. Isomorphism of Dedekind complete totally ordered fields. Let $\mathbb{K}_1 = (\mathbb{K}_1, +_1, \cdot_1, \leq_1)$ and $\mathbb{K}_2 = (\mathbb{K}_2, +_2, \cdot_2, \leq_2)$ be two complete ordered fields. A bijective function $\varphi : \mathbb{K}_1 \to \mathbb{K}_2$ is said to be an **increasing isomorphism**[6] (of fields) if φ is an isomorphism of fields and

$$\forall x, y \in \mathbb{K}_1,\ \ x <_1 y \text{ in } \mathbb{K} \Rightarrow \varphi(x) <_2 \varphi(y) \text{ in } \mathbb{K}_2.$$

The main result of this subsection reads as follows.

Theorem 3.3. *There exists a unique $\mathscr{D}$–complete totally ordered field up to increasing isomorphism.*

Proof. Let $\mathbb{K}_1 = (\mathbb{K}_1, +_1, \cdot_1, \leq_1)$ and $\mathbb{K}_2 = (\mathbb{K}_2, +_2, \cdot_2, \leq_2)$ be two complete totally ordered fields. For every $x \in \mathbb{K}_1$ let us define the set

$$A(x) := \{q : q \in \mathbb{Q} \wedge q \leq_1 x\}.$$

Since $\mathbb{K}_1$ is an Archimedean field, the set $A(x)$ is bounded from above in $\mathbb{Q}$. Consequently, the set[7] $A(x)$ is also bounded from above in $\mathbb{K}_2$. Since $\mathbb{K}_2$ is complete the function $\varphi : \mathbb{K}_1 \to \mathbb{K}_2$ given by

$$\varphi(x) := \sup_{\mathbb{K}_2} A(x) = \sup_{\mathbb{K}_2}\{q : q \in \mathbb{Q} \wedge q \leq_1 x\},$$

is well defined[8].

Since $\mathbb{Q}$ is dense in $\mathbb{K}_1$, if $x <_1 y$ then $A(x) \subset A(y)$, *i.e.*, the function φ is increasing. Moreover, if $x \in \mathbb{Q}$ then $x = \max_{\mathbb{K}_2} A(x)$ and $\varphi(x) = x$.
Analogously, it is possible to define an increasing function $\psi : \mathbb{K}_2 \to \mathbb{K}_1$ such that $\psi|_{\mathbb{Q}} = id_{\mathbb{Q}}$. Hence, the compositions $\psi \circ \varphi : \mathbb{K}_1 \to \mathbb{K}_1$ and $\varphi \circ \psi : \mathbb{K}_2 \to \mathbb{K}_2$ are increasing functions such that $\psi|_{\mathbb{Q}} = id_{\mathbb{Q}}$ and $\varphi|_{\mathbb{Q}} = id_{\mathbb{Q}}$. On account of Proposition **3.2** one has that $\psi \circ \varphi = id_{\mathbb{K}_1}$ and $\varphi \circ \psi = id_{\mathbb{K}_2}$. Then φ is an order preserving bijective function. Let us show that:

(1) $\varphi(x +_1 y) = \varphi(x) +_2 \varphi(y)$ for every $x, y \in \mathbb{K}_1$;
(2) $\varphi(x \cdot_1 y) = \varphi(x) \cdot_2 \varphi(y)$ for every $x, y \in \mathbb{K}_1$.

From now on, in order to simplify the notation, we use the symbol $+$ for sums and the symbol $\cdot$ for products (and usually we shall omit the latter). Their meaning should be clear from the context.

In order to prove (1) and (2), since $\mathbb{Q}$ is dense in $\mathbb{K}_1$ and $\mathbb{K}_2$, we will show that for every $\varepsilon > 0$ in $\mathbb{Q}$ and $x, y \in \mathbb{K}_1$ one has

$$\varphi(x) + \varphi(y) - \varepsilon < \varphi(x + y) < \varphi(x) + \varphi(y) + \varepsilon, \tag{2}$$

and

$$\varphi(x)\varphi(y) - \varepsilon < \varphi(xy) < \varphi(x)\varphi(y) + \varepsilon. \tag{3}$$

[6]The notion of increasing isomorphism defines an equivalence relation in the class of ordered fields. Theorem **3.3** ensures that all the complete totally ordered fields are in the same equivalence class. Whenever two complete totally ordered fields are (order) isomorphic, they can be considered to be essentially the same field with respect to operations and orders just by a renaming of the elements.

[7]More precisely, the isomorphic copy $j(A(x)) = \{j(q) : q \in \mathbb{Q} \wedge q \leq_1 x\}$ in $\mathbb{K}_2$.

[8]We notice that in the definition of φ, with abuse of notation, we simply identify the different copies of rational numbers in $\mathbb{K}_1$ and $\mathbb{K}_2$ as well as their subsets.

Again, the density of $\mathbb{Q}$ in $\mathbb{K}_1$ ensures that there are $p_1, p_2, q_1, q_2 \in \mathbb{Q}$ such that

$$x - \frac{\varepsilon}{2} < p_1 < x < p_2 < x + \frac{\varepsilon}{2} \quad \text{and} \quad y - \frac{\varepsilon}{2} < q_1 < y < q_2 < y + \frac{\varepsilon}{2}. \tag{4}$$

Now, we recall that φ is an injective increasing function with $\varphi|_{\mathbb{Q}} = id_{\mathbb{Q}}$. Thus, if $x < p_1 + \frac{\varepsilon}{2}$ it follows that $\varphi(x) < \varphi\left(p_1 + \frac{\varepsilon}{2}\right) = p_1 + \frac{\varepsilon}{2}$, *i.e.*, $\varphi(x) - \frac{\varepsilon}{2} < p_1$. Analogously, by (4), it follows that

$$\varphi(x) - \frac{\varepsilon}{2} < p_1; \quad p_2 < \varphi(x) + \frac{\varepsilon}{2}; \quad \varphi(y) - \frac{\varepsilon}{2} < q_1; \quad q_2 < \varphi(y) + \frac{\varepsilon}{2}.$$

Consequently,

$$\varphi(x) + \varphi(y) - \varepsilon < p_1 + q_1 \quad \text{and} \quad p_2 + q_2 < \varphi(x) + \varphi(y) + \varepsilon. \tag{5}$$

Now, since (4) yields $p_1 + q_1 < x + y < p_2 + q_2$, we deduce $p_1 + q_1 < \varphi(x+y) < p_2 + q_2$ and so, by (5), inequality (2) holds. To prove (3), without loss of generality, we suppose that x and y are positive in $\mathbb{K}_1$. Let $\delta \in \mathbb{Q}$ such that

$$\delta > 0; \quad \delta < x; \quad \delta < y; \quad \delta < \frac{\varepsilon}{2(x+y)}; \quad \delta^2 < \frac{\varepsilon}{2}.$$

The density of $\mathbb{Q}$ in $\mathbb{K}_1$ guarantees that such δ exists. Let $p_1, p_2, q_1, q_2 \in \mathbb{Q}$ such that

$$x - \delta < p_1 < x < p_2 < x + \delta \quad \text{and} \quad y - \delta < q_1 < y < q_2 < y + \delta. \tag{6}$$

Now, observe that $p_i, q_i, p_2 - \delta$ and $q_2 - \delta$ are positive in $\mathbb{Q}$. Thus, by (6)

$$p_1 q_1 < xy < p_2 q_2; \quad p_2 - \delta < x < p_1 + \delta; \quad q_2 - \delta < y < q_1 + \delta. \tag{7}$$

Consequently, by exploiting again the properties of φ, we have

$$p_1 q_1 < \varphi(xy) < p_2 q_2, \tag{8}$$

and

$$p_2 - \delta < \varphi(x) < p_1 + \delta; \quad q_2 - \delta < \varphi(y) < q_1 + \delta. \tag{9}$$

By (9) it follows that

$$(p_2 - \delta)(q_2 - \delta) < \varphi(x)\varphi(y) < (p_1 + \delta)(q_1 + \delta). \tag{10}$$

Now, by (8) and (10), one has

$$\varphi(xy) - \varphi(x)\varphi(y) < p_2 q_2 - (p_2 - \delta)(q_2 - \delta) = \delta(p_2 + q_2 - \delta) < \delta(x + y + \delta),$$

as well as

$$\varphi(xy) - \varphi(x)\varphi(y) > p_1 q_1 - (p_1 + \delta)(q_1 + \delta) = -\delta(p_1 + q_1 + \delta) > -\delta(x + y + \delta).$$

Hence, bearing in mind that $\delta < \frac{\varepsilon}{2(x+y)}$ and $\delta^2 < \frac{\varepsilon}{2}$, it follows that

$$|\varphi(xy) - \varphi(x)\varphi(y)| < \delta(x+y) + \delta^2 < \varepsilon.$$

In conclusion (3) is verified and the proof is now complete[9]. □

[9]See, for the same subject and related topics, the didactic notes of A. Zanardo, Course: Fondamenti della Matematica, A.Y. 2011/12 - La struttura dei numeri reali: costruzione e proprietà. Department of Mathematics - University of Padova, March 2012.

Thanks to Theorem **3.3** the following definition makes sense:
The unique $\mathscr{D}$–complete totally ordered field (up to increasing isomorphism) is denoted by $\mathbb{R} := (\mathbb{R}, +, \cdot, \leq)$ and it is said to be the **field of real numbers**.

3.3. Existence of $\sqrt{2}$. Let us prove that the field of rational numbers $\mathbb{Q} = (\mathbb{Q}, +, \cdot, \leq)$ is not $\mathscr{D}$–complete; Lanconelli 1998, Esempio 3.10.

Proposition 3.4. *Let X be the subset of $\mathbb{Q}$ defined by*

$$X := \{q : q \in \mathbb{Q} \wedge q \geq 0,\, q^2 < 2\}.$$

Then $X \neq \emptyset$, $\mathscr{U}_X \neq \emptyset$ and the least upper bound $\varrho := \sup_{\mathbb{Q}} X$ does not exist.

Proof. Since $1 \in X$, clearly X is nonempty. Moreover $2 \in \mathscr{U}_X$, indeed: if $q \in X$, then

$$q^2 < 2 \quad \Rightarrow \quad q^2 < 4 \quad \Rightarrow (q-2)(q+2) < 0 \quad \Rightarrow \quad q < 2.$$

Arguing by contradiction, suppose that there exists $\varrho := \sup_{\mathbb{Q}} X$, that is $\varrho = \min_{\mathbb{Q}} \mathscr{U}_X$ and $\varrho \geq 1$. The trichotomy property of $\leq$ ensures that only one of the following cases occurs

$$\varrho^2 < 2, \quad \varrho^2 = 2, \quad \varrho^2 > 2.$$

Case $\varrho^2 < 2$ - Since $\mathbb{Q}$ is Archimedean as proved in Devillanova and Molica Bisci 2021, Corollary 5.6, there exists $n \in \mathbb{N}$ such that

$$n > \max\left\{1, \frac{2\varrho + 1}{2 - \varrho^2}\right\}.$$

Now, we claim that $\varrho + \dfrac{1}{n} \in X$, that is

$$\varrho + \frac{1}{n} \in \mathbb{Q}, \quad \varrho + \frac{1}{n} \geq 0, \quad \text{and} \quad \left(\varrho + \frac{1}{n}\right)^2 < 2.$$

Indeed, since $n > 1$, it follows that $n^2 > n$ and

$$\begin{aligned}\left(\varrho + \frac{1}{n}\right)^2 &= \varrho^2 + \frac{2\varrho}{n} + \frac{1}{n^2} < \varrho^2 + \frac{2\varrho}{n} + \frac{1}{n} = \\ &= \varrho^2 + \frac{2\varrho + 1}{n} < \varrho^2 + 2 - \varrho^2 = 2.\end{aligned}$$

Hence $\varrho + \dfrac{1}{n} \in X$ against $\varrho := \sup_{\mathbb{Q}} X$.

