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ELEMENTS OF SET THEORY AND RECURSIVE ARGUMENTS

Giuseppe Devillanova [a] and Giovanni Molica Bisci [b] *

Abstract. In this paper we provide a self-contained introduction to some of the basic topics of Mathematical Analysis, comprising natural and unrestricted set theoretic methods. The note reflects partially the contents of a lecture given by the second author during the International Workshop on *New Horizons in Teaching Science* in Messina on June 2018. More precisely, following a quite new didactic approach, we recall here some basic facts on the Generalized Induction Principle as well as the Recursion Theorem, which plays a crucial role in the foundation of Mathematical Logic. Some elements of von Neumann, Gödel and Bernays (**NGB**) set theory are given in the last section. The note provides the preliminary tools that are essential in order to study the classical notion of Dedekind completeness.

1. Introduction

The theory of ordered fields has developed in step with Commutative Algebra and has played an essential role as the foundation of the field of real numbers; see Section **3** for some preliminary facts. Analytical aspects of this theory have became a new and important focus of research inspired by the work of Richard Dedekind. Along this direction, a rigorous treatment of the construction of the real field is given by ourselves in a recent expository note.

This approach requires checking many technical details in order to have a unified treatment of rational and irrational numbers.

To this aim in Section **4**, first of all, we introduce the set of naturals $\mathbb{N}$ according to the von Neumann model and the Dedekind–Peano postulates and we precise the operations on them, see the Recursion Theorem **4.4**. Subsequently, we consider the ring of integers $\mathbb{Z}$ and the field of rationals $\mathbb{Q}$. The crucial role of $\mathbb{Q}$ on the theory of Archimedean fields and their characterization has been explained in Proposition **5.4**. In particular, by a careful analysis of the limit notion induced by the order topology, we are able to prove that a totally ordered field $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ is Archimedean if and only if the sequence $(1/n)_n \subset \mathbb{Q}$ satisfies

$$\lim_{n\to\infty} \frac{1}{n} = 0 \text{ in } \mathbb{K};$$

see Theorem **5.8**.

In Section **5** we recall some basic facts on Commutative Algebra mainly focused on totally ordered fields. In particular, the order topology on a totally ordered field $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ has been introduced as a metrizable topology induced by the total order $\leq$ in $\mathbb{K}$. In this framework, the construction of the rational field $\mathbb{Q}$ has been carried out rigorously via equivalence relations starting by the notion of set in the von Neumann, Gödel and Bernays theory; see Section **6** for related comments and remarks. The Axiom of Choice, as given in Axiom VI - Section **6**, has been assumed along our approach; see also Theorem **6.6** in which some equivalent versions of Axiom VI are also presented for the sake of clarity.

The paper is addressed for the reader who has not studied Algebra and Mathematical Analysis previously, but who has some experience in mathematical reasoning. There are no specific prerequisites aside from a willingness to function at a certain level of abstraction and rigour.

2. The didactic approach

We have tried as much as possible to make our presentation self–contained, and we believe that readers who are familiar with only basic facts on Commutative Algebra and Mathematical Analysis will understand most of this note without having to read other textbooks on the same subject. In writing the paper, we tried to structure the material so that the manuscript could be used in different courses, and at a variety of different levels. For instance, these notes could serve as an introduction to undergraduate Abstract Algebra, as well as it could provide a credible alternative to the initial part of a first course in Mathematical Analysis. As general references on the topics of this note we refer to the monographs Cohen and Ehrlich 1963; Bourbaki 1964; Burrill 1967; Bourbaki 1970 and the references therein. We also mention the historical paper Landau 1951 on the Foundation of Mathematical Analysis, as well as the monumental books Whitehead and Russell 1927a, 1927b, 1910 on the Principia of Mathematics.

Notation: As customary, a metalinguistic sentence “If A, then B” will sometimes be abbreviated as $A \Rightarrow B$. We also have $\Leftarrow$ for the converse implication. Finally, for the sentence “if and only if” we use the shorter “iff” or the symbol $\Leftrightarrow$.

3. Relations

The following concepts will be crucial for our purposes.

3.1. Basic notions: equivalences and orders. Let $X \neq \emptyset$ be a set. For a 1–place relation (or a **unary relation**) on X we simply mean a subset of X also called a **property** on X. Now, let us consider the cartesian product

$$X^2 := X \times X = \{(x, y) : x, y \in X\}.$$

A subset $R \subseteq X^2$ is said to be a **relation** (or a **binary relation**[1]) on X. The relation

$$\Delta(X) := \{(x,x) : x \in X\}$$

is called **equality** on X. From now on, let R be a relation on X. The set

$$R^{-1} := \{(y,x) : (x,y) \in R\}$$

denotes the **inverse** relation of R. The following definitions will be useful in the sequel:

R is reflexive if and only if (briefly iff) $\Delta(X) \subseteq R$;
R is antireflexive iff $\Delta(X) \cap R = \emptyset$;
R is symmetric iff $R^{-1} = R$;
R is antisymmetric iff $R \cap R^{-1} \subseteq \Delta(X)$;
R is transitive iff $(x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R$.

R is an **equivalence** on X iff:

R is reflexive, symmetric and transitive.

R is a **preorder** on X iff:

R is reflexive and transitive.

R is an **order** on X iff:

R is an antisymmetric preorder.

R is a **total order** on X iff:

R is an order and $R \cup R^{-1} = X^2$ (Trichotomy).

R is a **strict order** on X iff:

R is antireflexive and transitive.

If R is a given order on X, then $S := R \setminus \Delta(X)$ is a strict order on X. Conversely, if S is a strict order on X then $R := S \cup \Delta(X)$ is an order on X. Moreover, if R is an order on X then R^{-1} is also an order on X. If R is an order on a set X, for any $Y \subseteq X$, one has that $R \cap (Y \times Y)$ is an order on Y, namely the **induced order** on Y. Finally, an order R on X will be denoted by $\leq$ and a strict order by $<$. More precisely, as customary, we say that[2] $x \leq y$ (resp. $x < y$) iff $(x,y) \in R$ (resp. $(x,y) \in R \setminus \Delta(X)$).

An **ordered set** (or **poset**) $X := (X, \leq)$ is a set X endowed by an order[3] $\leq$. Every finite poset[4] can be represented by a Hasse diagram[5]; see Curzio *et al.* 2014. A set X can be endowed by different orders.

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[1]In general, an n–place relation (or a relation with n arguments, or a n–ary relation) on X is a subset of $X^n := \underbrace{X \times \cdots \times X}_{n \text{ terms}}$. A 0–place relation on X is said to be a **sentence** on X.

[2]Let $\leq$ be an order on X. Without ambiguity, for every $x, y \in X$, $x \geq y$ (resp. $x > y$) simply means $y \leq x$ (resp. $y < x$).

[3]If $\leq$ is a total order on X, the Trichotomy property assume the form: for every $x, y \in X$, the following facts are mutually exclusive: $x < y$, $x = y$, $x > y$.

[4]A poset $X := (X, \leq)$ is said to be finite if X is a finite set.

[5]A Hasse diagram is a diagram used to represent a finite partially ordered set, in the form of a drawing of its "transitive reduction".

Let X be an ordered set. A subset $Y \subseteq X$ is said to be:

bounded above: if there is an element $b \in X$ such that

$$y \leq b \quad \text{for every } y \in Y.$$

The element b is called an **upper bound** (or a **majorant**) for Y.

bounded below: if there is an element $a \in X$ such that

$$a \leq y \quad \text{for every } y \in Y.$$

The element a is called a **lower bound** (or a **minorant**) for Y.

bounded: if Y has both an upper bound and a lower bound.

An element $e' := \inf_X Y \in X$ is called a **greatest lower bound** (or an **infimum**) of $Y \subseteq X$ if the following facts hold:

e' is a lower bound for Y;
$a \leq e'$ for all lower bounds a of Y.

An element $e'' := \sup_X Y \in X$ is called a **least upper bound** (or a **supremum**) of $Y \subseteq X$ if the following facts hold:

e'' is an upper bound for Y;
$e'' \leq b$ for every upper bound b of Y.

A subset $Y \subseteq X$ has at most one greatest lower bound $\inf_X Y$ and at most one least upper bound $\sup_X Y$. A **lattice** is an ordered set in which every two elements set has a unique supremum (also called **join**) and a unique infimum (also called **meet**)[6]. Lattices can also be characterized as algebraic structures satisfying certain axiomatic identities; see Curzio *et al.* 2014 for an explicit abstract construction.

If $\inf_X Y$ exists and $\inf_X Y \in Y$ we say that Y has **minimum** in X and we write $\min_X Y$ instead of $\inf_X Y$. If $\sup_X Y$ exists and $\sup_X Y \in Y$ we say that Y has **maximum** in X and we write $\max_X Y$ instead of $\sup_X Y$. Clearly

$$m := \min_X Y \Leftrightarrow Y \ni m \leq y \text{ for every } y \in Y;$$

$$M := \max_X Y \Leftrightarrow y \leq M \in Y \text{ for every } y \in Y.$$

An element $y \in Y$ is said to be:

minimal iff $x \leq y$ in Y implies that $x = y$;
maximal iff $x \geq y$ in Y implies that $x = y$.

The minimum (maximum) of a set $Y \subseteq X$ is a minimal (maximal) element in Y. The converse in general is not true. If X is a totally ordered set the notion of minimum (maximum) of $Y \subseteq X$ is equivalent to the notion of minimal (maximal) element in Y.

[6] An example of lattice is given by the natural numbers, partially ordered by divisibility, for which the unique supremum is the least common multiple and the unique infimum is the greatest common divisor. We notice that a notion of "completeness" can be given on posets: a **complete lattice** is a poset in which every subset have both a supremum and an infimum.

3.2. Relations and functions. The notion of relation on a set X is a special case of the general concept of relation between classes. A relation R between the classes X and Y is a subclass $R \subseteq X \times Y$. The classes

$$Dom(R) := \{x : \exists y(y \in Y \wedge (x,y) \in R)\},$$

and

$$Cod(R) := \{y : \exists x(x \in X \wedge (x,y) \in R)\},$$

are called respectively, **domain** and **codomain** (or **image**) of R. We are interested on relations between sets. Hence, let X and Y be sets and [7] $R \subseteq X \times Y$. If $x \in X$ and $y \in Y$, the symbol $x\mathscr{R}y$ is commonly used instead of $(x,y) \in R$. By using the notions recalled in Subsection **6.1**, given a predicate $P(x,y)$ the set

$$E := \{(x,y) : (x,y) \in X \times Y \ \wedge \ P(x,y)\},$$

is a relation between the sets X and Y and E is said to be the **graph** of P. Conversely, given a relation $R \subseteq X \times Y$, we have the predicate $(x,y) \in R$; see, among others, Prodi 1970, Chapter 0. Now, let us consider $T \subseteq X \times Y$. The sets

$$T^{-1} := \{(y,x) : (x,y) \in T\},$$

and

$$T^c := \{(x,y) : (x,y) \notin T\},$$

denote respectively the **inverse** and **complementary** relation of T. If $R \subseteq X \times Y$ and $S \subseteq X \times Y$, the following facts hold:

if $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$;
if $R \subseteq S$ then $S^c \subseteq R^c$;
$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ and $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$;
$(R \cap S)^c = R^c \cup S^c$ and $(R \cup S)^c = R^c \cap S^c$.

Furthermore, let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, where X, Y, Z are sets. We may define the **composition** relation $S \circ R \subseteq X \times Z$ of R and S as follows

$$(x,z) \in S \circ R \Leftrightarrow \exists y(y \in Y : (x,y) \in R \ \wedge \ (y,z) \in S).$$

We notice that the converse composition $R \circ S$ needs $Z \subseteq X$ to be defined, moreover even in such a case, generally, $S \circ R \neq R \circ S$. On the other hand, the following equality holds

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1}.$$

A **function** $f : X \to Y$ between the sets[8] X and Y is a relation given by a subset $Gr(f) \subseteq X \times Y$ such that, following the standard notation:

$$\forall\, x \in X\, \exists!\, y \in Y : (x,y) \in Gr(f),$$

i.e., $Dom(Gr(f)) = X$ and $(x,y) \in Gr(f) \wedge (x,z) \in Gr(f) \Rightarrow y = z$. We call the set $Gr(f)$ the *graph* of f, while $Dom(f) := Dom(Gr(f))$ and $f(X) := Cod(Gr(f))$

[7] If $R = \emptyset$ we have the **empty** relation to $X \times Y$.

[8] The notion of function remains valid also for classes. Moreover if $X = \emptyset$ and Y is a set, then the relation $R = \emptyset \subseteq X \times Y$ is a function.

are called, respectively, the *domain* and the *image* of f. Abusing of notation, most of the times, we write $y = f(x)$ instead of $(x, y) \in Gr(f)$.[9]

The following are standard definitions:

If $Z \subseteq X$, the **restriction** of f to Z is the function, denoted by $f|_Z$, defined by $Gr(f|_Z) := Gr(f) \cap (Z \times Y)$;
$f : X \to Y$ is **surjective** iff $f(X) = Y$;
$f : X \to Y$ is **injective** iff

$$\forall\, x, y \in X \text{ and } x \neq y \Rightarrow f(x) \neq f(y);$$

$f : X \to Y$ is **bijective** iff f is injective and surjective. In such a case, the inverse relation f^{-1} is a function $f^{-1} : Y \to X$, namely the **inverse** function of f;
If $f : X \to Y$ and $g : Y \to Z$ are functions, the **composite function** $g \circ f : X \to Z$ is defined by

$$(g \circ f)(x) = g(f(x)), \quad \forall x \in X;$$

Note that

$$Gr(g \circ f) = Gr(g) \circ Gr(f).$$

The set $\Delta(X)$ is the graph of the **identity function** $id_X : X \to X$ defined by $id_X(x) = x$ for every $x \in X$.

Finally, let us consider an equivalence relation $\mathscr{R}$ on a set X. The set[10]

$$X/\mathscr{R} := \{[x] : x \in X\},$$

is said to be the **quotient set** of X modulo $\mathscr{R}$, where

$$[x] := \{y : y \in X \,\wedge\, y\mathscr{R}x\} \subseteq X.$$

is the **equivalence class** of $x \in X$ (under $\mathscr{R}$).

Now, fixing $x, y \in X$, the following statements are equivalent: $x\mathscr{R}y$; $[x] = [y]$; $[x] \cap [y] \neq \emptyset$. More precisely, the quotient set of X forms a **partition**[11] of X. Conversely, every partition $(X_\alpha)_{\alpha \in J}$ of X comes from an equivalence relation on X usually denoted by $\sim$ defined by setting $x \sim y$ in X iff there exists $\alpha \in J$ such that $x, y \in X_\alpha$. Whenever the set X possesses some algebraic structure and the equivalence relation $\mathscr{R}$ is compatible with this structure, the quotient set often inherits a similar structure from X. The abstract constructions, as well as some of their special cases, presented briefly here will be crucial in the sequel.

We refer to the books Curzio *et al.* 2014 and Lombardo Radice 1965 as classical references on the subjects treated in this section.

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[9]From now on we assume the reader to be familiar with the basic notions and standard notations for functions defined on sets; see, among others, Pagani and Salsa 2015, Chapter I and De Marco 1986, Chapter 0.

[10]We notice that the canonical projection map $\pi : X \to X/\mathscr{R}$, defined by $\pi(x) := [x]$ for every $x \in X$, is surjective. Then, thanks to the Axiom of Replacement, the quotient $X/\mathscr{R}$ is a set.

[11]A partition of a set X is a set of nonempty subsets of X such that every element $x \in X$ is in exactly one of these subsets.

4. Some algebraic structures

A **commutative** (or **Abelian**) **ring** $A = (A, +_A, \cdot_A)$ with unity 1_A is a nonempty set A endowed by two operations in A, namely sum $+_A : A \times A \to A$ and product $\cdot_A : A \times A \to A$, such that:

- Axioms of the sum -

$(x +_A y) +_A z = x +_A (y +_A z)$ for every $x, y, z \in A$ (Associativity of sum);
$x +_A y = y +_A x$ for every $x, y \in A$ (Commutativity of sum);
There exist $0_A \in A$ such that $x +_A 0_A = x$ (Identity of sum);
For every $x \in A$ there exists $-x \in A$ such that $x +_A (-x) = 0_A$ (Additive inverses).

- Axioms of the product -

$(x \cdot_A y) \cdot_A z = x \cdot_A (y \cdot_A z)$ for every $x, y, z \in A$ (Associativity of product);
$x \cdot_A y = y \cdot_A x$ for every $x, y \in A$ (Commutativity of product);
There exist $1_A \in A \setminus \{0_A\}$ such that $x \cdot_A 1_A = x$ (Identity of product).

- Distributivity Axioms -

$x \cdot_A (y +_A z) = (x \cdot_A y) +_A (x \cdot_A z)$ for every $x, y, z \in A$;
$(x +_A y) \cdot_A z = (x \cdot_A z) +_A (y \cdot_A z)$ for every $x, y, z \in A$.

We notice that the distributivity Axioms are not redundant only for not commutative rings (when $\cdot_A$ is not commutative). The ring axioms yield:

The additive identity 0_A of A is unique.
Given $x \in A$, the additive inverse $-x \in A$ is unique.
The multiplicative identity 1_A of A is unique.

The pair $A = (A, +_A)$ is said to be an additive **commutative group**. The following definitions are also used:

A **semigroup** is an algebraic structure with a unique associative binary operation.

A **monoid** is an algebraic structure with a unique associative binary operation and an identity element.

A **semiring** $(A, +_A, \cdot_A)$ is an algebraic structure similar to a ring, but without the requirement that each element must have an additive inverse and such that $0_A \cdot_A x = x \cdot_A 0_A = 0_A$ for every $x \in A$.

A **field** $\mathbb{K} = (\mathbb{K}, +_\mathbb{K}, \cdot_\mathbb{K})$ is a set $\mathbb{K} \neq \emptyset$ endowed by two operations in $\mathbb{K}$, namely sum $+_\mathbb{K} : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ and product $\cdot_\mathbb{K} : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$, such that $\mathbb{K} = (\mathbb{K}, +_\mathbb{K}, \cdot_\mathbb{K})$ is a commutative ring with unity $1_\mathbb{K}$ and such that

for every $x \in \mathbb{K} \setminus \{0_\mathbb{K}\}$ there exists $x^{-1} \in \mathbb{K}$ such that $x \cdot_\mathbb{K} x^{-1} = 1_\mathbb{K}$ (Multiplicative inverses),

i.e., $(\mathbb{K} \setminus \{0_\mathbb{K}\}, \cdot_\mathbb{K})$ is a multiplicative commutative group. In other words, a field $\mathbb{K}$ is a commutative ring with unity $1_\mathbb{K}$ in which every element $x \in \mathbb{K} \setminus \{0_\mathbb{K}\}$ admits a multiplicative inverse $x^{-1} \in \mathbb{K}$.

Direct consequence of the field axioms are the next facts:

The additive identity $0_\mathbb{K}$ of $\mathbb{K}$ is unique.
Given $x \in \mathbb{K}$, the additive inverse $-x \in \mathbb{K}$ is unique.
The multiplicative identity $1_\mathbb{K}$ of $\mathbb{K}$ is unique.

Given $x \in \mathbb{K} \setminus \{0_{\mathbb{K}}\}$, the inverse $x^{-1} \in \mathbb{K}$ is unique[12].
$0_{\mathbb{K}} \cdot_{\mathbb{K}} x = 0_{\mathbb{K}}$ for every $x \in \mathbb{K}$.
If $x +_{\mathbb{K}} z = y +_{\mathbb{K}} z$, then $x = y$.
If $x \cdot_{\mathbb{K}} z = y \cdot_{\mathbb{K}} z$ and $z \neq 0_{\mathbb{K}}$, then $x = y$.

Moreover, it is easily seen that every field $\mathbb{K}$ is an **integral domain**[13], *i.e.*, if $x \cdot_{\mathbb{K}} y = 0_{\mathbb{K}}$ and $y \neq 0_{\mathbb{K}}$, then $x = 0_{\mathbb{K}}$. Indeed $x = x \cdot_{\mathbb{K}} 1_{\mathbb{K}} = x \cdot_{\mathbb{K}} (y \cdot_{\mathbb{K}} y^{-1}) = (x \cdot_{\mathbb{K}} y) \cdot_{\mathbb{K}} y^{-1} = 0_{\mathbb{K}} \cdot_{\mathbb{K}} y^{-1} = 0_{\mathbb{K}}$.

An **ideal** of a commutative ring with unit $(A, +_A, \cdot_A)$ is a set $I \subseteq A$ such that:

If $x, y \in I$ then $x +_A y \in I$,
For every $a \in A$ and $x \in I$, one has $a \cdot_A x \in I$.

Note that if I is an ideal of A, then $(I, +_I)$ is a subgroup[14] of the group $(A, +_A)$, where $+_I$ is the restriction of $+_A$ to $I \times I$.

The quotient ring A/I: Let I be an ideal in a ring $(A, +_A, \cdot_A)$. Then, consider the equivalence relation $\sim_I$ given by

$$\forall x, y \in A,\ x \sim_I y \Leftrightarrow x +_A (-y) \in I.$$

Any equivalence class $[x]$ is the set $\{x +_A y \ : \ y \in I\}$ and it is denoted by $x + I$. These equivalence classes are often called **cosets** of the ideal I. If $x + I = y + I$, that is $x +_A (-y) \in I$ we say that *x is equivalent to y* modulo I. The set of equivalence classes is denoted by A/I and is a ring when equipped with the following two operations:

$$(x + I) + (y + I) := (x +_A y) + I,$$

and

$$(x + I) \cdot (y + I) := x \cdot_A y + I,$$

for every $x + I, y + I \in A/I$.

A total order $\leq_S$ on an algebraic structure (either a ring or a field) $S = (S, +_S, \cdot_S)$ with neutral (additive) element 0_S, is said to be **compatible**, with respect to the sum $+_S$ and the product $\cdot_S$, if $x \leq_S y$ implies

$$x +_S z \leq_S y +_S z \quad \text{for every } z \in S$$

and

$$x \cdot_S t \leq_S y \cdot_S t \quad \text{for every } t \geq_S 0_S \text{ in } S.$$

For instance, a **totally ordered field** is a field $\mathbb{K} = (\mathbb{K}, +_{\mathbb{K}}, \cdot_{\mathbb{K}})$ endowed by a total order $\leq_{\mathbb{K}}$ compatible with $+_{\mathbb{K}}$ and $\cdot_{\mathbb{K}}$. When no ambiguity arises, we shall simplify the notation by dropping the label $\mathbb{K}$.

Morphisms. Let $A := (A, +_1, \cdot_1)$ and $R := (R, +_2, \cdot_2)$ be two commutative rings with unit. A function $\varphi : A \to R$ is said to be a ring homomorphism if:

$\varphi(x +_1 y) = \varphi(x) +_2 \varphi(y)$ for every $x, y \in A$;

[12]The inverse x^{-1} is also denoted by $1/x$.

[13]An **integral domain** is a nonzero commutative ring in which the product of any two nonzero elements is nonzero.

[14]A substructure of an algebraic structure A (*i.e.*, a subgroup, subring, subfield, and so on) is a subset of A that is closed under any operations and contains all distinguished elements; see Curzio *et al.* 2014 for precise definitions and characterizations.

$\varphi(x \cdot_1 y) = \varphi(x) \cdot_2 \varphi(y)$ for every $x, y \in A$;
$\varphi(1_A) = 1_R$.

A ring homomorphism $\varphi : A \to R$ is said to be:

monomorphism if φ is injective;
epimorphism if φ is surjective;
isomorphism if φ is bijective.

We refer to Lombardo Radice 1965; De Marco 1986; Waerden 1991 for some basic properties of ring homomorphisms. The **Cathegory of rings**, denoted by **Ring** is the category whose objects are rings (with unit) and whose **morphisms** are ring homomorphisms; see Hartshorne 1977; Curzio *et al.* 2014 for a careful analysis of these concepts.

Characterization in totally ordered fields of Inf and Sup. It is easily seen that the following characterization holds in a totally ordered field $\mathbb{K}$.

Proposition 4.1. *Let $\mathbb{K} = (\mathbb{K}, +, \cdot)$ be a totally ordered field and let $Y \subseteq \mathbb{K}$ be a nonempty set. Furthermore, let $e' \in \mathbb{K}$. Then $e' := \inf_{\mathbb{K}} Y$ iff*

(*i*) $e' \leq y$ *for every* $y \in Y$;
(*ii*) $\forall \varepsilon > 0, \exists y \in Y : y < e' + \varepsilon$.

The counterpart of Proposition **4.1** for the supremum reads as follows.

Proposition 4.2. *Let $\mathbb{K} = (\mathbb{K}, +, \cdot)$ be a totally ordered field and let $Y \subseteq \mathbb{K}$ be a nonempty set. Furthermore, let $e'' \in \mathbb{K}$. Then $e'' := \sup_{\mathbb{K}} Y$ iff*

(*j*) $y \leq e''$ *for every* $y \in Y$;
(*jj*) $\forall \varepsilon > 0, \exists y \in Y : e'' - \varepsilon < y$.

We will only focus on the characterization of the supremum proving Proposition **4.2**. Similar arguments ensure the validity of Proposition **4.1**.

Proof of Proposition 4.2. Assume that $e'' = \sup_{\mathbb{K}} Y$. Then $x \leq e''$ for every $x \in Y$. Moreover, let $\varepsilon > 0$. Since $e'' - \varepsilon < e''$, one has that $e'' - \varepsilon$ is not an upper bound of Y. Thus, there exists $y \in Y$ such that $e'' - \varepsilon < y$. Conversely, assume that conditions (j) and (jj) hold. By (j) one clearly has that e'' is an upper bound of Y. We still have to prove that if $x \in \mathbb{K}$ is an upper bound of Y then $e'' \leq x$. Arguing by contradiction, assume that there exists an upper bound $x \in \mathbb{K}$ of Y and $x < e''$. Thus, let $\varepsilon := e'' - x > 0$. By (jj) there exists $y \in Y$ such that $e'' - \varepsilon < y$, *i.e.*, $x < y$ that is an absurd. The proof is complete. □

4.1. Total orders on a field $\mathbb{K}$. Given a totally ordered field $(\mathbb{K}, +, \cdot, \leq)$ let use define **the cone of nonnegative elements** as follows:

$$C_+ := \{x \in \mathbb{K} : x \geq 0\},$$

where 0 denotes the neutral element of $\mathbb{K}$ with respect to the additive law. The set C_+ satisfies the properties:

(h_1) $C_+ \cap (-C_+) = \{0\}$;
(h_2) $C_+ + C_+ \subseteq C_+$, where

$$C_+ + C_+ := \{x + y : x, y \in C_+\};$$

(h_3) $C_+ \cdot C_+ \subseteq C_+$, where

$$C_+ \cdot C_+ := \{x \cdot y : x, y \in C_+\};$$

(h_4) $C_+ \cup (-C_+) = \mathbb{K}$.

Conversely, given a set $C_+ \subset (\mathbb{K}, +, \cdot)$ such that (h_1)–(h_4) hold, the relation $\leq$ defined by[15]:

$$\forall x, y \in \mathbb{K},\ x \leq y \Leftrightarrow y - x \in C_+,$$

is a total order on $\mathbb{K}$ compatible with the operations of sum and product. A similar characterization can be given for a **strict total order**[16] defined on a field $\mathbb{K}$. Indeed, if $<$ is a strict total order on $\mathbb{K}$, define **the cone of strictly positive elements** as follows:

$$C_+^* := \{x \in \mathbb{K} : x > 0\}.$$

The set C_+^* satisfies the properties:

(h_1^*) $C_+^* \cap (-C_+^*) = \emptyset$;
(h_2) $C_+^* + C_+^* \subseteq C_+^*$, where

$$C_+^* + C_+^* := \{x + y : x, y \in C_+^*\};$$

(h_3^*) $C_+^* \cdot C_+^* \subseteq C_+^*$, where

$$C_+^* \cdot C_+^* := \{x \cdot y : x, y \in C_+^*\};$$

(h_4^*) $C_+^* \cup (-C_+^*) = \mathbb{K} \setminus \{0\}$.