Case $\varrho^2 = 2$ - Since $\varrho \in \mathbb{Q}$ and $\varrho > 0$ there are two **coprime**[10] integers $m, n \in \mathbb{N}^*$ such that $\varrho = \dfrac{m}{n}$. Hence

$$\varrho^2 = 2 \iff \frac{m^2}{n^2} = 2 \iff m^2 = 2n^2 \Rightarrow \exists p \in \mathbb{N} \text{ such that } m = 2p.$$

Consequently

$$\frac{m^2}{n^2} = 2 \quad \Rightarrow \frac{4p^2}{n^2} = 2 \quad \Rightarrow n^2 = 2p^2 \quad \Rightarrow n \text{ is even.}$$

[10]The integers m and n are coprime if their greatest common divisor is 1.

Thus m and n are both even. We clearly have an absurd since m and n are coprime. Case $\varrho^2 > 2$ - Since $\mathbb{Q}$ is Archimedean, there exists $n \in \mathbb{N}^*$ such that $n > \max\left\{\frac{1}{\varrho}, \frac{2\varrho}{\varrho^2-2}\right\}$. Thus

$$\varrho - \frac{1}{n} \in \mathbb{Q}, \quad \varrho - \frac{1}{n} > 0, \quad \text{and} \quad \left(\varrho - \frac{1}{n}\right)^2 > 2.$$

Indeed

$$\left(\varrho - \frac{1}{n}\right)^2 = \varrho^2 - \frac{2\varrho}{n} + \frac{1}{n^2} > \varrho^2 - \frac{2\varrho}{n} > \varrho^2 - \varrho^2 + 2 = 2.$$

We claim that $\varrho - \frac{1}{n} \in \mathscr{U}_X$. Indeed, since $\varrho - \frac{1}{n} > 0$, for any $q \in X$ one has

$$\varrho - \frac{1}{n} > q \Leftrightarrow \left(\varrho - \frac{1}{n}\right)^2 > q^2,$$

taking into account that, for every $q \in X$

$$q^2 < 2 < \left(\varrho - \frac{1}{n}\right)^2.$$

Hence $\varrho - \frac{1}{n} \in \mathscr{U}_X$ against $\varrho := \sup_{\mathbb{Q}} X$. This completes the proof. □

An element $x \in \mathbb{R} \setminus \mathbb{Q}$ is said to be **irrational**[11].

Let X be a nonempty set of $\mathbb{R}$. If X is unbounded from below, *i.e.*, $\mathscr{L}_X = \emptyset$, we write $\inf_{\mathbb{R}} X = -\infty$. Analogously, if X is unbounded from above, *i.e.*, $\mathscr{U}_X = \emptyset$, we write $\sup_{\mathbb{R}} X = +\infty$. Let X and Y subsets of $\mathbb{R}$. Furthermore, let $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$, with $\mu > 0$. Define

$$-X := \{-x : x \in X\}, \quad \text{and} \quad X + \lambda := \{x + \lambda : x \in X\},$$

as well as

$$\mu X := \{\mu x : x \in X\}.$$

The following properties hold whenever X and Y are bounded or unbounded, empty or nonempty:

if $X \subset Y$, then $\sup_{\mathbb{R}} X \leq \sup_{\mathbb{R}} Y$ and $\inf_{\mathbb{R}} X \geq \inf_{\mathbb{R}} Y$ (monotonicity);
$\sup_{\mathbb{R}}(-X) = -\inf_{\mathbb{R}} X$, and $\inf_{\mathbb{R}}(-X) = -\sup_{\mathbb{R}} X$ (reflection);
$\sup_{\mathbb{R}}(X + \lambda) = \sup_{\mathbb{R}} X + \lambda$, and $\inf_{\mathbb{R}}(X + \lambda) = \inf_{\mathbb{R}} X + \lambda$ (translation);

The extended real number system. The extended real number system $\tilde{\mathbb{R}}$ consists of the real field $\mathbb{R}$ and two symbols, $+\infty$ and $-\infty$; that is $\tilde{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. We preserve the original order in $\mathbb{R}$, and define

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$.

[11] We notice that $\mathbb{R} \setminus \mathbb{Q} \neq \emptyset$. Indeed $\sqrt{2} := \sup_{\mathbb{R}}\{q : q \in \mathbb{Q} \wedge q \geq 0,\ q^2 < 2\}$ exists since $\mathbb{R}$ is $\mathscr{D}$–complete and by Proposition **3.4** clearly $\sqrt{2} \notin \mathbb{Q}$.

It is then clear that $+\infty$ is an upper bound of every subset of the extended real number system, and that every nonempty subset has a least upper bound. The extended real number system does not form a field, but it is customary to make the following conventions:

if $x \in \mathbb{R}$ then

$$x + (+\infty) = +\infty, \quad x + (-\infty) = -\infty, \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0;$$

if $x > 0$ then

$$x \cdot (+\infty) = +\infty, \quad \text{and} \quad x \cdot (-\infty) = -\infty;$$

if $x < 0$ then

$$x \cdot (+\infty) = -\infty, \quad \text{and} \quad x \cdot (-\infty) = +\infty.$$

(Note that no precise meaning is *a priori* given in the case $x = 0$.) If $X \subseteq \mathbb{R}$ is a nonempty set unbounded from above (in $\mathbb{R}$), then $\sup_{\tilde{\mathbb{R}}} X = +\infty$. Analogously, if $Y \subset \mathbb{R}$ is a nonempty[12] set unbounded from below (in $\mathbb{R}$), then $\inf_{\tilde{\mathbb{R}}} Y = -\infty$. When it is desired to make quite explicit the distinction between real numbers, on the one hand, and the symbols $+\infty$ and $-\infty$, on the other, the former are called *finite*; see Rudin 1976. As a topological space $\tilde{\mathbb{R}}$ is a compactification of the usual line, *i.e.*, a compact space containing $\mathbb{R}$ as a dense subspace, and it is homeomorphic to a bounded closed interval of $\mathbb{R}$; recall that a neighbourhood of $+\infty$ (resp. $-\infty$) in $\tilde{\mathbb{R}}$ is any subset of $\tilde{\mathbb{R}}$ containing a half–line $(x, +\infty]$ (resp. $[-\infty, x)$) for some $x \in \mathbb{R}$, where

$$(x, +\infty] := \{y \ : y \in \tilde{\mathbb{R}} \wedge x < y \leq +\infty\}$$

and

$$[-\infty, x) := \{y \ : y \in \tilde{\mathbb{R}} \wedge -\infty \leq y < x\}.$$[13]

Let $X, Y \subseteq \tilde{\mathbb{R}}$ be two nonempty sets and let

$$X + Y := \{x + y : x \in X \wedge y \in Y\}.$$

Then, the following addition rules hold:

$\sup_{\tilde{\mathbb{R}}}(X + Y) = \sup_{\tilde{\mathbb{R}}} X + \sup_{\tilde{\mathbb{R}}} Y$;
$\inf_{\tilde{\mathbb{R}}}(X + Y) = \inf_{\tilde{\mathbb{R}}} X + \inf_{\tilde{\mathbb{R}}} Y$;

provided that also the right–hand sides are defined; in particular, this always holds when X and Y are nonempty real subsets.

Completion: Assume that $(X, \leq)$ is not a $\mathscr{D}$–complete totally ordered set. Then it is possible to construct a $\mathscr{D}$–complete totally ordered set[14] $\widehat{X}$ such that X is dense in $\widehat{X}$; see De Marco 1986, Appendix B.1.5; the construction of $\widehat{X}$ is based on the notion of Dedekind cuts as in the Cantor model of the reals given in Subsection **5.2**. Analogously, if $(X, \leq)$ is not complete with respect to the order topology, then X can be canonically embedded in a complete metric space $\overline{X}^{|\cdot|}$; in this case the

[12] In order to avoid contradiction, we assume: $\inf_{\tilde{\mathbb{R}}} \emptyset = +\infty$ and $\sup_{\tilde{\mathbb{R}}} \emptyset = -\infty$.

[13] We also define: $[x, +\infty] := \{y \ : y \in \tilde{\mathbb{R}} \wedge x \leq y \leq +\infty\}$ and $[-\infty, x] := \{y \ : y \in \tilde{\mathbb{R}} \wedge -\infty \leq y \leq x\}$.

[14] We notice that the completion $\widehat{\mathbb{K}}$ of a field $\mathbb{K}$ cannot be a field; see Example **4.4**.

construction of $\overline{X}^{|\cdot|}$ can be done by adapting the procedure given in Subsection **5.1** trough equivalence classes of Cauchy sequences; see Cecconi and Stampacchia 1983, Capitolo 1 - Appendice: Completamento di Spazi Metrici. The following property, that is peculiar of non–Archimedean fields, has been proved in Massaza 1969/70 and will be used in the sequel.

Lemma 3.5. *Let $\mathbb{K}$ be a non–Archimedean field and let $\sum_{n=0}^{\infty} x_n$ be any series*[15] *over $\mathbb{K}$. Then the series $\sum_{n=0}^{\infty} x_n$ is convergent in $\overline{\mathbb{K}}^{|\cdot|}$, i.e., there exists $S \in \overline{\mathbb{K}}^{|\cdot|}$ such that*

$$\lim_{n\to\infty} \sum_{k=0}^{n} x_k = S \quad \text{in } \mathbb{K},$$

iff

$$\lim_{n\to\infty} x_n = 0 \quad \text{in } \overline{\mathbb{K}}^{|\cdot|}.$$

As a consequence, a series $\displaystyle\sum_{n=0}^{\infty} x_n$ is convergent in $\overline{\mathbb{K}}^{|\cdot|}$ iff $\displaystyle\sum_{n=0}^{\infty} |x_n|$ is convergent in $\overline{\mathbb{K}}^{|\cdot|}$ (absolute convergence).

It is clear that the $\mathscr{D}$–complete totally ordered set $\widehat{X}$ is also Cauchy complete. Finally, we emphasize that in the current literature there are several and different notions of completion. For instance, the extended real line $\tilde{\mathbb{R}}$ constructed above can be viewed as the **Dedekind–MacNeille completion**[16] of the totally ordered field $\mathbb{Q}$.

Finally, we emphasize that the Zermelo's theorem (see Devillanova and Molica Bisci 2021, Theorem 6.6) ensures that a total order exists on the complex field $\mathbb{C} := \mathbb{R}^2$ endowed by the operations defined by setting for every $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$

$$(x_1, y_1) +_{\mathbb{C}} (x_2, y_2) := (x_1 + x_2, y_1 + y_2),$$

and

$$(x_1, y_1) \cdot_{\mathbb{C}} (x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$

However, thanks to the results presented in Devillanova and Molica Bisci 2021, Section 4, it is easily seen that no total order exists on $\mathbb{C}$ which is compatible with the field structure. For instance, this fact can be viewed as a direct consequence of Devillanova and Molica Bisci 2021, Theorem 4.3 since the equation

$$x_1^2 + x_2^2 = 0 \ \text{ in } \mathbb{C}$$

admits the nontrivial solution $(x_1, x_2) = (1, i) \in \mathbb{C}^2$. The above remarks clarify that the theory of ordered fields cannot be applied to study metric and analytic aspects of the Euclidean space $\mathbb{R}^d$, with $d \geq 2$.

[15] Over an ordered field $\mathbb{K}$ the notion of series can be given as usual.

[16] Given a poset $X := (X, \leq)$, the Dedekind–MacNeille completion of X is the smallest complete lattice containing a subset order–isomorphic with X.

4. Characterizations of Completeness

In most textbooks, the set of real numbers is commonly taken to be a totally ordered Dedekind complete field. Exploiting this definition, one can then establish the basic properties of the real field such as: the Bolzano–Weierstrass property, the Monotone Convergence property, the Cantor completeness, and the sequential (Cauchy) completeness; see Hall 2010. In the sequel, on account of Theorem **3.1**, we prove the equivalence between the Dedekind completeness property given in Subsection **3.1** and some meaningful well–known notions present in the literature.

4.1. A fundamental characterization result. The aim of this subsection is to show, as in Cohen and Ehrlich 1963, the equivalence between some notions of completeness present in the classical literature. To this aim we recall a basic definition. Let $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ be a totally ordered field. The pair (X, Y) is a **cut** in $\mathbb{K}$ if X and Y are two nonempty sets of $\mathbb{K}$ such that:

(1) $X \cup Y = \mathbb{K}$;
(2) if $x \in X$, $y \in Y$, then $x \leq y$.

The sets X and Y are called, respectively, the **lower class** and the **upper class** of the cut. A cut is a **gap** if its lower class has no maximum, and its upper class has no minimum. We notice that if (X, Y) is a cut in $\mathbb{K}$, then the sets X and Y are intervals in $\mathbb{K}$.

The main result reads as follows.

Theorem 4.1. *Let $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ be a totally ordered field. The next facts are equivalent:*

(k_1) *$\mathbb{K}$ is Archimedean and (Cauchy) complete;*
(k_2) *For every nonempty $X \subseteq \mathbb{K}$, if $\mathscr{U}_X \neq \emptyset$ then $\sup_{\mathbb{K}} X$ exists;*
(k_3) *There are no gaps in $\mathbb{K}$;*
(k_4) *For every nonempty $X \subseteq \mathbb{K}$, if $\mathscr{L}_X \neq \emptyset$ then $\inf_{\mathbb{K}} X$ exists;*
(k_5) *If $X \in \mathscr{B}_{\mathbb{K}}$ is closed in $\mathbb{K}$ and $\mathscr{W}$ is a family of open intervals which covers*[17] *the set X, then $\mathscr{W}$ has a finite subfamily $\mathscr{S}$ which covers X (Heine–Borel property);*
(k_6) *If $X \in \mathscr{B}_{\mathbb{K}}$ is infinite, then $DX \neq \emptyset$ (Bolzano–Weierstrass property);*
(k_7) *$\mathbb{K}$ is Archimedean, and every nested sequence of closed intervals in $\mathbb{K}$ has a nonempty intersection (Cantor property).*[18]

Proof. The proof is divided in several steps:

$(k_1) \Rightarrow (k_2)$ - Let X be a nonempty subset of $\mathbb{K}$ such that $\mathscr{U}_X \neq \emptyset$ and fix $b \in \mathbb{K}$ which is an upper bound for X. Furthermore, let $\bar{x} \in X \setminus \mathscr{U}_X$. Since $\mathbb{K}$ is Archimedean, there is, for each $n \in \mathbb{N}^*$, some $\bar{m} \in \mathbb{N}^*$ such that $\bar{x} + \dfrac{\bar{m}}{n} \geq b$ in $\mathbb{K}$.

[17] A family $\mathscr{W}$ of sets of $\mathbb{K}$ that covers $X \subset \mathbb{K}$ (usually called a *covering* of X) is a set $\mathscr{W}$ of subsets of $\mathbb{K}$ such that for each $x \in X$ there is some $W_x \in \mathscr{W}$ with $x \in W_x$.

[18] If for each $n \in \mathbb{N}^*$, X_n is a closed interval in $\mathbb{K}$, and $X_{n+1} \subseteq X_n$, then $\bigcap_{n \in \mathbb{N}^*} X_n \neq \emptyset$.

Hence, $\bar{x} + \dfrac{\bar{m}}{n}$ is an upper bound for X. Thus, for each $n \in \mathbb{N}^*$, the set

$$B_n = \left\{ m :\ \bar{x} + \frac{m}{n} \in \mathscr{U}_X \right\} \subset \mathbb{N}^*$$

is nonempty and, by Devillanova and Molica Bisci 2021, Theorem 4.5, there exists $m_n := \min_{\mathbb{N}^*} B_n$. Then, for each $n \in \mathbb{N}^*$

$$y_n := \bar{x} + \frac{m_n}{n} \in \mathscr{U}_X$$

and, by construction,

$$x_n := y_n - \frac{1}{n} = \bar{x} + \frac{m_n - 1}{n} \leq x \text{ in } \mathbb{K} \text{ for some } x \in X.$$

Hence, we get two sequences $(x_n)_n$ and $(y_n)_n$ such that for any $m, n \in \mathbb{N}^*$

$$x_m < y_n,$$
$$x_m - x_n < y_n - \left(y_n - \frac{1}{n} \right) = \frac{1}{n},$$

so that

$$\begin{aligned} |x_m - x_n| &= \max\{x_m - x_n, x_n - x_m\} \\ &\leq \max\left\{ \frac{1}{n}, \frac{1}{m} \right\} \text{ in } \mathbb{K} \text{ for every } m, n \in \mathbb{N}^*. \end{aligned}$$

Since $\mathbb{K}$ is Archimedean, it follows that $\lim_{k\to\infty} 1/k = 0$. Consequently, $(x_n)_n$ is a Cauchy sequence in $\mathbb{K}$ and by (k_1) there exists $x_o \in \mathbb{K}$ such that $\lim_{n\to\infty} x_n = x_o$ in $\mathbb{K}$. We claim that $x_o = \sup_{\mathbb{K}} X$. Indeed:

$x_o \in \mathscr{U}_X$ - Arguing by contradiction assume that $x_o < x$ for some $x \in X$. Since $\lim_{n\to\infty} x_n = x_o$ in $\mathbb{K}$ and $\lim_{n\to\infty} 1/n = 0$, there is some $n \in \mathbb{N}$ such that

$$x_n - x_o \leq |x_n - x_o| < \frac{x - x_o}{2} \quad \text{and} \quad \frac{1}{n} < \frac{x - x_o}{2} \text{ in } \mathbb{K}.$$

Hence

$$y_n = x_n + \frac{1}{n} < \left(x_o + \frac{x - x_o}{2} \right) + \frac{x - x_o}{2} = x \text{ in } \mathbb{K}.$$

Since $x \in X$, and $y_n \in \mathscr{U}_X$ a contradiction is obtained.

If $x_o^* \in \mathscr{U}_X$, then $x_o \leq x_o^*$ in $\mathbb{K}$ - Arguing by contradiction, assume that there exists $x_o^* \in \mathscr{U}_X$ such that $x_o > x_o^*$ in $\mathbb{K}$. Then, for some $n \in \mathbb{N}^*$,

$$x_o - x_n \leq |x_o - x_n| < x_o - x_o^* \text{ in } \mathbb{K}.$$

Hence, $x_o^* < x_n$ in $\mathbb{K}$ and, consequently, by the construction of x_n, $x_o^* < x_n \leq x$ in $\mathbb{K}$ for some $x \in X$. This is impossible, since $x_o^* \in \mathscr{U}_X$.

$(k_2) \Rightarrow (k_3)$ - Let (X, Y) be a cut in $\mathbb{K}$. Then X is a nonempty subset of $\mathbb{K}$ such that $\emptyset \neq Y \subset \mathscr{U}_X$. Therefore, by (k_2), there exists $e'' := \sup_{\mathbb{K}} X$. Now, since $X \cup Y = \mathbb{K}$, either $e'' = \max_{\mathbb{K}} X$ or $e'' = \min_{\mathbb{K}} Y$ so that (X, Y) is not a gap.

$(k_3) \Rightarrow (k_4)$ - Suppose H is a nonempty subset of $\mathbb{K}$ and $\mathscr{L}_H \neq \emptyset$. Let

$X := \mathscr{L}_H$, and
$Y := \mathbb{K} \setminus X = \mathbb{K} \setminus \mathscr{L}_H$.

We claim that (X, Y) is a cut in $\mathbb{K}$. Indeed, $X \cup Y = \mathbb{K}$ by construction; $X \neq \emptyset$ since $X = \mathscr{L}_H$; $Y \neq \emptyset$ since $h + 1 \in Y$ for every h in $H \neq \emptyset$. Finally, if $x \in X$ and $y \in Y$, then $x < y$. Indeed, suppose otherwise that $y \leq x$, since $x \in X = \mathscr{L}_H$ we should have $y \leq x \leq h$ for every $h \in H$, *i.e.*, $y \in \mathscr{L}_H = X$ and so $X \cap Y \neq \emptyset$ which contradicts the definition of X and Y.

Now, since, by (k_3), (X, Y) cannot be a gap two cases occur:
either $\exists \max_{\mathbb{K}} X$ or $\exists \min_{\mathbb{K}} Y$. In the first case $\exists \max_{\mathbb{K}} X = \exists \max_{\mathbb{K}} \mathscr{L}_H = \inf_{\mathbb{K}} H$ and the the thesis is obtained, while the second case is ruled out since $\min_{\mathbb{K}} Y$ have to belong to both Y and $\mathscr{L}_Y = X$ getting in contradiction to the fact that $X \cap Y = \emptyset$.