Conversely, given a set $C_+^* \subset (\mathbb{K}, +, \cdot)$ such that (h_1^*)–(h_4^*) hold, the relation $<$ defined by:

$$\forall x, y \in \mathbb{K},\ x < y \Leftrightarrow y - x \in C_+^*,$$

is a strict total order on $\mathbb{K}$ compatible with the operations of sum and product.

A field $\mathbb{K} = (\mathbb{K}, +, \cdot)$ that admits a total order compatible with the operations $(+, \cdot)$ is said to be **orderable**. The set of natural numbers $\mathbb{N}$ will be defined in the next subsection. A field $\mathbb{K}$ is **formally real** if, for every $n \in \mathbb{N} \setminus \{0\}$, the equation

$$\sum_{k=1}^{n} x_k^2 = 0 \ \text{ in } \mathbb{K}$$

has only the trivial solution, that is $x_k = 0$ for each $k \in \{1, ..., n\}$ in $\mathbb{K}$. The following result holds.

Theorem 4.3. *A field $\mathbb{K}$ is orderable iff $\mathbb{K}$ is formally real.*

See Waerden 1991, Chapter 11 for a detailed proof.

[15] Let $G = (G, +)$ be an Abelian group. Given $x, y \in G$, in order to simplify the notation, from now on we simply denote by $x - y$ the element $x + (-y)$.

[16] A strict order in a set X is total if, for any $x, y \in X$, either $(x < y \ aut\, y < x)$ or $(x, y) \in \Delta(X)$.

4.2. The set of natural numbers $\mathbb{N}$. Our initial aim is to find some suitable sets to serve as *natural numbers*. Following a construction due to von Neumann in Neumann 1923 it is usual to take:

$$0 := \emptyset, \quad 1 := \{0\}, \quad 2 := 1 \cup \{1\} = \{0,1\}, \quad 3 := 2 \cup \{2\} = \{0,1,2\}, ...$$

and so on[17], the *successor* $s(n)$ of n is defined as

$$s(n) := n \cup \{n\}.$$

For instance $s(3) := 3 \cup \{3\} = \{0,1,2\} \cup \{3\} = \{0,1,2,3\}$, namely 4. The discussion above provides a precise formal definition for any given natural number. It does not provide a precise formal notion of the set of (all) natural numbers. For this purpose we require the *Axiom of infinity*. This asserts the existence of at least a set X (said to be an *inductive* set) such that:

(i_1) $\emptyset \in X$;
(i_2) $x \in X \Rightarrow x \cup \{x\} \in X$.

Finally, in order to give a precise definition of the set of natural numbers, let us consider the class $\mathscr{F}$ of all the sets X such that (i_1) and (i_2) holds. Since $\mathscr{F} \neq \emptyset$, thanks to the axiom of infinity, we set $\omega := \bigcap_{X \in \mathscr{F}} X$. If X satisfies (i_1) and (i_2), clearly $\omega \subseteq X$. Thus ω is a set as consequence of the Comprehension Axiom (see Subsection **6.1**). The set ω is a model for the **Dedekind–Peano Postulates**: There exists a set $\mathbb{N}$ and a function $\sigma : \mathbb{N} \to \mathbb{N}$ such that:

(p_1) $0 \in \mathbb{N}$;
(p_2) $\sigma : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ is injective;
(p_3) **The induction principle** holds: If $X \subseteq \mathbb{N}$ such that
$0 \in X$;
$\forall n : n \in X \Rightarrow \sigma(n) \in X$,
then $X = \mathbb{N}$.

According to the classical Ax–Kochen's categoricity theorem, it is possible to show that the model ω for the Peano postulates is unique up to identifications; see Curzio *et al.* 2014 for a detailed proof. Hence, we can define $\omega = \bigcap_{X \in \mathscr{F}} X$ to be the set of **natural numbers**. Instead of ω, as customary, for our purpose, we use the symbol $\mathbb{N}$ in the axiomatic theory. A direct and meaningful consequence of the induction principle is the following **(Weak) Recursion Theorem**[18] valid in the Dedekind–Peano Arithmetic; see Cohen and Ehrlich 1963, Theorem 1.2.

Theorem 4.4. *Let X be a nonempty set. If $g : X \to X$ is a function and $a \in X$, there exists a unique function (sequence in X) $f : \mathbb{N} \to X$ such that:*

(r_1) $f(0) = a \in X$;
(r_2) $f(\sigma(n)) = g(f(n))$ *for every* $n \in \mathbb{N}$.

Proof. We divide the proof in two steps: Existence and Uniqueness.

Existence - Let $\mathscr{C}$ be the set of all subsets S of $\mathbb{N} \times X$ such that

[17] The iterative procedure that generates the natural numbers in the von Neumann model is mathematically formalized by the recursive theorem recalled in Theorem **4.4**.

[18] A generalized Recursion Theorem can be found in Cohen and Ehrlich 1963, page 19. The recursion theorem is still valid if, X is a class rather than a set.

(s_1) $(0,a) \in S$, and
(s_2) $(\sigma(n), g(b)) \in S$ whenever $(n,b) \in S$.

Since $\mathbb{N} \times X$ satisfies the above assumptions, the set $\mathscr{C}$ is nonempty. Let us consider

$Gr(f) := \bigcap_{S \in \mathscr{C}} S \subseteq \mathbb{N} \times X$.

Clearly $Gr(f)$ satisfies (s_1) and (s_2). Hence $Gr(f) \in \mathscr{C}$ and $Gr(f) \subset S$ for every $S \in \mathscr{C}$.

We show that $Gr(f)$ is the graph of a function f for which (r_1) and (r_2) hold. To this goal, define the set

$$M := \{n : n \in \mathbb{N} \wedge \exists! b \in X,\ (n,b) \in Gr(f)\}.$$

We show that M is an inductive set:

- Claim 1 - $0 \in M$. Since $Gr(f) \in \mathscr{C}$ one has $(0,a) \in Gr(f)$ by (s_1). Arguing by contradiction, assume that $(0,b) \in Gr(f)$ for some $b \neq a$. Let $Gr(f)_b := Gr(f) \setminus \{(0,b)\}$. Since $(0,b) \neq (0,a) \in Gr(f)$, $(0,a) \in Gr(f)_b$. If $(n,c) \in Gr(f)_b$ then $(\sigma(n), g(c)) \in Gr(f)$ and since $\sigma(n) \neq 0$ one has $(\sigma(n), g(c)) \in Gr(f)_b$. Consequently $Gr(f)_b \in \mathscr{C}$ and $Gr(f) \subset Gr(f)_b = Gr(f) \setminus \{(0,b)\}$ that is an absurd.

- Claim 2 - If $n \in M$ then $\sigma(n) \in M$. Indeed, let $n \in M$. Then there is exactly one $b \in X$ such that $(n,b) \in Gr(f)$. Since $Gr(f) \in \mathscr{C}$, $(\sigma(n), g(b)) \in Gr(f)$. Arguing by contradiction, suppose that $(\sigma(n), c) \in Gr(f)$ for some $c \neq g(b)$, and let $Gr(f)_c := Gr(f) \setminus \{(\sigma(n), c)\}$. Since $(\sigma(n), c) \neq (0,a) \in Gr(f)$, $(0,a) \in Gr(f)_c$. If $(m,d) \in Gr(f)_c$, then $(\sigma(m), g(d)) \neq (\sigma(n), c)$. Otherwise, $\sigma(m) = \sigma(n)$, $g(d) = c \neq g(b)$, so that (by (p_2)) $m = n$, $d \neq b$, and (n,b), $(n,d) \in Gr(f)$, contrary to the assumption that $n \in M$. Consequently, it follows that $(\sigma(m), g(d)) \in Gr(f)_c$. Hence $Gr(f)_c \in \mathscr{C}$, and $Gr(f) \subset Gr(f)_c = Gr(f) \setminus \{(\sigma(n), c)\}$ that is an absurd.

By Claims 1 and 2, the induction principle (p_3) yields $M = \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, there is exactly one $b \in X$ such that $(n,b) \in Gr(f)$. Hence, let $f : \mathbb{N} \to X$ be the function defined by $Gr(f)$. Since $(0,a) \in Gr(f)$ one has $f(0) = a$. Moreover, by definition, $f(\sigma(n)) = g(b)$ if and only if $f(n) = b$. Then, $f(\sigma(n)) = g(f(n))$ for every $n \in \mathbb{N}$. In conclusion (r_1) and (r_2) are verified.

Uniqueness - Let $h : \mathbb{N} \to X$ be such that

$$h(0) = a \quad \text{and} \quad h(\sigma(n)) = g(h(n)),$$

for every $n \in \mathbb{N}$.

Let $T \subseteq \mathbb{N}$ such that $h = f$ on T. Now, since $h(0) = f(0)$, it follows that $0 \in T$. Moreover, let $n \in T$ be such that $h(n) = f(n)$. Then

$$h(\sigma(n)) = g(h(n)) = g(f(n)) = f(\sigma(n)).$$

Hence, the induction principle (p_3) yields $T = \mathbb{N}$. In conclusion, one has $h = f$ on $\mathbb{N}$, *i.e.*, the uniqueness of the function f is achieved. □

By using the Recursion Theorem **4.4** we define in $\mathbb{N}$, the associative and commutative laws of sum $+_{\mathbb{N}}$ and multiplication $\cdot_{\mathbb{N}}$. More precisely, for a fixed $m \in \mathbb{N}$ and every $n \in \mathbb{N}$:

$0 +_{\mathbb{N}} m = m$;
$\sigma(n) +_{\mathbb{N}} m = \sigma(n +_{\mathbb{N}} m)$;

and

$0 \cdot_{\mathbb{N}} m = 0;$
$\sigma(n) \cdot_{\mathbb{N}} m = (n \cdot_{\mathbb{N}} m) +_{\mathbb{N}} m.$

See Cohen and Ehrlich 1963, Theorems 1.3 and 1.6. Finally, in $\mathbb{N}$ define the total order $\leq_{\mathbb{N}}$ as follows

$$\forall\, m, n \in \mathbb{N}, n \leq_{\mathbb{N}} m \iff \forall X \subseteq \mathbb{N} \text{ stable}^{19} \text{under } \sigma, \text{if } n \in X \text{ then } m \in X.$$

The order $\leq_{\mathbb{N}}$ is compatible with addition $+_{\mathbb{N}}$ and multiplication $\cdot_{\mathbb{N}}$, that is: for every $n \leq_{\mathbb{N}} m$ one has

$$n +_{\mathbb{N}} t \leq_{\mathbb{N}} m +_{\mathbb{N}} t,$$

and

$$n \cdot_{\mathbb{N}} t \leq_{\mathbb{N}} m \cdot_{\mathbb{N}} t,$$

for every $t \in \mathbb{N}$.

Clearly, in the von Neumann model the role of the injective map σ of the Dedekind–Peano postulates is played by the successor function $s : \mathbb{N} \to \mathbb{N}$ such that $s(n) = n \cup \{n\} = n +_{\mathbb{N}} 1$. In such a case $n \leq_{\mathbb{N}} m$ simply means $n \subseteq m$.

We notice that, as a consequence of Theorem **4.4**, the **factorial function** $! : \mathbb{N} \to \mathbb{N}$ is well defined recursively by setting:

$0! = 1;$
$(n + 1)! = (n + 1) \cdot_{\mathbb{N}} n!$ for every $n \in \mathbb{N}$.

The structure $(\mathbb{N}, +_{\mathbb{N}}, \cdot_{\mathbb{N}}, \leq_{\mathbb{N}})$ is an ordered **semiring**. Simplifying the notation, we write $\mathbb{N} = (\mathbb{N}, +, \cdot, \leq)$ when no confusion arises. Moreover, we shall also denote $\mathbb{N}^* := (\mathbb{N} \setminus \{0\}, +, \cdot, \leq)$.

A characterization of the induction principle reads as follows.

Theorem 4.5. *The following facts are equivalent:*

(a_1) *The induction principle given in* (p_3);
(a_2) *The* ***well–ordering principle****: every nonempty set* $X \subseteq \mathbb{N}$ *admits minimum.*

Proof. $(a_1) \Rightarrow (a_2)$ - Let $X \subseteq \mathbb{N}$ be a nonempty set and, arguing by contradiction, suppose that X has no minimum. Thus, consider the set $\mathbb{N} \setminus X$. Clearly $0 \in \mathbb{N} \setminus X$, otherwise $0 = \min_{\mathbb{N}} X$. Now, let $n > 0$. Since X has no minimum, it follows that $\{0, 1, ..., n\} \cap X = \emptyset$ and $n + 1 \in \mathbb{N} \setminus X$. Consequently, the induction principle (p_3) yields $\mathbb{N} \setminus X = \mathbb{N}$, *i.e.*, $X = \emptyset$, which is an absurd.
$(a_2) \Rightarrow (a_1)$ - Let $X \subseteq \mathbb{N}$ be a nonempty set such that:

(i_1) $0 \in X$;
(i_2) $\forall n : n \in X \Rightarrow n + 1 \in X$.

Arguing by contradiction suppose that $X \neq \mathbb{N}$. Since $Y := \mathbb{N} \setminus X \neq \emptyset$, by (a_2), there exists $m_Y := \min_{\mathbb{N}} Y \in Y$. Clearly, by (i_1), $m_Y \neq 1$ and $m_Y - 1 \notin Y$. Consequently $m_Y - 1 \in X$ and, by (i_2), it follows that $m_Y \in X$. Since $X \cap Y = \emptyset$, we have a contradiction. □

Finally, we emphasize that (p_3) can be formulated in different (weak) forms. A typical example is given below.