$(k_4) \Rightarrow (k_5)$ - Let $X \in \mathscr{B}_{\mathbb{K}}$ be closed in $\mathbb{K}$, and let $\mathscr{J}$ be a set of open intervals J which covers X. Since X is bounded, there are elements $x_1, y_1 \in \mathbb{K}$ such that $X \subset [x_1, y_1]$. Since X is closed, $[x_1, y_1] \setminus X$ is open, so, for every $y \in [x_1, y_1] \setminus X$, there exists an open interval J_y such that $y \in J_y$ and $X \cap J_y = \emptyset$. Let us consider the set

$$\mathscr{H} := \{J_y : y \in [x_1, y_1] \setminus X, \text{ and } J_y \cap X = \emptyset\}.$$

Then the set $\mathscr{G} := \mathscr{J} \cup \mathscr{H}$ covers the interval $[x_1, y_1]$. Now, let

$$L := \{x : x \in [x_1, y_1] \text{ and } [x, y_1] \text{ is covered by a finite subset of } \mathscr{G}\}.$$

We claim that:

- (c_1) $L \neq \emptyset$, in particular $y_1 \in L$;
- (c_2) There exists $e'_L := \inf_{\mathbb{K}} L$;
- (c_3) $e'_L \in L$;
- (c_4) $e'_L = x_1$.

(c_1) Since $[y_1, y_1]$ is covered by some open interval $J \in \mathscr{J}$ if $y_1 \in X$, or by $J_{y_1} \in \mathscr{H}$ if $y_1 \notin X$, it follows that $y_1 \in L$, and so $L \neq \emptyset$.
(c_2) Since $x_1 \leq x$ for every $x \in L$, it follows that $\mathscr{L}_L \neq \emptyset$. Hence, by (k_4) there exists $e'_L := \inf_{\mathbb{K}} L$.
(c_3) We prove that $e'_L \in L$, *i.e.*, $[e'_L, y_1]$ is covered by a finite subset of $\mathscr{G}$. Since $L \subseteq [x_1, y_1]$, $e'_L \in [x_1, y_1]$ and there is an open interval $I_1 = (a, b) \in \mathscr{G}$ such that $e'_L \in I_1$. Since $e'_L = \inf_{\mathbb{K}} L$, there is some $z_1 \in L$ such that $a < e'_L < z_1 < b$, *i.e.*, such that $[e'_L, z_1] \subset (a, b) = I_1$. Now, since $z_1 \in L$, there is a finite set $\{I_2, \ldots, I_m\} \subset \mathscr{G}$ which covers the interval $[z_1, y_1]$ and, as a consequence, the finite set $\{I_1, I_2, \ldots, I_m\} \subset \mathscr{G}$ covers $[e'_L, y_1]$.
(c_4) We prove that $e'_L = x_1$. Arguing by contradiction, assume $e'_L \neq x_1$, *i.e.*, that $x_1 < e'_L$. Since, with the same notation as above, $a < e'_L$, $\max\{x_1, a\} < e'_L$, and, since $\mathbb{K}$ is dense (in itself), see Devillanova and Molica Bisci 2021, Proposition 5.4, there is an element $z_1 \in \mathbb{K}$ such that

$$\max\{x_1, a\} < z_1 < e'_L < b \leq y_1. \tag{11}$$

Since $x_1 \leq \max\{x_1, a\}$, by (11) it follows that $z_1 \in [x_1, y_1]$. On the other hand, since $a \leq \max\{x_1, a\}$, inequality (11) implies that $[z_1, y_1]$ is covered by $\{I_1, I_2, \ldots, I_m\}$, where $I_1 = (a, b)$. Thus, $z_1 \in L$. This is impossible since $z_1 < e'_L = \inf_{\mathbb{K}} L$.

We have shown that $[x_1, y_1]$, and hence the subset X of $[x_1, y_1]$, is covered by the sets $\{I_1, I_2, \dots, I_m\} \subset \mathscr{G} = \mathscr{J} \cup \mathscr{H}$. Since no interval in $\mathscr{H}$ contains a point of X, one has that $\mathscr{J} \cap \{I_1, I_2, \dots, I_m\}$ is a finite subset of $\mathscr{J}$ which covers X.

$(k_5) \Rightarrow (k_6)$ - Let $X \in \mathscr{B}_{\mathbb{K}}$ be an infinite subset of $\mathbb{K}$, and suppose, by contradiction, that $DX = \emptyset$. Then X is closed. Suppose $x \in X$. Since $x \notin DX$, there is an open interval J_x such that $X \cap J_x = \{x\}$. The set $\mathscr{K} = \{J_x : x \in X\}$ covers X. Since $X \in \mathscr{B}_{\mathbb{K}}$ and is closed in $\mathbb{K}$, by (k_5), there is a finite subset $\mathscr{Z}$ of $\mathscr{K}$ which covers X. If $\mathscr{Z} = \{J_{x_1}, \dots, J_{x_n}\}$ then, for every $x \in X$, there is some $k \in \{1, ..., n\}$ such that $x \in J_{x_k}$, *i.e.*, $x = x_k$. Hence $X = \{x_1, \dots, x_n\}$ is a finite set, against to our assumption.

$(k_6) \Rightarrow (k_7)$ - Assume that (k_6) holds. We divide the proof in two steps:

1 - $\mathbb{K}$ is Archimedean;

2 - Every nested sequence $(J_n)_n$ of closed intervals $J_n := [a_n, b_n]$ in $\mathbb{K}$ has a nonempty intersection.

Step 1 - Arguing by contradiction, suppose that $\mathbb{K}$ is not Archimedean. Hence, there exist $a, b \in \mathbb{K}$ such that $0 < a < b$ in $\mathbb{K}$ and $a \le na < b$ for every $n \in \mathbb{N}^*$. Therefore, the set $\{na : n \in \mathbb{N}\}$ is bounded, infinite with no accumulation point against (k_6).

Step 2 - Since $(J_n)_n := ([a_n, b_n])_n$ is a nested sequence we have $J_{n+1} \subseteq J_n$ for each $n \in \mathbb{N}^*$, so

$$a_n \le a_{n+1} \le b_{n+1} \le b_n \quad \text{for every } n \in \mathbb{N}^*.$$

Let $X := \{a_n : n \in \mathbb{N}^*\}$. Since $a_1 \le a_n \le b_1$ for every $n \in \mathbb{N}^*$, X is bounded in $\mathbb{K}$. If for some $\bar{n} \in \mathbb{N}^*$, $a_j = a_n$ for every $j \ge \bar{n}$, then $a_n \le a_{\bar{n}} \le b_n$ for every $n \in \mathbb{N}^*$, so that $a_{\bar{n}} \in \bigcap_{n \in \mathbb{N}^*} J_n$ giving the thesis. If, otherwise, for each $n \in \mathbb{N}^*$ there is some $j \in \mathbb{N}^*$ such that $a_j > a_n$, the set X has no greatest element and it is therefore an infinite subset of $\mathbb{K}$. By the hypothesis, the bounded, infinite set X admits an accumulation point $x \in \mathbb{K}$. If $a_{\bar{n}} > x$ for some $\bar{n} \in \mathbb{N}^*$, then $a_j \ge a_{\bar{n}} > x$ for every $j \ge \bar{n}$. Thus, if $0 < \varepsilon < a_{\bar{n}} - x$, the interval $(x - \varepsilon, x + \varepsilon)$ contains only a finite number of points of X (indeed $x + \varepsilon < a_{\bar{n}} \le a_j$ for all $j \ge \bar{n}$), against the fact that $x \in DX$. Hence $a_n \le x$ for every $n \in \mathbb{N}^*$. If $b_{\bar{n}} < x$ for some $\bar{n} \in \mathbb{N}^*$, then $a_j \le b_{\bar{n}} < x$ for every $j \in \mathbb{N}$. Thus, if $0 < \varepsilon < x - b_{\bar{n}}$, the interval $(x - \varepsilon, x + \varepsilon)$ contains only a finite number of points of X (indeed $a_j \le b_{\bar{n}} < x - \varepsilon$ for all $j \ge \bar{n}$), against the fact that $x \in DX$. Hence $x \le b_n$ for each $n \in \mathbb{N}^*$. But then $a_n \le x \le b_n$ for each $n \in \mathbb{N}^*$, and $x \in \bigcap_{n \in \mathbb{N}^*} J_n$ giving the thesis.

$(k_7) \Rightarrow (k_1)$ - Under assumption (k_7) we only need to prove that $\mathbb{K}$ is Cauchy complete. Let $(x_n)_n$ be a Cauchy sequence in $\mathbb{K}$. Then, for each $k \in \mathbb{N}^*$ there is some $n_k \in \mathbb{N}^*$ such that

$$x_{n_k} - \frac{1}{k} < x_n < x_{n_k} + \frac{1}{k}$$

for every $n \ge n_k$. For each $j \in \mathbb{N}^*$ let

$$p_j := \max\{n_k : k \in \mathbb{N}^* \wedge k \le j\};$$

so that

$$x_{n_k} - \frac{1}{k} < x_n < x_{n_k} + \frac{1}{k} \quad \text{for all } k \leq j \text{ and } n \geq p_j.$$

Then, set

$$a_j := \max\left\{x_{n_k} - \frac{1}{k} : k \in \mathbb{N}^* \wedge k \leq j\right\} \in \mathbb{K};$$

$$b_j := \min\left\{x_{n_k} + \frac{1}{k} : k \in \mathbb{N}^* \wedge k \leq j\right\} \in \mathbb{K};$$

we get $a_j < x_{p_j} < b_j$, so that

$$a_j \leq a_{j+1} < x_n < b_{j+1} \leq b_j \qquad \text{for every } j \in \mathbb{N}^* \text{ and } n \geq p_{j+1}.$$

Moreover, set $J_j := [a_j, b_j]$ for each $j \in \mathbb{N}^*$, we get $J_{j+1} \subseteq J_j$ for every $j \in \mathbb{N}^*$. Then, by (k_7) there exists $\lambda \in \mathbb{K}$ such that

$$\lambda \in \bigcap_{j \in \mathbb{N}^*} J_j.$$

We claim that $\lim_{n\to\infty} x_n = \lambda$ in $\mathbb{K}$. Indeed, it is easily seen that

$$x_{n_j} - \frac{1}{j} \leq a_j \leq \lambda \leq b_j \leq x_{n_j} + \frac{1}{j} \qquad \text{for every } j \in \mathbb{N}^*.$$

Moreover,

$$x_{n_j} - \frac{1}{j} < x_n < x_{n_j} + \frac{1}{j} \qquad \text{for every } n \geq p_j.$$

Hence, by combining the previous inequalities, we get

$$|x_n - \lambda| = \max\{x_n - \lambda, \lambda - x_n\} \leq \frac{2}{j} \qquad \text{for every } j \text{ and } n \geq p_j.$$

Now, let $\varepsilon > 0$ in $\mathbb{K}$. Since $\mathbb{K}$ is Archimedean, there is some $\bar{\jmath} \in \mathbb{N}^*$ such that $\bar{\jmath}\varepsilon > 2$. Hence $|x_n - \lambda| < \varepsilon$ for every $n \geq p_j$, *i.e.*, $\lim_{n\to\infty} x_n = \lambda$ in $\mathbb{K}$ as claimed. □

Additional characterizations of $\mathscr{D}$–completeness can be found in Hall 2010. We emphasize that in a totally ordered Archimedean field $\mathbb{K}$ the $\mathscr{D}$–completeness is equivalent to the fact that every bounded sequence admits a convergent subsequence (Equivalently: every bounded and monotone sequence in $\mathbb{K}$ admits limit in $\mathbb{K}$).