Theorem 4.6. *Let $P(n)$ be a predicate in $\mathbb{N}$. Assume that:*

(g_1) *There exists $n_0 \in \mathbb{N}$ such that $P(n_0)$ is true*;
(g_2) $\forall n \in \mathbb{N}$, $n \geq n_0 :$ *$P(n)$ true $\Rightarrow P(n+1)$ true.*

Then $P(n)$ is true for every $n \geq n_0$.

Proof. Consider the set

$$X := \{n : n \in \mathbb{N} \wedge P(n + n_0)\} \subseteq \mathbb{N}.$$

By (g_1) and (g_2) it follows that (p_3) holds. Hence, we have $X = \mathbb{N}$, so that $P(n+n_0)$ is true for every $n \in \mathbb{N}$. □

For related topics we refer the reader to, among others, Giusti 1988, Theorem 6.1.

4.3. The Generalized Induction and Reduction Principles. In this subsection we present a powerful generalization of the classical induction principle given in Subsection **4.2**.

Let X be a nonempty set and let $k \in \mathbb{N}^*$. For every $i = 1, \ldots, k$ let

$$f_i : \underbrace{X \times \cdots \times X}_{r_i \text{ terms}} \to X$$

be a function of *arity* $r_i \in \mathbb{N}^*$ and set

$$\mathscr{F} := \{f_i : i = 1, \ldots, k\}.$$

Moreover, let $A \subset X$ with $A \neq \emptyset$. We shall say that $B \subseteq X$ is $(A, \mathscr{F})$-*inductive* if and only if:

(1) $A \subseteq B$;
(2) B is *closed with respect to every* $f_i \in \mathscr{F}$; *i.e.*, if for all $i = 1, \ldots, k$, $f_i(b_1, \ldots, b_{r_i}) \in B$ for every $b_1, \ldots, b_{r_i} \in B$.

Let $\mathscr{I}$ be the set of $(A, \mathscr{F})$-inductive sets of X.

Define

$$I(A, \mathscr{F}) := \bigcap_{B \in \mathscr{I}} B. \tag{1}$$

Since $X \in \mathscr{I}$ by (1) it follows that $I(A, \mathscr{F}) \neq \emptyset$. The set[20] $I(A, \mathscr{F})$ is said to be *the set generated from A by the functions in $\mathscr{F}$*.

The **generalized induction principle** reads as follows.

Theorem 4.7. *Let A and $\mathscr{F}$ be as above and let $T \subseteq X$ such that:*
Base induction: $A \subseteq T$;
Inductive step: $\forall i = 1, \ldots, k$ *if* $\alpha_1, \ldots, \alpha_{r_i} \in T$ *it follows that* $f_i(\alpha_1, \ldots, \alpha_{r_i}) \in T$.
Then $I(A, \mathscr{F}) \subseteq T$.

Proof. Note that the base induction and the inductive step guarantee that $T \in \mathscr{I}$, *i.e.*, $T \subseteq X$ is $(A, \mathscr{F})$-inductive. Consequently, by (1), $I(A, \mathscr{F}) \subseteq T$ as claimed. □

[20]We emphasize that $I(A, \mathscr{F})$ is $(A, \mathscr{F})$-inductive, and in particular that for every $i = 1, ..., k$ and $x_1, ..., x_{r_i} \in I(A, \mathscr{F})$, $f_i(x_1, ..., x_{r_i}) \in I(A, \mathscr{F})$.

It is easily seen that Theorem **4.6** is a particular case of Theorem **4.7**.

Finally, by introducing some new notion as in Enderton 2001, Chapter 1, a generalized recursion theorem holds (see Theorem **4.9**).

Definition 4.8. We say that $I(A, \mathscr{F})$ is *freely generated* from A by $\mathscr{F}$ if and only if:

(1) $f_i|_{I(A,\mathscr{F})}$ is one-to-one for every $i \in \{1, ..., k\}$;
(2) for every $i, j \in \{1, ..., k\}$ with $i \neq j$

$$f_i|_{I(A,\mathscr{F})}(I(A, \mathscr{F})) \cap f_j|_{I(A,\mathscr{F})}(I(A, \mathscr{F})) = \emptyset,$$

and, for every $i \in \{1, ..., k\}$

$$f_i|_{I(A,\mathscr{F})}(I(A, \mathscr{F})) \cap A = \emptyset,$$

where, the symbol $f_i|_{I(A,\mathscr{F})}$ denotes the restriction of f_i to $I(A, \mathscr{F})$.

Let $V \neq \emptyset$ equipped with k operations

$$F_i : \underbrace{V \times \cdots \times V}_{r_i \text{ terms}} \to V \quad (i = 1, ..., k),$$

with arity r_i. The main result below says that if $I(A, \mathcal{F})$ is freely generated from A by $\mathcal{F}$, any function $v : A \to V$ always admits a unique homomorphism extension

$$\bar{v} : (I(A, \mathscr{F}), f_1, ..., f_k) \to (V, F_1, ..., F_k).$$

More precisely, the following **Generalized Recursion Theorem** holds.

Theorem 4.9. *Let $k \in \mathbb{N}^*$ and for all $i \in \{1, \ldots, k\}$ let f_i be a function of arity $r_i \in \mathbb{N}^*$ on X. Let $\emptyset \neq A \subset X$ and $\mathscr{F} = \{f_i : i = 1, \ldots, k\}$. If $I(A, \mathscr{F})$ is freely generated from A by $\mathscr{F}$, then any function $v : A \to V$, taking values on a nonempty set V equipped by k operations F_i, each one with arity r_i, admits a unique* extension

$$\bar{v} : I(A, \mathscr{F}) \to V$$

such that:

(*i*) *for x in A, $\bar{v}(x) = v(x)$;*
(*ii*) *for every $i = 1, ..., k$*

$$\bar{v}(f_i(x_1, ..., x_{r_i})) = F_i(\bar{v}(x_1), ..., \bar{v}(x_{r_i})),$$

whenever $x_1, ..., x_{r_i} \in I(A, \mathscr{F})$.

The graph of the function $\bar{v}$ given by the thesis of the above theorem will be constructed by taking the union of the graphs related to functions which we call *acceptable* functions with respect to v. More precisely, we shall say that a function f is *acceptable* with respect to v if and only if the domain $Dom(f)$ of f is a subset of $I(A, \mathscr{F})$, the image $Im(f) := f(Dom(f)) \subseteq V$, and the following conditions hold:

(i_1) if $x \in A \cap Dom(f)$ then $f(x) = v(x)$.
(i_2) for every $i, \ldots, k$,

$$\text{if } f_i(x_1, \ldots, x_{r_i}) \in Dom(f) \text{ then } (x_1, \ldots, x_{r_i}) \in (Dom(f))^{r_i}$$

and

$$f(f_i(x_1, \ldots, x_{r_i})) = F_i(f(x_1), \ldots, f(x_{r_i})).$$

Note that, due to Definition **4.8** - Part (2), not only condition (i_1) but also (i_2) is trivially satisfied by $f = v$, namely the function v is acceptable for itself.

PROOF OF THEOREM **4.9**. Fix on $V \neq \emptyset$ the operations F_i and let $v : A \to V$ be a fixed function. Let $\mathscr{C}_v$ be the collection of all acceptable functions with respect to v, and let

$$G := \bigcup_{f \in \mathscr{C}_v} Gr(f) \tag{2}$$

be the union of the graphs of each acceptable function $f \in \mathscr{C}_v$. Thus

$$\begin{aligned} (x, z) \in G \text{ iff } & (x, z) \in Gr(f) \text{ for some } f \in \mathscr{C}_v \\ \text{iff } & z = f(x) \text{ for some } f \in \mathscr{C}_v. \end{aligned} \tag{3}$$

We claim that G is the graph of a function $\bar{v} : I(A, \mathscr{F}) \to V$ for which conditions (i) and (ii) hold.

The argument is set–theoretic, and comprises four steps:

(1) G is the graph of a function $\bar{v} : C \to V$, where $A \subseteq C \subseteq I(A, \mathscr{F})$.

Set

$$\begin{aligned} S &:= \{x \in I(A, \mathscr{F}) : \text{ for \textbf{at most} one } z \in V, (x, z) \in G\} \\ &= \{x \in I(A, \mathscr{F}) : \text{ every } h \in \mathscr{C}_v \text{ defined at } x \text{ agree in } x\}, \end{aligned} \tag{4}$$

we shall verify that $S = I(A, \mathscr{F})$ by proving that S is $(A, \mathscr{F})$-inductive. Firstly let us fix x in A and assume that f and g are acceptable functions defined at x. Then, by (i_1), both $f(x)$ and $g(x)$ must equal $v(x)$, so we get $f(x) = g(x)$. This shows that $x \in S$; therefore, since x was an arbitrary member of A we have $A \subseteq S$. Secondly we must check that S is closed under the operations $f_i \in \mathscr{F}$ for every $i = 1, \dots, k$. So, suppose that some points $x_1, \dots, x_{r_i}$ are in S; we ask whether $f_i(x_1, ..., x_{r_i})$ is in S. So suppose that f and g are acceptable functions defined at $f_i(x_1, ..., x_{r_i})$; we prove that they agree there. Indeed, by condition (i_2) it follows that

$$\begin{aligned} f(f_i(x_1, ..., x_{r_i})) = F_i(f(x_1), ..., f(x_{r_i})), \text{ and} \\ g(f_i(x_1, ..., x_{r_i})) = F_i(g(x_1), ..., g(x_{r_i})). \end{aligned} \tag{5}$$

Since $x_1, ..., x_{r_i}$ are in S, we have $f(x_1) = g(x_1)$,...,$f(x_{r_i}) = g(x_{r_i})$. Consequently, (5) implies that $f(f_i(x_1, ..., x_{r_i})) = g(f_i(x_1, ..., x_{r_i}))$. This shows that $f_i(x_1, ..., x_{r_i}) \in S$. Hence S is closed under $f_i \in \mathscr{F}$ for every $i = 1, ..., k$. Thus S is $(A, \mathscr{F})$-inductive and since $S \subseteq I(A, \mathscr{F})$ we get $S = I(A, \mathscr{F})$.

By recalling that v is acceptable for itself we deduce by (3) that $(x, v(x)) \in G$ for all $x \in Dom(v)$, and in particular, by exploiting the definition of graph of a function, there exists a (single-valued) function $\bar{v} : C \to V$, where[21] $A \subseteq C \subseteq S = I(A, \mathscr{F})$, such that $G = Gr(\bar{v})$. In particular[22], we can set

$$\bar{v}(x) := f(x) \in V \text{ for all } x \in Dom(f) \text{ whatever } f \in \mathscr{C}_v, \tag{6}$$

[21]We emphasize that $A \subseteq C$ since, for every $x \in A$, the function whose graph is given by $\{(x, v(x))\}$ is acceptable with respect to v. Indeed, in such a case, condition (i_1) trivially holds and (i_2) is (vacuously) verified taking into account that $f_i|_{I(A,\mathscr{F})}(A)) \cap \{x\} = \emptyset$ for every $i = 1, ..., k$.

[22]We notice that $(x, \bar{v}(x)) \in Gr(\bar{v})$ iff there exists an acceptable function f with respect to v such that $\bar{v}(x) = f(x)$.

on account of (4).

(2) $\bar{v}$ is an acceptable function with respect of v, *i.e.*, $\bar{v} \in \mathscr{C}_v$.

This follows fairly easily from the definition of G and the fact that $\bar{v}$ is a function. More precisely, by construction, $\bar{v}$ is a function with $A \subseteq C := Dom(\bar{v}) \subseteq I(A, \mathscr{F})$ and $\bar{v}(C) \subseteq V$. It remains to check that $\bar{v}$ satisfies conditions (i_1) and (i_2).

Condition (i_1) trivially follows by (6) since $A \cap Dom(\bar{v}) = A = Dom(v)$ and v is a function which is acceptable with respect to itself. Now, we have to prove (i_2) that is: if, for fixed i, $f_i(x_1, ..., x_{r_i}) \in Dom(\bar{v})$ then $(x_1, ..., x_{r_i}) \in (Dom(\bar{v}))^{r_i}$ and

$$\bar{v}(f_i(x_1, ..., x_{r_i})) = F_i(\bar{v}(x_1), ..., \bar{v}(x_{r_i})).$$

Hence, let $f_i(x_1, ..., x_{r_i}) \in Dom(\bar{v})$. Again, by definition of $\bar{v}$ (see also (2) and (6)), there must be some acceptable f with respect to v such that

$$\bar{v}(f_i(x_1, ..., x_{r_i})) = f(f_i(x_1, ..., x_{r_i})),$$

therefore $f_i(x_1, ..., x_{r_i}) \in Dom(f)$ and, since f satisfies (i_2), we have that $(x_1, ..., x_{r_i}) \in (Dom(f))^{r_i}$ and

$$f(f_i(x_1, ..., x_{r_i})) = F_i(f(x_1), ..., f(x_{r_i})).$$

Now, since $f \in \mathscr{C}_v$ and $(x_1, \ldots, x_{r_i}) \in (Dom(f))^{r_i}$, by (6), one has

$$\bar{v}(x_1) = f(x_1), ..., \bar{v}(x_{r_i}) = f(x_{r_i}),$$

so that $(x_1, ..., x_{r_i}) \in (Dom(\bar{v}))^{r_i}$ and

$$\begin{aligned}\bar{v}(f_i(x_1, ..., x_{r_i})) &= f(f_i(x_1, ..., x_{r_i})) \\ &= F_i(f(x_1), ..., f(x_{r_i})) \\ &= F_i(\bar{v}(x_1), ..., \bar{v}(x_{r_i})).\end{aligned}$$

(3) $\bar{v}$ is defined throughout $I(A, \mathscr{F})$, *i.e.*, $C := Dom(\bar{v}) = I(A, \mathscr{F})$.