4.2. Comparisons properties. By Devillanova and Molica Bisci 2021, Corollary 5.6, and Theorem **4.1** next result holds.

Corollary 4.2. *The rational field* $\mathbb{Q} = (\mathbb{Q}, +, \cdot, \leq)$ *is not Cauchy complete.*

Proof. By Devillanova and Molica Bisci 2021, Corollary 5.6, $\mathbb{Q}$ is Archimedean, and so if we assume, by contradiction, that $\mathbb{Q}$ is also Cauchy complete, $\mathbb{Q}$ should be, by Theorem **4.1**, $\mathscr{D}$–complete against Proposition **3.4**. □

Finally, in order to complete the picture of the possible combinations of the above recalled properties, we exhibit a meaningful example that naturally appears also in many questions arising from Algebraic and Differential Geometry.

Example 4.3. Let $\mathbb{R}((X))$ be the set of formal Laurent series $\sum_{j\in\mathbb{Z}} r_j X^j$, where $r_j \in \mathbb{R}$ for each $j \in \mathbb{Z}$ and such that for negative values of j only a finite number of coefficients r_j are different from zero. In $\mathbb{R}((X))$ define the following operations:

Sum:

$$\sum_{j\in\mathbb{Z}} r_j X^j + \sum_{j\in\mathbb{Z}} s_j X^j := \sum_{j\in\mathbb{Z}} (r_j + s_j) X^j,$$

(Cauchy) Product:

$$\sum_{j\in\mathbb{Z}} r_j X^j \cdot \sum_{j\in\mathbb{Z}} s_j X^j := \sum_{j\in\mathbb{Z}} \left(\sum_{p+q=j} r_p s_q \right) X^j.$$

According to the notations and remarks given in Devillanova and Molica Bisci 2021 the field of formal Laurent series $\mathbb{R}((X))$ can be viewed as the field of fractions of the formal series ring $\mathbb{R}[[X]]$. Moreover, in $\mathbb{R}((X))$ let us define the strict order

$$\sum_{j\in\mathbb{Z}} r_j X^j < \sum_{j\in\mathbb{Z}} s_j X^j$$

if there is $k \in \mathbb{Z}$ such that

$$r_j = s_j \quad \text{for every} \quad j < k$$

and

$$r_k < s_k.$$

Then $\mathbb{R}((X)) = (\mathbb{R}((X)), +, \cdot, <)$ is a Cauchy complete totally ordered field; see Morgan 1968, Theorems 1.7, 1.8, 2.14. However, $\mathbb{R}((X))$, as structured above, is not Archimedean; see Morgan 1968 for details.

Example 4.4. Let $\mathbb{Q}(X) := (\mathbb{Q}(X), +, \cdot)$ be the field of fractions of the polynomial ring $\mathbb{Q}[X]$ endowed by the canonical operations of sum and product between rational functions. We define a total order in $\mathbb{Q}(X)$ by setting, for every quotient of polynomials:

$$\frac{f(X)}{g(X)} > 0$$

if the function $X \mapsto f(X)/g(X)$ is positive for X sufficiently large in $\mathbb{Q}$. Hence, on the basis of the results contained in Devillanova and Molica Bisci 2021, Section 4, standard computations ensures that $(\mathbb{Q}(X), \leq)$ is a totally ordered field. However, the field $\mathbb{Q}(X)$ is not Archimedean. Indeed, given $X \in \mathbb{Q}(X)$ and $n \in \mathbb{N}$ the polynomial $f(X) := n - X$ is negative for X large enough in $\mathbb{Q}$. Thus, there is no $n \in \mathbb{N}$ such that $n > X$. We claim that $\mathbb{Q}(X)$ is not $\mathscr{D}$–complete. Indeed, let us consider the series

$$\sum_{n=0}^{\infty} \binom{1/2}{n} X^{-n}, \quad \text{where} \quad \binom{1/2}{n} := \frac{1}{n!} \prod_{h=0}^{n-1} \left(\frac{1}{2} - h \right).$$

Let for every $n \in \mathbb{N}$ $s_n(X) := \sum_{k=0}^{n} \binom{1/2}{k} X^{-k}$ be the partial sum of the above series. Now, observe that

$$\lim_{n\to\infty} \binom{1/2}{n} X^{-n} = 0 \quad \text{in } \mathbb{Q}(X),$$

where $0 \in \mathbb{Q}(X)$ denotes the identically zero rational function. Thus, by Lemma **3.5**, the series $\sum_{n=0}^{\infty} \binom{1/2}{n} X^{-n}$ converges in the metric completion $\overline{\mathbb{Q}(X)}^{|\cdot|}$, *i.e.*, the sequence $(s_n)_n \subset \mathbb{Q}(X) \subset \overline{\mathbb{Q}(X)}^{|\cdot|}$ converges[19], *i.e.*, exists $\lambda(X) \in \overline{\mathbb{Q}(X)}^{|\cdot|}$ such that

$$\lim_{n\to\infty} s_n(X) = \lim_{n\to\infty} \sum_{k=0}^{n} \binom{1/2}{k} X^{-k} = \lambda(X). \tag{12}$$

Let us prove that $\lambda(X) \notin \mathbb{Q}(X)$. To this goal, let us observe that

$$\lim_{n\to\infty} (s_n(X))^2 = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \binom{1/2}{k} X^{-k} \right)^2 = 1 + \frac{1}{X} \quad \text{in } \overline{\mathbb{Q}(X)}^{|\cdot|}.$$

Thus

$$\lambda(X) = \sqrt{1 + \frac{1}{X}} \notin \mathbb{Q}(X)$$

as claimed. In conclusion, the Cauchy sequences $(s_n)_n \subset \mathbb{Q}(X)$ of the partial sums of $\sum_{n=0}^{\infty} \binom{1/2}{n} X^{-n}$ does not converge in $\mathbb{Q}(X)$. Consequently, the field $\mathbb{Q}(X)$ cannot be Cauchy complete. We also notice that the field $\mathbb{R}((X)) = (\mathbb{R}((X)), +, \cdot)$ endowed with the standards operations of sum and product $(+, \cdot)$, as well as of the total order introduced above, is $\mathscr{D}$–complete. In such a case the completion $\widehat{\mathbb{R}((X))}$ cannot be endowed by a natural structure of totally ordered field. Indeed, the assumption, by contradiction, that $\widehat{\mathbb{R}((X))}$ is a $\mathscr{D}$–complete totally ordered field implies, by Theorem **4.1**, that $\widehat{\mathbb{R}((X))}$ and, as a consequence $\mathbb{R}((X))$ too, is Archimedean, and this is not true.

We also mention the papers Corgnier and Valabrega 2017a, 2017b, 2013, 2015 for related topics.

5. Real field models

We prove the existence of a complete ordered field (unique up to isomorphism) exhibiting different classical **models**[20].

[19]Note that $(s_n)_n$ is a Cauchy sequence in $\mathbb{Q}(X)$ since $(s_n)_n$ is a Cauchy sequence in $\overline{\mathbb{Q}(X)}^{|\cdot|}$ by (12).

[20]A model for an axiomatic formal system is an interpretation of the primitive terms for which the axioms became true propositions.

5.1. Construction by Cauchy sequences. (G. Cantor (1873)) Let $\mathbb{Q}^{\mathbb{N}}$ be the ring of sequences in $\mathbb{Q}$ and $\mathbb{Q}^{\mathbb{N}}(\mathscr{C})$ be the subring of the Cauchy sequence in $\mathbb{Q}$. Furthermore, let $\mathscr{J}$ be the ideal of $\mathbb{Q}^{\mathbb{N}}(\mathscr{C})$ given by

$$\mathscr{J} := \left\{(x_n)_n : (x_n)_n \in \mathbb{Q}^{\mathbb{N}}(\mathscr{C}) \wedge \lim_{n\to\infty} x_n = 0 \text{ in } \mathbb{Q}\right\}.$$

Define on $\mathbb{Q}^{\mathbb{N}}(\mathscr{C})$ the following equivalence relation:

$$\forall (x_n)_n, (y_n)_n \in \mathbb{Q}^{\mathbb{N}}(\mathscr{C}),\ \ (x_n)_n \sim_{\mathscr{J}} (y_n)_n \Leftrightarrow (x_n - y_n)_n \in \mathscr{J}.$$

Consider the quotient ring

$$\mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J} := \left\{(x_n)_n + \mathscr{J} : (x_n)_n \in \mathbb{Q}^{\mathbb{N}}(\mathscr{C})\right\}$$

endowed by the natural operations:

$$((x_n)_n + \mathscr{J}) + ((y_n)_n + \mathscr{J}) := (x_n + y_n)_n + \mathscr{J},$$

and

$$((x_n)_n + \mathscr{J}) \cdot ((y_n)_n + \mathscr{J}) := (x_n \cdot y_n)_n + \mathscr{J},$$

for every $(x_n)_n + \mathscr{J}, (y_n)_n + \mathscr{J} \in \mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J}$. Since $\mathscr{J}$ is a maximal ideal in $\mathbb{Q}^{\mathbb{N}}(\mathscr{C})$ the quotient $\mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J}$ is a field. More precisely, $(\mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J}, +, \cdot)$ is a totally ordered field endowed by the order $\leq_{\mathscr{J}}$ defined by setting for every $(x_n)_n + \mathscr{J}, (y_n)_n + \mathscr{J} \in \mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J}$

$$(x_n)_n + \mathscr{J} \leq_{\mathscr{J}} (y_n)_n + \mathscr{J} \Leftrightarrow \text{ either } (y_n - x_n)_n \text{ is positive, or } (y_n - x_n)_n + \mathscr{J} = \mathscr{J}.$$

The field $\mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J}$ is Dedekind complete:

As observed before the quotient ring $(\mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J}, +, \cdot, \leq_{\mathscr{J}})$ is a totally ordered field, hence it contains an isomorphic copy of $\mathbb{Q}$. More precisely, it is easily seen that the function

$$\gamma : \mathbb{Q} \to \mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J} \quad \text{ given by } \ \gamma(x) := (x)_n + \mathscr{J},$$

is an increasing (injective) field homomorphism[21]. Let us prove, on account of Theorem **4.1**, that the field $\mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J}$ is Archimedean and Cauchy complete.

Archimedeaness - Let $(x_n)_n + \mathscr{J} \in \mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J}$. Since $(x_n)_n \in \mathbb{Q}^{\mathbb{N}}(\mathscr{C})$ the sequence $(x_n)_n$ is bounded in $\mathbb{Q}$, in particular there exists $k \in \mathbb{N}$ such that $k \geq x_n$ for every $n \in \mathbb{N}$. Thus $\gamma(k) \geq_{\mathscr{J}} (x_n)_n + \mathscr{J}$ in $\mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J}$, *i.e.*, $\mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J}$ is an Archimedean field bearing in mind Devillanova and Molica Bisci 2021, Proposition 5.5 - Part (ii).

Cauchy completeness - For every $(x_n)_n + \mathscr{J} \in \mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J}$, if $k \in \mathbb{N}$ one has

$$(x_n)_n - \gamma\left(\frac{1}{k+1}\right) <_{\mathscr{J}} (x_n)_n + \gamma\left(\frac{1}{k+1}\right).$$

Hence, since $\mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J}$ is Archimedean, by Devillanova and Molica Bisci 2021, Proposition 5.5, there exists $q \in \mathbb{Q}$, depending on $k \in \mathbb{N}$, such that

$$|((x_n)_n + \mathscr{J}) - \gamma(q)| <_{\mathscr{J}} \gamma\left(\frac{1}{k+1}\right). \tag{13}$$

[21]Here $(x)_n$ denotes the (constant) sequence whose general term is $x_n = x$ for every $n \in \mathbb{N}$.

Therefore, if $((x^{(k)})_n + \mathscr{I})_k$ is a Cauchy sequence in $\mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{I}$, by (13) for every $k \in \mathbb{N}$ there exists $q_k \in \mathbb{Q}$ such that

$$|((x_n^{(k)})_n + \mathscr{I}) - \gamma(q_k)| <_{\mathscr{I}} \gamma\left(\frac{1}{k+1}\right). \tag{14}$$

Now, we are able to prove that:

(1) $(q_k)_k \in \mathbb{Q}^{\mathbb{N}}(\mathscr{C})$, moreover
(2) $(q_k)_k + \mathscr{I} \in \mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{I}$ and $(q_k)_k + \mathscr{I} = \lim_{h\to\infty} \gamma(q_h)$ in $\mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{I}$,

where we recall that for any $n \in \mathbb{N}$ $\gamma(q_h) := (q_h)_k$ is the constant sequence of constant value q_h. More precisely:

(1) - Let $\varepsilon > 0$ in $\mathbb{Q}$ there exists $\nu_\varepsilon \in \mathbb{N}$ such that for every $h, k \in \mathbb{N}$ with $h, k \geq \nu_\varepsilon$ one has $|((x_n^{(k)})_n + \mathscr{I}) - ((x_n^{(h)})_n + \mathscr{I})| \leq \gamma(\varepsilon/3)$. Without loss of generality we can suppose $1/(\nu_\varepsilon + 1) < \varepsilon/3$. Thus, by (14), for every $h, k \geq \nu_\varepsilon$ it follows that

$$\begin{aligned}
|\gamma(q_k) - \gamma(q_h)| \leq_{\mathscr{I}} & \ |\gamma(q_k) - ((x_n^{(k)})_n + \mathscr{I})| \\
& + |((x_n^{(k)})_n + \mathscr{I}) - ((x_n^{(h)})_n + \mathscr{I})| \\
& + |\gamma(q_h) - ((x_n^{(h)})_n + \mathscr{I})| \\
& <_{\mathscr{I}} \gamma\left(\frac{1}{\nu_\varepsilon + 1}\right) + \gamma\left(\frac{\varepsilon}{3}\right) + \gamma\left(\frac{1}{\nu_\varepsilon + 1}\right) \\
& <_{\mathscr{I}} \gamma\left(\frac{\varepsilon}{3}\right) + \gamma\left(\frac{\varepsilon}{3}\right) + \gamma\left(\frac{\varepsilon}{3}\right) \\
& = \gamma(\varepsilon).
\end{aligned}$$

The above inequality immediately leads to $|q_k - q_h| < \varepsilon$ for every $h, k \geq \nu_\varepsilon$, *i.e.*, $(q_k)_k$ is a Cauchy sequence in $\mathbb{Q}$ as claimed.

(2) - By (1) it is clear that $(q_k)_k + \mathscr{I} \in \mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{I}$. Hence, let us show that

$$\lim_{h\to\infty} \gamma(q_h) = (q_k)_k + \mathscr{I} \quad \text{in} \quad \mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{I},$$

i.e., by recalling that $\gamma(q_h) := (q_h)_k$ is a constant sequence,

$$(q_k - q_h)_k \in \mathscr{I} \quad \textit{i.e.}, \quad (q_k)_k \sim_{\mathscr{I}} (q_h)_k \tag{15}$$

for every $h \in \mathbb{N}$ sufficiently large.
Indeed, since $(q_k)_k \in \mathbb{Q}^{\mathbb{N}}(\mathscr{C})$, let us fix $\varepsilon > 0$ in $\mathbb{Q}$ and let $\nu_\varepsilon \in \mathbb{N}$ such that $|q_k - q_h| \leq \varepsilon$ for every $h, k \geq \nu_\varepsilon$. Consequently, for any large enough h, $|((q_k)_k + \mathscr{I}) - \gamma(q_h)| \leq_{\mathscr{I}} \gamma(\varepsilon)$ for every $k \geq \nu_\varepsilon$ concluding the proof of (2).

Now, observe that (14) yields

$$\lim_{k\to\infty} (((x_k^{(h)})_k + \mathscr{I}) - \gamma(q_h)) = \mathscr{I} \quad \text{in} \quad \mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{I},$$

i.e.,

$$(x_k^{(h)} - q_h)_k \in \mathscr{I} \quad \textit{i.e.}, \quad (x_k^{(h)})_k \sim_{\mathscr{I}} (q_h)_k$$

for every $h \in \mathbb{N}$ sufficiently large.

In conclusion, the above relation and (15) give, by transitivity of $\sim_{\mathscr{J}}$, that $(x_k^{(h)})_k \sim_{\mathscr{J}} (q_k)_k$, *i.e.*,

$$\lim_{h\to\infty}((x_k^{(h)})_k + \mathscr{J}) = (q_k)_k + \mathscr{J} \in \mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J},$$

i.e., the Cauchy sequence $((x_n^{(k)})_n + \mathscr{J})_k$ converges in $\mathbb{Q}^{\mathbb{N}}(\mathscr{C})/\mathscr{J}$. The proof is now complete; see also De Marco 1986, Sections B.2.13 and B.2.14.

5.2. Construction by Dedekind cuts. (R. Dedekind (1872); E.A. Maier and D. Maier's (1973)) We present a brief sketch of the construction of the real numbers starting from $\mathbb{Q}$ by using Dedekind cuts. This is the same approach used in Principles of Mathematical Analysis, Rudin 1976; see Appendix, Chapter 1 for a complete and detailed proof. The elements of $\mathbb{R}$ are some subsets of $\mathbb{Q}$ called cuts. On the collection of these subsets, we define an order, an addition, and a multiplication. We show that $\mathbb{R}$ endowed with this relation and these two operations is an ordered field. Each rational number can be identified with a specific cut, in such a way that $\mathbb{Q}$ can be viewed as a subfield of $\mathbb{R}$.

Let $\mathbb{Q} = (\mathbb{Q}, +, \cdot, \leq)$ be the rational field. A set $Y \subset \mathbb{Q}$ is a **cut**[22] in $\mathbb{Q}$ if:

(S_1) $Y \neq \emptyset$ and $Y \neq \mathbb{Q}$;
(S_2) If $r \in Y$ and $q \in \mathbb{Q}$, with $q < r$, then $q \in Y$;
(S_3) If $y \in Y$, there exists $x \in Y$ such that $x > y$.

For instance, if $r \in \mathbb{Q}$, the set $S(r) := \{q : q \in \mathbb{Q} \wedge q < r\}$ is a cut in $\mathbb{Q}$. Every set of the form $S(r)$ is said to be a **rational cut** in $\mathbb{Q}$. Examples of rational cuts are $S(0)$ and $S(1)$ that will be denoted respectively by $0_{\mathscr{T}}$ and $1_{\mathscr{T}}$, *i.e.*,

$$0_{\mathscr{T}} := \{q : q \in \mathbb{Q} \wedge q < 0\} \quad \text{and} \quad 1_{\mathscr{T}} := \{q : q \in \mathbb{Q} \wedge q < 1\}.$$

Now, let $\mathscr{T}$ be the set of cuts in $\mathbb{Q}$ equipped with the total order:

$$\forall\, X, Y \in \mathscr{T},\ X \leq_{\mathscr{T}} Y \Leftrightarrow X \subseteq Y.$$

For every $X, Y \in \mathscr{T}$ with $X >_{\mathscr{T}} 0_{\mathscr{T}}$ and $Y >_{\mathscr{T}} 0_{\mathscr{T}}$, define

$$XY := \{q : q \in \mathbb{Q} \wedge q \leq rs \text{ for some } r \in X \wedge s \in Y\}.$$

The set $(\mathscr{T}, \leq_{\mathscr{T}})$ is a totally ordered field endowed by the following two operations: the sum $+_{\mathscr{T}} : \mathscr{T} \times \mathscr{T} \to \mathscr{T}$ given by:

$$\forall\, X, Y \in \mathscr{T},\ X +_{\mathscr{T}} Y := \{x + y : x \in X \wedge y \in Y\},$$

so that the additive inverse of any $X \in \mathscr{T}$ is the cut

$$-X := \{q : q \in \mathbb{Q} \wedge q < -r \text{ for some } r \in X\},$$

and the product $\cdot_{\mathscr{T}} : \mathscr{T} \times \mathscr{T} \to \mathscr{T}$ given by setting

$$X \cdot_{\mathscr{T}} 0_{\mathscr{T}} = 0_{\mathscr{T}} \cdot_{\mathscr{T}} X = 0_{\mathscr{T}} \quad \text{for every } X \in \mathscr{T}$$

[22] A comparison with the notion of cut introduced in Section **4** is necessary. The interested reader should check the details.

and

$$X \cdot_{\mathscr{T}} Y := \begin{cases} XY & \text{if } X >_{\mathscr{T}} 0_{\mathscr{T}} \text{ and } Y >_{\mathscr{T}} 0_{\mathscr{T}} \\ (-X)(-Y) & \text{if } X <_{\mathscr{T}} 0_{\mathscr{T}} \text{ and } Y <_{\mathscr{T}} 0_{\mathscr{T}} \\ -((-X)Y) & \text{if } X <_{\mathscr{T}} 0_{\mathscr{T}} \text{ and } Y >_{\mathscr{T}} 0_{\mathscr{T}} \\ -(X(-Y)) & \text{if } X >_{\mathscr{T}} 0_{\mathscr{T}} \text{ and } Y <_{\mathscr{T}} 0_{\mathscr{T}}, \end{cases} \quad \forall X, Y \in \mathscr{T} \setminus \{0_{\mathscr{T}}\}.$$

So, for every cut $X \in \mathscr{T} \setminus \{0_{\mathscr{T}}\}$, the multiplicative inverse $X^{-1} \in \mathscr{T} \setminus \{0_{\mathscr{T}}\}$ is given by:

$$X^{-1} := \{q : q \in \mathbb{Q} \wedge q < r^{-1} \text{ for some } r \in X\}.$$

$\mathscr{D}$–Completeness of the field $\mathscr{T}$ - To this goal we prove that the field $\mathscr{T}$ has the least upper bound property, see Theorem **3.1** - Part (b). Let $\mathscr{Q}$ be a nonempty subset of $\mathscr{T}$, and assume that there exists an upper bound Z for $\mathscr{Q}$ in $\mathscr{T}$ (*i.e.*, $X \subseteq Z$ for all $X \in \mathscr{Q}$). Define $S := \bigcup_{X \in \mathscr{Q}} X$. Thus $q \in S$ iff $q \in X$ for some $X \in \mathscr{Q}$. We shall prove that $S \in \mathscr{T}$ and that $S = \sup_{\mathscr{T}} \mathscr{Q}$.