Since $A \subseteq Dom(\bar{v}) \subseteq S = I(A, \mathscr{F})$ it suffices to show[23] that $Dom(\bar{v})$ is $(A, \mathscr{F})$-inductive and in particular that it is closed under $f_i \in \mathscr{F}$ for every $i = 1, ..., k$, namely that for every $i = 1, ..., k$, if $x_1, ..., x_{r_i} \in Dom(\bar{v})$ then $f_i(x_1, ..., x_{r_i}) \in Dom(\bar{v})$. Fix $i \in \{1, ..., k\}$. Arguing by contradiction, let $x_1, ..., x_{r_i} \in Dom(\bar{v})$, be such that $f_i(x_1, ..., x_{r_i}) \notin Dom(\bar{v})$ and let

$$H_i := Gr(\bar{v}) \cup \{(f_i(x_1, ..., x_{r_i}), F_i(\bar{v}(x_1), ..., \bar{v}(x_{r_i}))\}. \tag{7}$$

It is easily seen that H_i is the graph of a function h_i such that $A \subseteq C' := Dom(h_i) \subseteq I(A, \mathscr{F})$ and $h(C') \subseteq V$.

Indeed, one has $Dom(H_i) = Dom(\bar{v}) \cup \{f_i(x_1, ..., x_{r_i})\} \subseteq I(A, \mathscr{F})$ with $f_i(x_1, ..., x_{r_i}) \notin Dom(\bar{v})$. Then, for any $x \in DomH_i$ we set

$$h_i(x) := \begin{cases} F_i(\bar{v}(x_1), \ldots, \bar{v}(x_{r_i})) \in V & \text{if } x = f_i(x_1, \ldots, x_{r_i}), \\ \bar{v}(x) \in V & \text{if } x \in Dom(\bar{v}). \end{cases} \tag{8}$$

[23]It is also here that the assumption of freeness is used.

We shall prove that h_i is acceptable for v by proving that it satisfies (i_1) and (i_2).

Part (i_1) - If $x \in A \cap Dom(h_i) = A$ then $x \neq f_i(x_1, ..., x_{r_i})$, by freeness, see Definition **4.8** - Part (2). Hence $x \in Dom(\bar{v})$ and we have by (8) $h_i(x) = \bar{v}(x) = v(x)$ since $\bar{v} \in \mathscr{C}_v$ (as proved at step 2).

Part (i_2) - We shall prove that if $f_j(y_1, ..., y_{r_j}) \in Dom(h_i)$ then $(y_1, ..., y_{r_j}) \in (Dom(h_i))^{r_j}$ and

$$h_i(f_j(y_1, ..., y_{r_j})) = F_j(h_i(y_1), ..., h_i(y_{r_j}))$$

for every $j = 1, ..., k$. Fixed j, assume that $f_j(y_1, ..., y_{r_j}) \in Dom(h_i)$ for some $y_1, ..., y_{r_j}$ in $C' = Dom(h_i)$. The following possibilities occur:

Case 1 - $j = i$ and $f_i(y_1, ..., y_{r_i}) \in Dom(\bar{v})$. Then, since $\bar{v} \in \mathscr{C}_v$ we have, see (i_2) that $(y_1, ..., y_{r_i}) \in (Dom(\bar{v}))^{r_i} \subseteq (Dom(h_i))^{r_i}$ and, see (8),

$$\begin{aligned} h_i(f_i(y_1, ..., y_{r_i})) &= \bar{v}(f_i(y_1, ..., y_{r_i})) \\ &= F_i(\bar{v}(y_1), ..., \bar{v}(y_{r_i})) \\ &= F_i(h_i(y_1), ..., h_i(y_{r_i})), \end{aligned}$$

since $h_i = \bar{v}$ on $Dom(\bar{v})$;

Case 2 - $j = i$ and $f_i(y_1, ..., y_{r_i}) = f_i(x_1, ..., x_{r_i})$. Then, by freeness (see Definition **4.8** - Part (1)), we have $x_1 = y_1$,...,$x_{r_i} = y_{r_i}$, so that $(y_1, \ldots, y_{r_i}) = (x_1, \ldots, x_{r_i}) \in (Dom(\bar{v}))^{r_i} \subseteq (Dom(h_i))^{r_i}$. Then, by (8) and since $\bar{v} \in \mathscr{C}_v$ we get

$$\begin{aligned} h_i(f_i(y_1, ..., y_{r_i})) &= h_i(f_i(x_1, ..., x_{r_i})) = \bar{v}(f_i(x_1, ..., x_{r_i})) \\ &= F_i(\bar{v}(x_1), ..., \bar{v}(x_{r_i})) = F_i(h_i(x_1), ..., h_i(x_{r_i})), \end{aligned}$$

owing to $h_i = \bar{v}$ on $Dom(\bar{v})$;

Case 3 - Finally suppose $j \neq i$ and that $f_j(y_1, ..., y_{r_j}) \in Dom(h_i)$ for some $y_1, ..., y_{r_j}$ in $I(A, \mathscr{F})$. Then, again by freeness, we have

$$f_j(y_1, ..., y_{r_j}) \neq f_i(x_1, ..., , x_{r_i}).$$

Thus, $f_j(y_1, ..., y_{r_j}) \in Dom(\bar{v})$ by (7). Consequently, since $\bar{v} \in \mathscr{C}_v$, it follows that $(y_1, ..., y_{r_j}) \in (Dom(\bar{v}))^{r_j} \subseteq (Dom(h_i))^{r_j}$ and

$$\begin{aligned} h_i(f_j(y_1, ..., y_{r_j})) &= \bar{v}(f_j(y_1, ..., y_{r_j})) \\ &= F_j(\bar{v}(y_1), ..., \bar{v}(y_{r_j})) \\ &= F_j(h_i(y_1), ..., h_i(y_{r_j})), \end{aligned}$$

bearing in mind again that $h_i = \bar{v}$ on $Dom(\bar{v})$.

Thus $h_i \in \mathscr{C}_v$ and, on account of (7), we get $H_i \subseteq Gr(\bar{v})$, so that $f_i(x_1, ..., x_{r_i}) \in Dom(\bar{v})$ against the assumption. In conclusion $Dom(\bar{v})$ is $(A, \mathscr{F})$-inductive and, as already pointed out, it is enough to guarantee that $Dom(\bar{v}) = I(A, \mathscr{F})$. $\bar{v}$ is unique.

(4) $\bar{v}$ is unique.

Given two functions $\bar{v}_1, \bar{v}_2$ as in the thesis, the set Z on which they agree, *i.e.*,

$$Z := \{x \in I(A, \mathscr{F}) : \bar{v}_1(x) = \bar{v}_2(x)\}$$

is $(A, \mathscr{F})$-inductive. Indeed $A \subseteq Z$ since if $x \in A$ then $\bar{v}_1(x) = \bar{v}_1(x) = v(x)$. Moreover, fixed i, if $x_1, ..., x_{r_i} \in Z$ then $x_1, ..., x_{r_i} \in I(A, \mathscr{F})$ and $\bar{v}_1(x_j) = \bar{v}_2(x_j)$ for every $j = 1, ..., r_i$. Since $I(A, \mathscr{F})$ is closed under the operations in $\mathscr{F}$ then $f_i(x_1, ..., x_{r_i}) \in I(A, \mathscr{F})$ and

$$\begin{aligned} \bar{v}_1(f_i(x_1, ..., x_{r_i})) &= F_i(\bar{v}_1(x_1), ..., \bar{v}_1(x_{r_i})) \\ &= F_i(\bar{v}_2(x_1), ..., \bar{v}_2(x_{r_i})) = \bar{v}_2(f_i(x_1, ..., x_{r_i})) \end{aligned}$$

for every $i = 1, ..., k$. Hence $f_i(x_1, ..., x_{r_i}) \in Z$ for every $i = 1, ..., k$. In conclusion Z is $(A, \mathscr{F})$-inductive and consequently $\bar{v}$ is unique as claimed.

The proof is now complete. □

From an abstract point of view, the conclusion of this theorem says that any function v of A into V can be extended to a unique (structure) homomorphism $\bar{v}$ from the algebraic structure $(I(A, \mathscr{F}), \mathscr{F})$ into $(V, F_1, ..., F_k)$.

The proof of Theorem **4.9** follows the strategy and the fruitful ideas developed in Enderton 2001, Chapter 1 where a special case of the above main result has been proved.

We end this subsection giving a different meaningful *bottom up* construction of the set $I(A, \mathscr{F})$. More precisely, the next result is a direct consequence of the induction principle given in Subsection **4.2**.

Proposition 4.10. *With the above notations, let* $C_0 := A$ *and*

$$C_{j+1} := C_j \cup \bigcup_{i=1}^{k} f_i\left(C_j^{r_i}\right), \qquad \textit{where} \qquad C_j^{r_i} := \underbrace{C_j \times \cdots \times C_j}_{r_i}, \tag{9}$$

for any $j \in \mathbb{N}$. *Set*

$$C^\star := \bigcup_{j \in \mathbb{N}} C_j.$$

Then

$$I(A, \mathscr{F}) = C^\star.$$

Proof. To this aim we shall prove that $C^\star$ is $(A, \mathscr{F})$-inductive. Clearly $A = C_0 \subseteq C^\star$. Moreover, for $i = 1, ..., k$, if $x_1, ..., x_{r_i} \in C^\star$ then $f_i(x_1, ..., x_{r_i}) \in C^\star$. Indeed, see (9), if $x_1, ..., x_{r_i} \in C^\star$, there exists $m \in \mathbb{N}$ such that $f_i(x_1, ..., x_{r_i}) \in C_m$ and, consequently, $f_i(x_1, ..., x_{r_i}) \in C^\star$ as claimed. So $I(A, \mathscr{F}) \subseteq C^*$. Conversely, let us prove that $C^\star \subseteq I(A, \mathscr{F})$. Indeed, let

$$C_0 \cup C_1 \cup C_2 \cup ... \cup C_\ell = \bigcup_{j=0}^{\ell} C_j = C_\ell$$

and let us proceed by induction on $\ell \in \mathbb{N}$. For $\ell = 0$ (base induction) $C_0 = A \subseteq I(A, \mathscr{F})$. Now, we show (inductive step) that, for fixed $\ell \in \mathbb{N}$, if $\bigcup_{j=0}^{\ell} C_j = C_\ell \subseteq I(A, \mathscr{F})$ then $\bigcup_{j=0}^{\ell+1} C_j = C_{\ell+1} \subseteq I(A, \mathscr{F})$. Indeed, for every $i = 1, ..., k$, if $x_1, ..., x_{r_i} \in C_\ell \subseteq I(A, \mathscr{F})$ then $f_i(x_1, ..., x_{r_i}) \in I(A, \mathscr{F})$ since $I(A, \mathscr{F})$ is closed under the operations in $\mathscr{F}$. Consequently $C_{\ell+1} \subseteq I(A, \mathscr{F})$ as claimed. Hence the induction principle stated by Theorem **4.6** yields that $\bigcup_{j \in \mathbb{N}} C_j = C^\star \subseteq I(A, \mathscr{F})$. $\square$

The above result is crucial in Propositional Logic in order to mathematically describe the w.f.f. (well formed formulas) of given length; see Enderton 2001, Chapter 1.

4.4. The ring of integers $\mathbb{Z}$. Let $(\mathbb{N}, +, \cdot, \leq)$ be the set (semiring) of natural numbers endowed by the operations of sum $+$, product $\cdot$ and total order $\leq$ introduced in Subsection **4.2**. In $\mathbb{N}^2 := \mathbb{N} \times \mathbb{N}$ consider the equivalence relation Σ given by

$$\forall (m,n), (s,t) \in \mathbb{N}^2,\ (m,n)\Sigma(s,t) \Leftrightarrow m + t = n + s.$$

The quotient set $\mathbb{Z} := \mathbb{N}^2/\Sigma$, endowed by the operations defined by setting for every $[(m,n)], [(s,t)] \in \mathbb{Z}$

$$[(m,n)] +_{\mathbb{Z}} [(s,t)] := [(m+s, n+t)],$$

and

$$[(m,n)] \cdot_{\mathbb{Z}} [(s,t)] := [(ms+nt, ns+mt)],$$

is a commutative ring, and, more precisely, an integral domain with unity $[(1,0)]$. It is easy to check that

$$\mathbb{Z} = \{[(0,n)] : n \in \mathbb{N}^*\} \sqcup \{[(n,0)] : n \in \mathbb{N}\},$$

where $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and $\sqcup$ is the disjoint union symbol[24]. Define

$$[(m,n)] \leq_{\mathbb{Z}} [(s,t)] \Leftrightarrow m + t \leq n + s$$

for every $[(m,n)], [(s,t)] \in \mathbb{Z}$. The relation $\leq_{\mathbb{Z}}$ gives a total order on $\mathbb{Z}$. The totally ordered integral domain $\mathbb{Z} = (\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, \leq_{\mathbb{Z}})$ is called the ring of integers. Simplifying the notation, we write $\mathbb{Z} = (\mathbb{Z}, +, \cdot, \leq)$ when no confusion arises.

4.5. The field of rationals $\mathbb{Q}$. Let $(\mathbb{Z}, +, \cdot, \leq)$ be the commutative ring of integers endowed by the operations of sum $+$, product $\cdot$ and total order $\leq$ defined in the previous section. In $\mathbb{Z} \times \mathbb{N}^*$ consider the equivalence relation $\sim$ given by

$$\forall (m,n), (s,t) \in \mathbb{Z} \times \mathbb{N}^*,\ (m,n) \sim (s,t) \Leftrightarrow m \cdot t = n \cdot s.$$

The quotient set $\mathbb{Q} := \mathbb{Z} \times \mathbb{N}^* / \sim$, endowed by the operations defined by setting for every $[(m,n)], [(s,t)] \in \mathbb{Q}$

$$[(m,n)] +_{\mathbb{Q}} [(s,t)] := [(t \cdot m + s \cdot n, n \cdot t)],$$

and

$$[(m,n)] \cdot_{\mathbb{Q}} [(s,t)] := [(m \cdot s, n \cdot t)].$$

[24] $A \sqcup B := A \Delta B := (A \cup B) \setminus (A \cap B)$.