$S \in \mathscr{T}$, *i.e.*, (S_1)–(S_3) are verified - Since $\mathscr{Q}$ is nonempty, there exists a cut $X \in \mathscr{Q}$. Since $X \subseteq S, X \neq \emptyset$, it follows that $S \neq \emptyset$. Moreover, since $X \subseteq Z$ for every $X \in \mathscr{Q}$ it follows that $S = \bigcup_{X \in \mathscr{Q}} X \subseteq Z \neq \mathbb{Q}$. Consequently $S \neq \mathbb{Q}$ and (S_1) is verified. To prove (S_2), pick $r \in S$. Then $r \in X$, for some $X \in \mathscr{Q}$. If $q < r$, then $q \in X$, hence $q \in S$ and (S_2) is proved. Finally, in order to prove (S_3), let $y \in S$. Hence, there exists $X \in \mathscr{Q}$ such that $y \in X$. Since X is a cut in $\mathbb{Q}$, there exists $x \in X$ such that $x > y$. Hence, also (S_3) is verified since $X \subseteq S$.

$S = \sup_{\mathscr{T}} \mathscr{Q}$ - Clearly $X \leq_{\mathscr{T}} S$ for every $X \in \mathscr{Q}$ (*i.e.*, S is an upper bound for $\mathscr{Q}$ in $\mathscr{T}$). Now, suppose $Y \leq_{\mathscr{T}} S$, $Y \neq S$. Then, there exists $q \in S$ such that $q \notin Y$. Since $q \in S$, $q \in X$ for some $X \in \mathscr{Q}$. Thus $Y \leq_{\mathscr{T}} X$, *i.e.*, Y is not an upper bound for $\mathscr{Q}$. In conclusion $S = \sup_{\mathscr{T}} \mathscr{Q}$ as claimed.

We end this subsection by observing that the rational cuts preserves sums, products, and order. In other words, the injective function $S(\cdot) : \mathbb{Q} \to \mathscr{T}$ which maps every rational number r into the rational cut $S(r) \in \mathscr{T}$ defined above is an increasing homomorphism of ordered fields:

$S(p+q) = S(p) + S(q)$ for every $p, q \in \mathbb{Q}$;
$S(pq) = S(p)S(q)$ for every $p, q \in \mathbb{Q}$;
If $p \leq q$, then $S(p) \leq_{\mathscr{T}} S(q)$.

See Rudin 1976, Chapter I, Appendix for additional comments and remarks. The construction of the real numbers from the rational numbers is an example of the Dedekind completion of a totally ordered set.[23]

5.3. Construction by Decimals. (S. Stevin (1585)) We follow the nice construction proposed in Pagani and Salsa 2015. Let D be the set of symbols (Stevin's representations) of the form

$$\alpha := \sigma_\alpha \alpha_0, \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_k \ldots$$

where:

1. $\sigma_\alpha \in \{+, -\}$;

[23]The Dedekind–MacNeille completion generalizes the concept of Dedekind completion from total orders to partial orders.

2. $\alpha_0 \in \mathbb{N}$;
3. $\alpha_i \in \{0, 1, 2, ..., 8, 9\}$ for every $i \in \mathbb{N}$.

If $\alpha_i = 0$ for every $i \in \mathbb{N}$ we simple write[24]

$$\theta := \sigma_\theta 0, 000...0... = 0, 000...0...$$

Two elements in D

$$\alpha := \sigma_\alpha \alpha_0, \alpha_1\alpha_2\alpha_3...\alpha_k... \quad \text{and} \quad \beta := \sigma_\beta \beta_0, \beta_1\beta_2\beta_3...\beta_k...$$

are equal if

1. $\sigma_\alpha = \sigma_\beta$;
2. $\alpha_0 = \beta_0$ (in $\mathbb{N}$);
3. $\alpha_i = \beta_i$ (in $\{0, 1, 2, ..., 8, 9\}$) for every $i \in \mathbb{N}^*$.

An element $\alpha \in D$ is said to be **periodic** if there exists $k_0 \in \mathbb{N}$ and $p \in \{0, 1, ..., 8, 9\}$ such that $\alpha_k = p$ for every $k \geq k_0$. Conventionally, to avoid contradictions, if $\alpha_k = 9$ for every $k \geq k_0$, we put

$$\alpha := \sigma_\alpha \alpha_0, \alpha_1\alpha_2\alpha_3...\alpha_{k_0-1}999... = \sigma_\alpha \alpha_0, \alpha_1\alpha_2\alpha_3...(\alpha_{k_0-1} + 1)$$

for every $\alpha \in D$. Moreover, if $\alpha = \sigma_\alpha \alpha_0, \alpha_1\alpha_2\alpha_3...\alpha_k... \in D$ we denote

$$-\alpha = -\sigma_\alpha \alpha_0, \alpha_1\alpha_2\alpha_3...\alpha_k...,$$

where

$$-\sigma_\alpha := \begin{cases} -1 & \text{if } \sigma_\alpha = +1 \\ +1 & \text{if } \sigma_\alpha = -1. \end{cases}$$

We say that $\alpha \in D$ is **positive** $\alpha >_D \theta$ (resp. **negative** $\alpha <_D \theta$) if $\sigma_\alpha = +$ (resp. $\sigma_\alpha = -$).

The **modulus** of $\alpha \in D$ is defined by

$$|\alpha| := \begin{cases} \alpha & \text{if } \alpha >_D \theta \\ \theta & \text{if } \alpha = \theta \\ -\alpha & \text{if } \alpha <_D \theta. \end{cases}$$

Bearing in mind the above conventions, on account of the results recalled in Devillanova and Molica Bisci 2021, Subsection 4.5, every periodic element $\alpha \in D$ can be identified with a rational number. Hence, the subset $\mathbb{Q}_D \subset D$ of periodic elements of D can be endowed by a natural structure of ordered field via the canonical identification with $(\mathbb{Q}, +, \cdot, \leq)$. The main idea is to extend the structure of ordered field defined on $\mathbb{Q}_D$ to the whole set D.

Lexicographic order in D: Let $\alpha, \beta \in D$

- Case 1: $\sigma_\alpha = \sigma_\beta = +$
  we set $\alpha >_D \beta$ iff $\alpha_0 > \beta_0$ or $\alpha_0 = \beta_0$ and there exists $m \in \mathbb{N}$ such that $\alpha_m > \beta_m$ and, if $m > 1$, $\alpha_k = \beta_k$ for $k \leq m - 1$.
- Case 2: $\sigma_\alpha = -$ and $\sigma_\beta = +$
  we set $\alpha <_D \beta$.

[24]To avoid ambiguity we identify $+0, 000...0...$ and $-0, 000...0...$ with $0, 000...0...$

- Case 3: $\sigma_\alpha = \sigma_\beta = -$
  we set $\alpha >_D \beta$ iff $|\beta| >_D |\alpha|$.

Denote by

$$D_+ := \{\alpha : \alpha \in D \ \wedge \alpha \geq_D \theta \text{ for every } n \in \mathbb{N}\}$$

the subsets of D of nonnegative Stevin's representations with respect to the order $\leq_D$.

Stabilized sequences of Stevin's representations: let $(\alpha^{(n)})_{n\geq 1} \subset D_+$ given by

$$\begin{aligned} \alpha^{(1)} &:= +\alpha_0^{(1)}, \alpha_1^{(1)}\alpha_2^{(1)}...\alpha_k^{(1)}... \\ \alpha^{(2)} &:= +\alpha_0^{(2)}, \alpha_1^{(2)}\alpha_2^{(2)}...\alpha_k^{(2)}... \\ \alpha^{(3)} &:= +\alpha_0^{(3)}, \alpha_1^{(3)}\alpha_2^{(3)}...\alpha_k^{(3)}... \\ &\vdots \\ \alpha^{(n)} &:= +\alpha_0^{(n)}, \alpha_1^{(n)}\alpha_2^{(n)}...\alpha_k^{(n)}... \\ &\vdots \end{aligned}$$

If for every $k \geq 0$ the sequence $(\alpha_k^{(n)})_{n\geq 1} \subset \mathbb{N}$ is definitively constant[25], the sequence $(\alpha^{(n)})_{n\geq 1} \subset D_+$ is said to be **stabilized**. Clearly, if $(\alpha^{(n)})_{n\geq 1} \subset D_+$ is a stabilized sequence, then there exists a unique $\gamma \in D_+$, given by

$$\gamma := +\gamma_0, \gamma_1\gamma_2...\gamma_k...,$$

in such a case we simply write $\alpha^{(n)} \rightsquigarrow \gamma$.

The following result will be crucial in the sequel.

Lemma 5.1. *Let $(\alpha^{(n)})_{n\geq 1} \subset D_+$ be a nondecreasing sequence. Assume that there exists $M \in D_+ \setminus \{\theta\}$ such that $|\alpha^{(n)}| \leq_D M$ for every $n \in \mathbb{N}^*$. Then $(\alpha^{(n)})_{n\geq 1}$ is stabilized*

$$\alpha^{(n)} \rightsquigarrow \gamma,$$

with $\alpha^{(n)} \leq_D \gamma \leq_D M$ for every $n \in \mathbb{N}^*$.

Proof. Let $(\alpha_0^{(n)})_{n\geq 1}$ be the sequence of integer coefficients of $(\alpha^{(n)})_n \subset D_+$. Since $|\alpha_0^{(n)}| \leq_D M$ for every $n \in \mathbb{N}$, and $(\alpha_0^{(n)})_{n\geq 1}$ is nonnegative and nondecreasing in $\mathbb{N}$, there exists $\gamma_0 \in \mathbb{N}$ such that $\alpha_0^{(n)} = \gamma_0$ for every $n \geq n_0$, with $\gamma_0 \leq_D M$. Arguing by induction: suppose that the first k terms of $(\alpha^{(n)})_{n\geq 1}$ are stabilized, *i.e.*, there exists $k \in \mathbb{N}$ such that

$$\alpha^{(n)} = +\gamma_0, \gamma_1\gamma_2...\gamma_k\alpha_{k+1}^{(n)}\alpha_{k+2}^{(n)}...$$

for every $n \geq n_k$ and

$$+\gamma_0, \gamma_1\gamma_2...\gamma_k \leq_D M.$$

[25]Fixed $k \geq 0$, the sequence $(\alpha_k^{(n)})_{n\geq 1}$ in $\mathbb{N}$ is said to be definitively constant if there exists $n_k \in \mathbb{N}$ such that $\alpha_k^{(n)} = \gamma_k$ for every $n \geq n_k$, where $\gamma_0 \in \mathbb{N}$ if $k = 0$ and $\gamma_k \in \{0, 1, ..., 8, 9\}$ whenever $k > 0$.