Defining on $\mathbb{Q}$ the total order

$$[(m,n)] \leq_{\mathbb{Q}} [(s,t)] \Leftrightarrow m \cdot t \leq n \cdot s$$

for every $[(m,n)],[(s,t)] \in \mathbb{Q}$, the structure $(\mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}, \leq_{\mathbb{Q}})$ is an ordered field, namely the **field of rational numbers**. As customary, from now on, we simply write $\dfrac{m}{n}$ (also denoted by m/n) instead of $[(m,n)]$.

Given an integral domain $A = (A, +, \cdot)$, we will construct a field, called the **fraction field** of A, that contains A and that does not have any subfield containing A. More precisely, in $A \times (A \setminus \{0\})$ consider the equivalence relation $\sim$ given by

$$\forall (a,b),(c,d) \in A \times (A \setminus \{0\}),\ (a,b) \sim (c,d) \Leftrightarrow a \cdot d = b \cdot c.$$

The set of equivalence classes $Q(A)$ associated with this equivalence relation can be made into a ring if we define the sum and product as follows:

$$[(a,b)] +_{Q(A)} [(c,d)] := [(a \cdot d + c \cdot b, b \cdot d)],$$

and

$$[(a,b)] \cdot_{Q(A)} [(c,d)] := [(a \cdot c, b \cdot d)],$$

for every $[(a,b)],[(c,d)] \in Q(A)$.

To show that these operations are well defined one must check that the sum and product does not depend on the choice of representatives. Thereafter it must be verified that the axioms for a ring hold. Moreover, every element different from the zero element has a multiplicative inverse, hence the ring $Q(A)$ is a field. All this is left as exercise to the interested reader. The map $\jmath : A \hookrightarrow Q(A)$, $\jmath : A \to Q(A)$, $x \in A \mapsto [(x,1)] \in Q(A)$ is a monomorphism of rings, and so there is a copy of A inside $Q(A)$. If we identify A such a copy $j(A)$ we can say that A is a subring of $Q(A)$. This means that addition and multiplication of elements in A are the same, regardless whether we do the operations in A or in $Q(A)$.

In particular, the field $\mathbb{Q}$ can be viewed as the **fraction field** of $\mathbb{Z}$ commonly denoted by $Q(\mathbb{Z})$. The field $\mathbb{Q}$ plays a fundamental role among all ordered fields, since $\mathbb{Q}$ may be isomorphically embedded in every totally ordered field $\mathbb{K}$ as we specify below.

The **characteristic** of a ring $A = (A, +, \cdot)$, denoted by $Char(A)$, is the smallest $n \in \mathbb{N}^*$ such that

$$n \cdot 1 := \underbrace{1 + 1 + ... + 1}_{n \text{ terms}} = 0,$$

if such a number n exists, and 0 otherwise[25]. Any ordered field $\mathbb{K}$, has characteristic equal to zero[26]. Moreover the field of rational numbers $\mathbb{Q}$ is isomorphic to the

[25]In other words, the characteristic of a ring A can be defined as the natural number $n \in \mathbb{N}$ such that $n\mathbb{Z}$ is the *kernel* of the (unique) ring homomorphism $n \in \mathbb{Z} \mapsto n \cdot 1 \in A$.

[26]Note that in a total ordered field $\mathbb{K}$ one has $1 > 0$, consequently $m \cdot 1 > 1$ for every $m \in \mathbb{N}^*$. If $\mathbb{K}$ has characteristic zero, the ring monomorphism $\kappa : n \in \mathbb{Z} \mapsto n \cdot 1 \in \mathbb{K}$ can be extended to a monomorphism of fields $\kappa : \mathbb{Q} \to \mathbb{K}$ such that $\kappa(m/n) = (m \cdot 1) \cdot (n \cdot 1)^{-1}$, for every $m/n \in \mathbb{Q}$. Hence, the fundamental field $\kappa(\mathbb{Q})$ is just the image of $\mathbb{Q}$ trough the monomorphism of fields κ. Furthermore, since $\mathbb{K}$ contains a copy of $\mathbb{N}$, the field $\mathbb{K}$ is an **infinite set**.

fundamental field of $\mathbb{K}$ defined by

$$\kappa(\mathbb{Q}) := \{(m \cdot 1) \cdot (n \cdot 1)^{-1} : m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\}\},$$

via the canonical field (increasing) isomorphism $j : \mathbb{Q} \to \kappa(\mathbb{Q})$ given by

$$j(m/n) = (m \cdot 1) \cdot (n \cdot 1)^{-1} \in \kappa(\mathbb{Q}), \quad \forall m/n \in \mathbb{Q}.$$

Thus, every ordered field $\mathbb{K}$ contains a copy of the rational field $\mathbb{Q}$: the fundamental field $\kappa(\mathbb{Q})$ is isomorphic to the field of fractions of $\mathbb{Z}$.

The well–ordering principle gives the existence of the **integer part** of any element $x \in \mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$, where $\mathbb{K}$ is a totally ordered (Archimedean) field; see De Marco 1986, Section 0.2.6, for the real case.

Proposition 4.11. *Suppose that $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ is a totally ordered field. Assume that*

($\star$) *For every $x \in \mathbb{K}$ there exists $n \in \mathbb{N}$ such that $n > x$.*

Then, for every $x \in \mathbb{K}$ there exists a unique $\lfloor x \rfloor \in \mathbb{Z}$ such that

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$$

Proof. Let us fix $x \in \mathbb{K}$ and define with abuse of notations:

$$M(x) := \{q : \, q \in \mathbb{Z} \wedge q > x\},$$

as well as

$$m(x) := \{p : p \in \mathbb{Z} \wedge p < x\}.$$

Thanks to assumption ($\star$), the sets $M(x)$ and $m(x)$ are nonempty[27]. Fixing $p \in m(x)$, set

$$D(p) := \{q - p : q \in M(x)\}.$$

Clearly $D(p) \neq \emptyset$ and $D(p) \subseteq \mathbb{N}$. Thus, the well–ordering principle ensures that exists $\min_{\mathbb{N}} D(p) = \bar{q} - p$, where $\bar{q} = \min_{\mathbb{Z}} M(x)$. Now, define $\lfloor x \rfloor := \bar{q} - 1$. Of course $\lfloor x \rfloor + 1 = \bar{q} > x$.

Moreover, also the other inequality $\lfloor x \rfloor \leq x$ holds true. Indeed, the inequality $\lfloor x \rfloor > x$ should give $\lfloor x \rfloor \in M(x)$ in contradiction to the fact that $\lfloor x \rfloor = \bar{q} - 1 < \bar{q} = \min_{\mathbb{Z}} M(x)$. □

An immediate consequence of the above result is the following one.

Corollary 4.12. *Let $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ be a totally ordered field for which condition ($\star$) in Proposition **4.11** holds. For every $x \in \mathbb{K}$ and $\delta > 0$, there exists a unique $n \in \mathbb{Z}$ such that*

$$n\delta \leq x < (n + 1)\delta. \tag{10}$$

Proof. By Proposition **4.11** there exists a unique $\lfloor x/\delta \rfloor \in \mathbb{Z}$ such that

$$\lfloor x/\delta \rfloor \leq x/\delta < \lfloor x/\delta \rfloor + 1.$$

Since $\delta > 0$ the conclusion is achieved for $n := \lfloor x/\delta \rfloor$. □

Finally, the next property will be useful in the sequel.

---

[27]Notice that if ($\star$) holds, then for every $x \in \mathbb{K}$, there exists $n \in \mathbb{N}$ such that $n > |x| := \max\{x, -x\}$.

Lemma 4.13. *Let $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ be a totally ordered field for which condition $(\star)$ in Proposition 4.11 holds. Then, for every $\delta > 0$ and for every $x, y \in \mathbb{K}$, with $y - x > \delta$, there exists $m \in \mathbb{Z}$ such that $x < m\delta < y$.*

Proof. Let $\delta > 0$ and $x, y \in \mathbb{K}$ with $y - x > \delta$. By Corollary **4.12** there exists a unique $n \in \mathbb{Z}$ such that (10) holds. We claim that $m := (n+1) \in \mathbb{Z}$ satisfies $x < m\delta < y$. Clearly $x < (n+1)\delta$ by the right hand side of (10). We only need to prove that $m\delta < y$. Arguing by contradiction, suppose that $m\delta = (n+1)\delta \geq y$. Now, the left hand side of (10) yields $-n\delta \geq -x$. Hence, the inequalities above immediately yields $y - x \leq \delta$ against our assumption. □

The fields $\mathbb{K}$ for which condition $(\star)$ in Proposition **4.11** holds will be studied and characterized in Subsection **5.3**.

5. Metric structure of totally ordered fields

For any totally ordered field $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$, there is a mapping of $\mathbb{K}$ into $\mathbb{K}$ which may be used to define a *distance* in A. Indeed, let $|\cdot| : \mathbb{K} \to \mathbb{K}$ be the function defined by setting

$$|x| := \max\{x, -x\} \quad \text{for every } x \in \mathbb{K}.$$

The image $|x|$ is called the **absolute value** of $x \in \mathbb{K}$.

A subset of $\mathbb{K}$ is called an **interval** if it coincides, for some $a, b \in \mathbb{K}$, with one of the following sets:

$$[a, b] := \{x \,:\, x \in \mathbb{K} \wedge a \leq x \leq b\}$$
$$(a, b) := \{x \,:\, x \in \mathbb{K} \wedge a < x < b\}$$
$$[a, b) := \{x \,:\, x \in \mathbb{K} \wedge a \leq x < b\}$$
$$(a, b] := \{x \,:\, x \in \mathbb{K} \wedge a < x \leq b\},$$

preserving the standard nomenclature. The elements $a, b \in \mathbb{K}$ are called **endpoints** of the interval. The element $b - a$ is called the **length** or **measure** of the interval. For every $a \in \mathbb{K}$ we also set[28]:

$$(-\infty, a] := \{x \,:\, x \in \mathbb{K} \wedge x \leq a\}$$
$$(-\infty, a) := \{x \,:\, x \in \mathbb{K} \wedge x < a\}$$
$$[a, \infty) := \{x \,:\, x \in \mathbb{K} \wedge x \geq a\}$$
$$(a, \infty) := \{x \,:\, x \in \mathbb{K} \wedge x > a\},$$

again preserving the classical nomenclature.

5.1. The order topology.

Let $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ be a totally ordered field. The total order $\leq$ induces a **metric** (or **distance**) $d : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ on $\mathbb{K}$ given by:

$$d(x, y) := |x - y| \qquad \text{for every } x, y \in \mathbb{K}.$$

It is easily seen that the properties of the total order $\leq$ ensure that the next result holds.

Proposition 5.1. *Let $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ be a totally ordered field. Then, for every $x, y, z \in \mathbb{K}$ one has:*

[28] The notation remains unchanged in an ordered set $(X, \leq)$.

(*i*) $d(x,y) \geq 0$;
(*ii*) $d(x,y) = 0$ *iff* $x = y$;
(*iii*) $d(x,y) = d(y,x)$;
(*iv*) $d(x,y) \leq d(x,z) + d(z,y)$.

By Proposition **5.1** it follows that every totally ordered field $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ can be viewed as a metric space $(\mathbb{K}, d)$, where d is the metric induced by the total order $\leq$.

For every $x \in \mathbb{K}$, and $\varepsilon > 0$ an **open ball** centered in x with radius ε is the set

$$B_\varepsilon(x) := \{y \in \mathbb{K} \,:\, d(x,y) < \varepsilon\},$$

i.e., $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$.

Let $X \subseteq \mathbb{K}$ be a nonempty set. We have the following definitions:
An element $x \in \mathbb{K}$ is said to be:

an **interior point** of X if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq X$;
an **exterior point** of X if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq \mathbb{K} \setminus X$;
a **boundary point** of X if for every $\varepsilon > 0$ there are $y, z \in B_\varepsilon(x)$ such that $y \in X$, $z \in \mathbb{K} \setminus X$;
an **accumulation point** (or **limit point**) for X if for every $\varepsilon > 0$ there exists $y \in B_\varepsilon(x) \setminus \{x\}$ such that $y \in X$, *i.e.*, $(B_\varepsilon(x) \setminus \{x\}) \cap X \neq \emptyset$;
an **isolated point** of X if there exists $\varepsilon > 0$ such that $X \cap B_\varepsilon(x) = \{x\}$.

As customary:

$\dot{X}$ is the set of interior points of X
∂X is the set of boundary points of X
$\overline{X} := X \cup \partial X$ is the **clousure** (or **adherence**) of X
DX is the set of accumulation points for X in $\mathbb{K}$; DX is said the **derived set** of X.

The family of open balls in $\mathbb{K}$ provides a **basis** of the order topology induced by $\leq$. We emphasize that

$$\partial X = \partial(\mathbb{K} \setminus X) \quad \text{and} \quad \overline{X} = X \cup DX.$$

For every $x \in \mathbb{K}$, $\varepsilon > 0$ the **closed ball** centered at x of radius ε is defined by

$$\overline{B}_\varepsilon(x) = \{y \in \mathbb{K} \,:\, d(x,y) \leq \varepsilon\},$$

i.e $\overline{B}_\varepsilon(x) := [x - \varepsilon, x + \varepsilon]$.

A set $X \subseteq \mathbb{K}$ is said to be

open iff $X = \dot{X}$;
closed iff $X = \overline{X}$.

Basic properties of open sets are summarized in the next result.

Proposition 5.2. *Let $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ be a totally ordered field. The following facts hold:*

(*i*) *$\mathbb{K}$ and $\emptyset$ are open sets*;
(*ii*) *If $(X_\alpha)_{\alpha \in J}$ is a family of open sets, then $\bigcup_{\alpha \in J} X_\alpha$ is an open set*;
(*iii*) *If $X_1, \dots, X_k$ $(k \in \mathbb{N}^*)$ are open sets, then $\bigcap_{i=1}^{k} X_i$ is an open set.*

Proposition **5.2** ensures that the family of open sets of $\mathbb{K}$, defined by the total order $\leq$, is an (open) **topology** on $\mathbb{K}$, namely the (open) **order topology** on $\mathbb{K}$.

We notice that in Proposition **5.2** the property (ii) holds for any family of open sets. On the contrary (iii) is valid, in general, only for finite families of open sets. For instance, the numerable family $(B_{1/n}(0))_n$ has intersection

$$\bigcap_{n\in\mathbb{N}} B_{1/n}(0) = \{0\},$$

that is not open in the order topology on $\mathbb{K}$.

The De Morgan's laws and Proposition **5.2** immediately yield the next result.

Proposition 5.3. *Let $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ be a totally ordered field. The following facts hold:*

(j) $\mathbb{K}$ *and* $\emptyset$ *are closed sets*;

(jj) *If* $(X_\alpha)_{\alpha\in J}$ *is a family of closed sets, then* $\bigcap_{\alpha\in J} X_\alpha$ *is a closed set*;

(jjj) *If* $X_1, \ldots, X_k$ $(k \in \mathbb{N}^*)$ *are closed sets, then* $\bigcup_{i=1}^{k} X_i$ *is a closed set.*

The metric structure $(\mathbb{K}, d)$ on an ordered space can be investigated studying the general results valid on abstract metric spaces; see, for instance, the classical monograph Cecconi and Stampacchia 1983 for a detailed introduction on the subject. We notice that the order topology can be introduced on every totally ordered set $(X, \leq)$.

5.2. Archimedean fields. Let $(X, \leq)$ be a totally ordered set. A subset $S \subseteq X$ is **dense** in X if S is dense $(\overline{S} = X)$ in the order topology, that is $S \cap A \neq \emptyset$, for every set A open in the order topology on X. In other words S is dense in X iff $\forall x, y \in X$, with $x < y$, there exists $c \in S$ such that $x < c < y$.

Proposition 5.4. *Every ordered field* $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ *is dense.*

Proof. If $x < y$ in $\mathbb{K}$, then

$$2x = x + x < x + y < y + y = 2y.$$

Since $2 = 1 + 1 > 0$ in $\mathbb{K}$, it follows that $1/2 > 0$ in $\mathbb{K}$, and that $x < \dfrac{x+y}{2} < y$. □

A totally ordered field $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ is called **Archimedean** iff for every $x > 0$ and $y \in \mathbb{K}$, there is some $n \in \mathbb{N}$ such that $n \cdot x > y$.[29]

As in De Marco 1986, Proposition B.2.7 we prove the following.

Proposition 5.5. *Let $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ be a totally ordered field. The next facts are equivalent:*

(i) $\mathbb{K}$ *is Archimedean*;

(ii) *For every* $y \in \mathbb{K}$ *there exists*[30] $n \in \mathbb{N}$ *such that* $n > y$;

(iii) $\mathbb{Q}$ *is dense in* $\mathbb{K}$.

[29] Equivalently the field $\mathbb{K}$ is Archimedean iff: for any $x, y \in \mathbb{K}$ such that $0 < x < y$ in $\mathbb{K}$, there is some $n \in \mathbb{N}$ such that $n \cdot x > y$.

[30] It is clear that (ii) can be replaced by the following equivalent condition: For every $y \in \mathbb{K}$, with $y > 0$ there exists $n \in \mathbb{N}$ such that $n > y$.

Proof. $(i) \Rightarrow (ii)$ $\mathbb{K}$ is Archimedean, thus (for $x = 1$) the conclusion immediately follows.

$(ii) \Rightarrow (i)$ Let $x > 0$ and $y \in \mathbb{K}$. Considering y/x, by (ii) there exists $n \in \mathbb{N}$ such that $n > y/x$. The conclusion is obtained.

$(iii) \Rightarrow (ii)$ Let $x, y \in \mathbb{K}$. By (iii), there is $p/q \in \mathbb{Q}$ such that $y < p/q < x$. If $p/q < 0$ the conclusion follows. Otherwise, if $p/q > 0$, then $y < p+1$. For $n := p+1$ the thesis is achieved.

$(ii) \Rightarrow (iii)$ Let $x < y$ in $\mathbb{K}$. By (ii) it follows that not only there exists $n \in \mathbb{N}$ such that $n > 1/(y-x)$ but also that $y - x > \delta$, where $\delta := 1/n$. Hence[31], by Lemma **4.13**, there exists $m \in \mathbb{Z}$ such that $x < m\delta < y$, that is $x < m/n < y$. In conclusion $\mathbb{Q}$ is dense in $\mathbb{K}$. ☐

An easy and direct consequence of Propositions **5.4** and **5.5** is the following meaningful fact.

Corollary 5.6. *The rational field* $\mathbb{Q} = (\mathbb{Q}, +, \cdot, \leq)$ *is Archimedean.*

5.3. Sequences in totally ordered fields. Let $\mathbb{K} = (\mathbb{K}, +\cdot, \leq)$ be a totally ordered field and let $(x_n)_n \subset \mathbb{K}$ be a sequence. Then we say that:

$(x_n)_n$ **converges** to $\lambda \in \mathbb{K}$ if for each $\varepsilon > 0$ in $\mathbb{K}$, there is some $\nu_\varepsilon \in \mathbb{N}$ such that

$$d(x_n, \lambda) < \varepsilon \text{ in } \mathbb{K} \quad \text{ for every } n \geq \nu_\varepsilon \text{ in } \mathbb{N};$$

We call λ a limit of $(x_n)_n$ in $\mathbb{K}$ and we shall write $\lambda = \lim_{n\to\infty} x_n$. A sequence $(x_n)_n$ has at most one limit in $\mathbb{K}$;

$(x_n)_n$ is a **Cauchy sequence** in $\mathbb{K}$ if

$$\forall \varepsilon > 0 \text{ in } \mathbb{K}, \exists \nu_\varepsilon \in \mathbb{N} : \forall n, m \geq \nu_\varepsilon \Rightarrow d(x_n, x_m) < \varepsilon \text{ in } \mathbb{K};$$

$(x_n)_n$ is **bounded** in $\mathbb{K}$ if there is some $\eta > 0$ in $\mathbb{K}$ such that

$$d(0, x_n) < \eta \text{ in } \mathbb{K} \quad \text{ for every } n \in \mathbb{N};$$

$(x_n)_n$ is **definitively positive** in $\mathbb{K}$ if, for some $\zeta > 0$ in $\mathbb{K}$ and some $\nu \in \mathbb{N}$,

$$x_n \geq \zeta \text{ in } \mathbb{K} \quad \text{ for every } n \geq \nu \text{ in } \mathbb{N};$$

$(x_n)_n$ is **monotone increasing (resp. decreasing)** in $\mathbb{K}$ if $x_n < x_{n+1}$ (resp. $x_n > x_{n+1}$) for every $n \in \mathbb{N}$. A sequence is said to be **monotone** if it is either monotone nondecreasing or monotone nonincreasing[32].

Let $(x_n)_n$ be a sequence in a totally ordered field $\mathbb{K}$. Let

$$n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \ldots$$

be any increasing sequence of positive integers. Then $(x_{n_k})_k \subset \mathbb{K}$ is a sequence and it is called a **subsequence** of $\mathbb{K}$. We notice that $n_k \geq k$ for every $k \in \mathbb{N}$ as a simple inductive argument ensures.

[31]Note that (ii) is exactly condition $(\star)$ in Proposition **4.11**.

[32]Note that a sequence $(x_n)_n$ in $\mathbb{K}$ is monotone nondecreasing (resp. nonincreasing) if $x_n \leq x_{n+1}$ (resp. $x_n \geq x_n$) for every $n \in \mathbb{N}$.

Let $(\mathbb{K}^{\mathbb{N}}, +, \cdot_C)$ be the commutative ring with unit[33] of all the sequences $x : \mathbb{N} \to \mathbb{K}$ with the operations of sum and product performed componentwise by setting for every $(x_n)_n, (y_n)_n \in \mathbb{K}^{\mathbb{N}}$

$$(x_n)_n + (y_n)_n := (x_n +_{\mathbb{K}} y_n)_n$$

and

$$(x_n)_n \cdot_C (y_n)_n := (c_n)_n,$$

where $c_n := \sum_{k=0}^{n} x_k \cdot_{\mathbb{K}} y_{n-k} = (x_0 \cdot_{\mathbb{K}} y_n) +_{\mathbb{K}} (x_1 \cdot_{\mathbb{K}} y_{n-1}) +_{\mathbb{K}} ... +_{\mathbb{K}} (x_n \cdot_{\mathbb{K}} y_0)$ for every $n \in \mathbb{N}$.

The commutative ring with unit $(\mathbb{K}^{\mathbb{N}}, +, \cdot_C)$ is canonically isomorphic to the ring of formal power series $\mathbb{K}[[X]] := (\mathbb{K}[[X]], +, \cdot)$ whose elements are the formal expressions of the form

$$\sum_{n=0}^{\infty} x_n X^n = x_0 + x_1 X + x_2 X^2 + ... + x_n X^n + ..., \text{ where } x_i \in \mathbb{K},$$

and the operations of sum and product are defined, for every $\sum_{n=0}^{\infty} x_n X^n, \sum_{n=0}^{\infty} y_n X^n \in \mathbb{K}[[X]]$, by

$$\sum_{n=0}^{\infty} x_n X^n + \sum_{n=0}^{\infty} y_n X^n := \sum_{n=0}^{\infty} (x_n +_{\mathbb{K}} y_n) X^n,$$

and

$$\left(\sum_{n=0}^{\infty} x_n X^n\right) \cdot \left(\sum_{n=0}^{\infty} y_n X^n\right) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} x_k \cdot_{\mathbb{K}} y_{n-k}\right) X^n.$$

Moreover, the subring of $(\mathbb{K}^{\mathbb{N}}, +, \cdot_C)$ given by

$$P[\mathbb{K}] := \{(x_n)_n : \text{ there exists } k \in \mathbb{N} \text{ such that } x_n = 0, \forall n \geq k\}$$

is isomorphic to the subring $\mathbb{K}[X] \subset \mathbb{K}[[X]]$ of polynomials with coefficients in $\mathbb{K}$, whose elements are the formal expressions of the form

$$\sum_{n=0}^{m} x_n X^n = x_0 + x_1 X + x_2 X^2 + ... + x_m X^m, \text{ where } x_i \in \mathbb{K}.$$

Since $\mathbb{K}$ is an integral domain, then $\mathbb{K}[[X]]$ is an integral domain[34].

Now, let us consider the ring $\mathbb{K}^{\mathbb{N}} := (\mathbb{K}^{\mathbb{N}}, +, \cdot)$ with the operations of sum and product between sequences performed pointwise as usual.

The following easy result holds.The following easy result holds.

Proposition 5.7. *Let $(x_n)_n, (y_n)_n \in \mathbb{K}^{\mathbb{N}}$ be sequences convergent in $\mathbb{K}$. Then:*

(*i*) $\lim_{n\to\infty} x_n$ *is unique in* $\mathbb{K}$;

[33] $\mathbb{K}^{\mathbb{N}}$ can be canonically identified with the product $\prod_{n\in\mathbb{N}} X_n$, with $X_n = \mathbb{K}$ for every $n \in \mathbb{N}$. In such a case the unit element in $(\mathbb{K}^{\mathbb{N}}, +, \cdot_C)$ is given by $(1, 0, 0, 0, ...)$.

[34] The above notions hold true if instead of a field $\mathbb{K}$ we consider a commutative ring (with unit) A. In such a case $A[[X]]$ is an integral domain if and only if A is an integral domain.

(*ii*) *If* $(x_{n_k})_k$ *is a subsequence of* $(x_n)_n$*, then* $(x_{n_k})_k$ *is convergent too*[35] *and* $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n$ *in* $\mathbb{K}$*;*

(*iii*) $(x_n)_n$ *is a Cauchy sequence in* $\mathbb{K}$*;*

(*iv*) $(x_n \pm y_n)_n$ *is a convergent sequence and* $\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n$ *in* $\mathbb{K}$*;*

(*v*) $(x_n \cdot y_n)_n$ *is a convergent sequence and* $\lim_{n\to\infty}(x_n \cdot y_n) = \left(\lim_{n\to\infty} x_n\right) \cdot \left(\lim_{n\to\infty} y_n\right)$ *in* $\mathbb{K}$*;*

(*vi*) *If* $y_n \neq 0$ *for every* $n \in \mathbb{N}$*, and* $\lim_{n\to\infty} y_n \neq 0$*, then* $(x_n/y_n)_n$ *is a convergent sequence and* $\lim_{n\to\infty}(x_n/y_n) = \dfrac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$ *in* $\mathbb{K}$*;*

(*vii*) *For every* $\alpha \in \mathbb{K}$*,* $(\alpha x_n)_n$ *converges and* $\lim_{n\to\infty} \alpha x_n = \alpha \lim_{n\to\infty} x_n$ *in* $\mathbb{K}$*.*

The above proposition ensures that every convergent sequence in $\mathbb{K}$ is a Cauchy sequence. The converse in general is not true. A field $\mathbb{K}$ in which every Cauchy sequence is convergent (in $\mathbb{K}$) is said to be **Cauchy complete**.

We emphasize that in a totally ordered field $\mathbb{K}$ in general it is not true that every bounded sequence admits a convergent subsequence as well as it is not true that every bounded and monotone sequence admits limit.