Since $(\alpha_{k+1}^{(n)})_{n\geq 1} \subset \{0,1,...,8,9\}$ is nondecreasing in $\mathbb{N}$, there exists $\gamma_{k+1} \in \{0,\ldots,9\}$ and $n_{k+1} \in \mathbb{N}$ such that

$$\alpha^{(n)} = +\gamma_0, \gamma_1\gamma_2...\gamma_k\gamma_{k+1}\alpha_{k+2}^{(n)}\alpha_{k+3}^{(n)}...$$

for every $n \geq n_{k+1}$ and

$$+\gamma_0, \gamma_1\gamma_2...\gamma_k\gamma_{k+1} \leq_D M.$$

Consequently, $\alpha^{(n)} \rightsquigarrow \gamma$ with $\gamma \leq_D M$.
Let us prove that $\alpha^{(n)} \leq_D \gamma$ for every $n \in \mathbb{N}^*$. Arguing by contradiction, suppose that $\alpha^{(\bar{n})} > \gamma$ for some $\bar{n} \in \mathbb{N}^*$. Thus

$$\alpha^{(\bar{n})} = +\gamma_0, \gamma_1\gamma_2...\gamma_k\alpha_{k+1}^{(\bar{n})}\alpha_{k+2}^{(\bar{n})}... >_D \gamma = +\gamma_0, \gamma_1\gamma_2...\gamma_k...$$

Consequently $\alpha_{k+1}^{(\bar{n})} > \gamma_{k+1}$. Now, since $(\alpha_{k+1}^{(m)})_{m\geq 1}$ is nondecreasing for every $m \geq \bar{n}$ we have a contradiction. This completes the proof. □

Let $\alpha, \beta \in D_+$, *i.e.*,

$$\alpha := +\alpha_0, \alpha_1\alpha_2\alpha_3...\alpha_k... \quad \text{and} \quad \beta := +\beta_0, \beta_1\beta_2\beta_3...\beta_k...$$

where: $\alpha_0, \beta_0 \in \mathbb{N}$ and $\alpha_i, \beta_i \in \{0,1,...,8,9\}$ for every $i \in \mathbb{N}$.

For every $n \in \mathbb{N}$ let us define the truncated Stevin's representations $\alpha^{[n]}, \beta^{[n]} \in \mathbb{Q}_D = (\mathbb{Q}_D, +, \cdot)$ given by:

$$\alpha^{[n]} := +\alpha_0, \alpha_1\alpha_2\alpha_3...\alpha_n000...0... \quad \text{and} \quad \beta^{[n]} := +\beta_0, \beta_1\beta_2\beta_3...\beta_n000...0...$$

Hence, the sequences

$$(\alpha^{[n]} + \beta^{[n]})_{n\geq 1} \quad \text{and} \quad ((\alpha^{[n]} \cdot \beta^{[n]})^{[n]})_{n\geq 1}$$

are nondecreasing and bounded from above in D, more precisely:

$$|\alpha^{[n]} + \beta^{[n]}| \leq_D \alpha_0 + \beta_0 + 2 \quad \text{and} \quad |(\alpha^{[n]} \cdot \beta^{[n]})^{[n]}| \leq_D (\alpha_0+1)(\beta_0+1),$$

for every $n \in \mathbb{N}^*$.

Thus, by Lemma **5.1**, it is possible to define: for every $\alpha, \beta \in D_+$

$\alpha +_D \beta := \sigma$ where $\sigma \in D_+$ and

$$\alpha^{[n]} + \beta^{[n]} \rightsquigarrow \sigma$$

$\alpha \cdot_D \beta = \zeta$ where $\zeta \in D_+$ and

$$(\alpha^{[n]} \cdot \beta^{[n]})^{[n]} \rightsquigarrow \zeta.$$

We notice that in this framework the element $\alpha - \beta \in D_+$ (with $\alpha >_D \beta$) is determined by

$$(\alpha^{[n]} - (\beta^{[n]} + 10^{-n})) \rightsquigarrow \eta.$$

The above operations of sum and product are extended to the whole D as follows: if $\alpha <_D \theta$ and $\beta <_D \theta$:

$$\alpha +_D \beta := -(|\alpha| +_D |\beta|)$$

$$\alpha \cdot_D \beta := |\alpha| \cdot_D |\beta|;$$

if $\alpha >_D \theta$ and $\beta <_D \theta$:

$$\alpha +_D \beta := \begin{cases} \alpha - |\beta| & \text{if } \alpha \geq_D |\beta| \\ -(|\beta| - \alpha) & \text{if } \alpha <_D |\beta| \end{cases}$$

$$\alpha \cdot_D \beta := -|\alpha| \cdot_D |\beta|;$$

if $\alpha <_D \theta$ and $\beta >_D \theta$:

$$\alpha +_D \beta := \begin{cases} \beta - |\alpha| & \text{if } \beta \geq_D |\alpha| \\ -(|\alpha| - \beta) & \text{if } \beta <_D |\alpha| \end{cases}$$

$$\alpha \cdot_D \beta := -|\alpha| \cdot_D |\beta|.$$

The structure $(D, +_D, \cdot_D, \leq_D)$ is a totally ordered field in which:

The neutral additive element $\theta \in D$ is the identically zero Stevin's representation:

$$\theta := 0,000...0...$$

For every $\alpha := \sigma_\alpha \alpha_0, \alpha_1\alpha_2...\alpha_k... \in D$, the inverse $-\alpha \in D$ is given by:

$$-\alpha := -\sigma_\alpha \alpha_0, \alpha_1\alpha_2...\alpha_k...$$

The neutral multiplicative element $1_D \in D$ is the representation:

$$1_D = 1,000...0...$$

The interested reader can easily check the details.

$\mathscr{D}$–Completeness of the field D - We show that if $X \subset D$ is a nonempty set such that $\mathscr{U}_X \neq \emptyset$, then there exists $\sup_D X$. Without loss of generality we can consider the case of an infinite set $X \subset D$. Indeed, the case in which X is finite is trivial. Since $\mathscr{U}_X \neq \emptyset$, there exists $y_0 \in D$ such that $y_0 \geq x$, for every $x \in X$. Note that if $y_0 \in X$ then $y_0 = \max_D X$ and the thesis follows. So, assume $y_0 \notin X$ and take $x_0 \in X$ (with $x_0 < y_0$). Set $c_0 := \frac{x_0+y_0}{2}$. If $c_0 \in X \cap \mathscr{U}_X$ the thesis holds by taking $\sup_D X = c_0$. So we have to face three cases:

- Case 1: $c_0 \in X \setminus \mathscr{U}_X$
- Case 2: $c_0 \in \mathscr{U}_X \setminus X$
- Case 3: $c_0 \notin X \cup \mathscr{U}_X$.

Then we shall take

$$x_1 := \begin{cases} c_0 \in X & \text{if Case 1 holds} \\ x_0 \in X & \text{if Case 2 holds} \\ \bar{x}_0 \in X & \text{if Case 3 holds} \end{cases} \quad \text{and} \quad y_1 := \begin{cases} y_0 \in \mathscr{U}_X & \text{if Case 1 holds} \\ c_0 \in \mathscr{U}_X & \text{if Case 2 holds} \\ y_0 \in \mathscr{U}_X & \text{if Case 3 holds} \end{cases}$$

where $\bar{x}_0$ is an element in X such that $x_0 < c_0 < \bar{x}_0$.

In any case we get an interval $[x_1, y_1] \subset [x_0, y_0]$ such that the left end point $x_1 \in X$ and the right end point $y_1 \in \mathscr{U}_X$. Now, we repeat the argument by taking $c_1 := \frac{x_1+y_1}{2}$. If $c_1 \in X \cap \mathscr{U}_X$ we have done, otherwise, by arguing as above, we are able to select an interval $[x_2, y_2] \subset [x_1, y_1]$ such that $x_2 \in X$ and $y_2 \in \mathscr{U}_X$. So, if the procedure does not stop in a finite number of steps, one inductively determines a nested sequence of intervals $([x_n, y_n])_n$ such that

$$x_n \in X, \quad y_n \in \mathscr{U}_X, \quad \text{and} \quad y_n - x_n \le \frac{y_0 - x_0}{2^n} \quad \text{for all } n \in \mathbb{N}. \tag{16}$$

Moreover, by construction, the sequence $(x_n)_n$ of the left end points is nondecreasing and bounded from above (by y_0). Hence, a careful analysis of Lemma **5.1** ensures that there exists $\bar{x} \in D$ for which $x_n \rightsquigarrow \bar{x}$. Since, by construction, for every $k \in \mathbb{N}$ and $n \in \mathbb{N}$, $x_n < y_k$ and $x_n \le \bar{x} \le y_k$ we deduce, in particular, that

$$x_n \le \bar{x} \le y_n, \quad \text{for every } n \in \mathbb{N}. \tag{17}$$

We shall get the thesis by showing that $\bar{x} = \sup_D X$. To this aim we shall use the characterizing properties of the supremum as given in Devillanova and Molica Bisci 2021, Proposition 4.2.

(j) $\bar{x} \ge x$ for every $x \in X$.

Indeed, arguing by contradiction, assume that there exists $x \in X$ and $\bar{x} < x$. Set $r := x - \bar{x} > 0$ and take $\bar{n} \in \mathbb{N}$ such that

$$\frac{y_0 - x_0}{2^{\bar{n}}} < r.$$

Now, (17) and (16) lead to

$$y_{\bar{n}} - \bar{x} \le y_{\bar{n}} - x_{\bar{n}} \le \frac{y_0 - x_0}{2^{\bar{n}}} < r = x - \bar{x},$$

i.e., $y_{\bar{n}} < x$ against the fact that each $y_n \in \mathscr{U}_X$ and $x \in X$.

(jj) $\forall \varepsilon > 0$, $\exists x \in X$: $x > \bar{x} - \varepsilon$.

Assume, arguing by contradiction, that there exists $\varepsilon > 0$ such that $x \le \bar{x} - \varepsilon$ for every $x \in X$. Then, by taking into account that $x_n \in X$ for all $n \in \mathbb{N}$, we deduce $x_n \le \bar{x} - \varepsilon$, *i.e.*, that $|\bar{x} - x_n| = \bar{x} - x_n \ge \varepsilon$ which is a contradiction to $x_n \rightsquigarrow \bar{x}$.

We refer the interested reader to Pagani and Salsa 2015, Chapter 2, Appendix A for a detailed proof of the results contained in this subsection.

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