Direct computations ensure that the following facts hold. Let $\mathbb{K}^{\mathbb{N}}(\mathscr{C}) \subset \mathbb{K}^{\mathbb{N}}$ be the set of Cauchy sequences in $\mathbb{K}$, then

$(x_n + y_n)_n \in \mathbb{K}^{\mathbb{N}}(\mathscr{C})$ for every $(x_n)_n, (y_n)_n \in \mathbb{K}^{\mathbb{N}}(\mathscr{C})$;
$(x_n \cdot y_n)_n \in \mathbb{K}^{\mathbb{N}}(\mathscr{C})$ for every $(x_n)_n, (y_n)_n \in \mathbb{K}^{\mathbb{N}}(\mathscr{C})$;
$(e_n)_n \in \mathbb{K}^{\mathbb{N}}(\mathscr{C})$ where $e_n = 1_{\mathbb{K}}$ for every $n \in \mathbb{N}$.

In conclusion, $\mathbb{K}^{\mathbb{N}}(\mathscr{C})$ is a subring of $\mathbb{K}^{\mathbb{N}}$ with respect to the induced operations of $\mathbb{K}^{\mathbb{N}}$ into $\mathbb{K}^{\mathbb{N}}(\mathscr{C})$. The above facts is crucially used in Devillanova and Molica Bisci 2021 to construct the Cantor model for reals. We notice that if $(x_n)_n \in \mathbb{K}^{\mathbb{N}}(\mathscr{C})$ then:

$(x_n)_n$ is bounded;
if there exists a subsequence $(x_{n_k})_k$ such that $\lim_{k\to\infty} x_{n_k} = \lambda$ in $\mathbb{K}$, then

$$\lim_{n\to\infty} x_n = \lambda.$$

Finally, let $\mathbb{K}$ be an Archimedean field. Consider in $\mathbb{K}$ the sequences of *rationals* $(1/n)_{n\geq 1}$ and $(1/p^n)_n$, where $p \in \mathbb{N} \setminus \{1\}$. As a consequence of Proposition **5.5** it follows that

$$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{p^n} = 0 \quad \text{in } \mathbb{K}.$$

More precisely, the following general result holds.

Theorem 5.8. *A totally ordered field* $\mathbb{K} = (\mathbb{K}, +, \cdot, \leq)$ *is Archimedean iff* $\lim_{n\to\infty} \dfrac{1}{n} = 0$ *in* $\mathbb{K}$*.*

[35] We notice that if $(x_n)_n$ is a sequence in $\mathbb{K}$ such that every subsequence converges to a fixed $\lambda \in \mathbb{K}$, then also $(x_n)_n$ converges to λ in $\mathbb{K}$.

Proof. Suppose $\mathbb{K}$ to be Archimedean and fix $\varepsilon \in \mathbb{K}$ and $\varepsilon > 0$. Then, there exists $m \in \mathbb{N}$ such that $1/m < \varepsilon$. If $n \geq m$, then $1/n \leq 1/m$. Consequently $n \geq m$ implies that $|1/n - 0| = 1/n \leq 1/m < \varepsilon$. Thus the sequence $(1/n)_n$ converges to zero.

Conversely, suppose that $(1/n)_{n\geq 1}$ converges to zero. If $y \in \mathbb{K}$ with $y > 0$, then $1/y > 0$ and so there exists $\nu \in \mathbb{N}$ such that for every $n \geq \nu$ one has $1/n < 1/y$. Then we get $n > y$ by taking $n \geq \nu$. So, by Proposition **5.5** - Part (*ii*) it follows that $\mathbb{K}$ is Archimedean. □

6. Elements of Set Theory

"En effet, l'analyse du mécanisme des démonstrations dans des textes mathématiques bien choisis a permis d'en dégager la structure, du double point de vue du vocabulaire et de la syntaxe. On arrive ainsi à la conclusion qu'un texte mathématique suffisamment explicite pourrait être exprimé dans une language conventionelle ne comportant qu'un petit nombre de "mots" invariables assemblés suivant une syntaxe qui consisterait en un petit nombre de règles inviolables: un tel text est dit formalisé"

NICOLAS BOURBAKI
ELÉMENTS DE MATHÉMATIQUES
THÉORIE DES ENSEMBLES

In 1874 George Cantor laid the foundations for what it is now known in literature as *Naive Set Theory*. This theory has been assigned axiomatised to the work of other mathematicians with the aim of overcoming the *paradoxes* that is generated within it.

There are three main axiomatizations (or theories) due to:

Skolem, Zermelo and Fraenkel (**ZF** or also **ZFC**);
von Neumann, Gödel and Bernays (**NGB**);
Morse and Kelley (**MK**).

The substantial difference in these theories concerns the way in which the problem of the distinction between the concept of *set* and that of *class* is approached. In the following we briefly present the scheme of axioms of the **NGB** theory[36]. We refer to Garling 2013 for a readable introduction on the most popular **ZFC** theory.

A universal class $\mathscr{U}$ called **Universe** of every subclasses and objects (or elements) is assigned giving axioms that gather information about $\mathscr{U}$, its objects, and its subclasses which are called simply classes. This means that all the classes X are subclasses of $\mathscr{U}$, which are included in the universal class $\mathscr{U}$. Some of the classes of $\mathscr{U}$ are themselves objects of $\mathscr{U}$ and are called "sets". Saying $x \in \mathscr{U}$ is equivalent to saying that x is an object of the universe $\mathscr{U}$.

[36]**NGB** can be correctly encoded as a *formal theory* on a *First–Order Language* - FOL for short - *with equality*, *i.e.*, a *formal system* in which it is possible to express sentences and deduce their consequences in a formal and mechanical way. We assume the familiarity of the reader with some basic fact of FOL such as: alphabet, logical symbols, non–logical symbols, formation rules, free and bounded variables; see Mendelson 2015.

6.1. Elementary NGB Theory. In the **NGB** theory one postulates the following rules of formation for classes, sets[37] and objects:

I. Axiom of Extensionality: Let X and Y be classes. They are equal if X and Y contain the same sets as objects, *i.e.*, the well formed formula $\forall z(z \in X \leftrightarrow z \in Y)$ is *provable*[38] in **NGB**. In this case we write

$$X = Y.$$

II. Axioms Schema of Abstraction: Let $\varphi(X, Y_1, ..., Y_m)$ be a *predicative well formed formula*[39]. Then its extension

$$Z := \{x : \varphi(x, Y_1, ..., Y_m)\}$$

is a class;
III. Axiom of Comprehension: If Y is a set, then any subclass[40] $X \subseteq Y$ is a set;
IV. Axiom of the Empty set: The empty[41] class $\emptyset$ is a set;
V. Axiom of Pairing: Let x, y be objects and set $\{x, y\}$ be the class (pair) given by

$$\{z : z = x \vee z = y\}.$$

The pair $\{x,\, y\}$ is a set;

[37]A set X is a class that is also an object. By following Bernays (1937–1954) and Gödel (1940), we shall use capital italic letters as variables and lower case letters as special restricted variables for sets.

[38]By using the standard notation valid in any formal theory we also write $\vdash \forall z(z \in X \leftrightarrow z \in Y)$.

[39]By a predicative well formed formula we mean a well formed formula (wff) in a first order language whose variables occur among $X_1, ..., X_n, Y_1, ..., Y_m$, and in which only set variables are quantified (*i.e.*, φ can be abbreviated *i.e.* "refrased" in such a way that only set variables are quantified); see the classical book Mendelson 2015, Chapter 4. More precisely, after Axiom V, by following Mendelson 2015, Proposition 4.4 (Class Existence Theorem), assigned a predicative wff $\varphi(X_1, ..., X_n, Y_1, ..., Y_m)$, one has that

$$\vdash \exists Z \forall x_1 ... \forall x_n((x_1, ..., x_n) \in Z \leftrightarrow \varphi(x_1, ..., x_n, Y_1, ..., Y_m)).$$

Moreover, on account of Axiom I, the above class Z is unique and we write

$$Z = \{(x_1, ..., x_n) : \varphi(x_1, ..., x_n, Y_1, ..., Y_m)\}.$$

We emphasize that in order to construct well formed formulas in a first order language, the notion of set is necessary. Fortunately, only a minimal part of **NGB** is needed to define the syntax as well as the semantic of first order languages. Consequently, **we assume this minimal part of Set Theory as foundation of the First Order Languages**; bootstrapping Hilbert argument. The remainder part can be formally constructed as shortly described in this subsection. In such a way circular definitions can be avoided in the formal construction of the **NGB** Theory.

[40]Given a class Y we say that X is a subclass of Y (and we write $X \subseteq Y$) if

$$\forall x(x \in X \rightarrow x \in Y).$$

We will write $X \subset Y$, saying that the class X is strictly included in Y, if $X \subseteq Y$ and $X \neq Y$ (*i.e.*, $\neg(X = Y)$).

[41]The empty class $\emptyset$ is the unique class that has no elements. More precisely,

$$\vdash \exists x \forall y(y \notin x).$$

Therefore, we can introduce a new individual constant $\emptyset$ by means of the following condition $\forall y(y \notin \emptyset)$.

VI. Axiom of Union[42]: If X is a set of sets, then

$$\bigcup X := \{x : \exists y(y \in X \wedge x \in y)\}$$

is a set;
VII. Axiom of the Objects: Every object is a set[43];
VIII. Axiom of Infinity: There exists a set X such that:

(i) $\emptyset \in X$,
(ii) If $x \in X$ then $x \cup \{x\} \in X$;

IX. Axiom of Power set: Given a set X, the class

$$\wp(X) := \{Y : Y \subseteq X\}$$

of subsets of X is a set;
X. Axiom of Replacement: Let $f : X \to Y$ be a function between the classes X and Y. If the domain X of f is a set, then its image

$$f(X) := \{y : \exists x(x \in X \wedge f(x) = y)\}$$

is a set;
XI. Axiom of Choice: If $(X_\alpha)_{\alpha \in J}$ is a disjoint family of nonempty sets, then there exists a set S such that $S \cap X_\alpha = \{x_\alpha\}$ for every $\alpha \in J$, where J is a set of indexes.
XII. Axiom of Foundation: For every set $X \neq \emptyset$ there exists $a \in X$ such that

$$X \cap a = \emptyset.$$

Facts:

The universe $\mathscr{U}$ contains classes;
Every class contains objects;
The classes that are also objects are called sets;
There are proper classes, *i.e.*, classes that are no sets;
$x \in Y$ is well defined if x is a set and Y is a class.

The **NGB** Axioms ensure that the following classes make sense:

$X \cup Y := \{x : x \in X \vee x \in Y\}$	(Union)
$X \cap Y := \{x : x \in X \wedge x \in Y\}$	(Intersection)
$X \setminus Y := \{x : x \in X \wedge x \notin Y\}$	(Difference)
$X^c := \mathscr{U} \setminus X$	(Complementary set)
$X \Delta Y := (X \setminus Y) \cup (Y \setminus X)$	(Symmetric difference)

where X and Y are classes.

The next facts holds true (**Boolean Algebra** of classes):

[42]Given a class of sets X (*i.e.*, a class whose elements are sets) we define $\bigcup X$ (union of X) as the class of those objects that belong to at least one element of X. Analogously, given a class of sets X, we also define the class (intersection) $\bigcap X$. More precisely: $x \in \bigcap X$ iff $x \in Y$ for every $Y \in X$. We notice that if $X = \emptyset$, then $\bigcap X = \mathscr{U}$.

[43]Axiom VI - on account of Axiom VII - can be rewritten as follows: if X is a set, then $\bigcup X$ is a set.

$X \cup Y = Y \cup X$	(Commutative of $\cup$)
$X \cap Y = Y \cap X$	(Commutative of $\cap$)
$(X \cup Y) \cup Z = X \cup (Y \cup Z)$	(Associative of $\cup$)
$(X \cap Y) \cap Z = X \cap (Y \cap Z)$	(Associative of $\cap$)
$(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$	(Distributive of $\cap$ w.r.t. $\cup$)
$(X \cap Y) \cup Z = (X \cup Z) \cap (Y \cup Z)$	(Distributive of $\cup$ w.r.t. $\cap$)
$X \cup \emptyset = X$ and $X \cap \emptyset = \emptyset$	(Identity)
$X \cup X^c = \mathscr{U}$ and $X \cap X^c = \emptyset$	(Completion)

where X, Y and Z are classes.

The usual paradoxes now no longer lead to contradictions but only yield the results that various classes are proper classes, *i.e.*, are not sets. For instance, let us consider the **Russell's class**

$$\Psi := \{x : x \notin x\}.$$

We have the following result.

Theorem 6.1. *The Russell's class is not a set.*

Proof. By definition an object $x \in \Psi$ iff $x \notin x$. Assume by contradiction that Ψ is an object, *i.e.*, Ψ is a set. Hence $\Psi \in \Psi$ iff $\Psi \notin \Psi$, that clearly is a contradiction. □

Clearly, Theorem **6.1** ensures that the class Ψ is not an object. A direct consequence of the above theorem is the next meaningful property.

Corollary 6.2. *The class C_{Obj} of all the objects is not a set.*

Proof. Every class X is included in C_{Obj}, in particular $\Psi \subseteq C_{Obj}$. Assume by contradiction that C_{Obj} is a set. By Axiom III the class Ψ is a set against Theorem **6.1**. □

The next facts are essentially consequences of Axiom X.

Proposition 6.3. *If X and Y are sets, then $X \times Y$ is a set.*

Proof. Let us fix $y \in Y$. By Axiom X there exists the set

$$\{(x, y) : x \in X\} = X \times \{y\}.$$

Moreover, the following class

$$\{X \times \{y\} : y \in Y\},$$

is a set again by Axiom X. Finally, since

$$\bigcup \{X \times \{y\} : y \in Y\} = \bigcup_{y \in Y} X \times \{y\} = X \times Y,$$

by Axiom VI, the class $X \times Y$ is a set as claimed. □

According to the classical notation due to Kuratowski, we may define[44]

$$(x, y) := \{\{x\}, \{x, y\}\}.$$

[44]The Kuratowski definition of pair immediately yields that $(x, y) \neq (y, x)$ whenever $x \neq y$.

In such a case Proposition **6.3** easily follows by Axioms IX and III instead of Axiom X owing to

$$X \times Y \subseteq \wp(\wp(X \cup Y)).$$

The following corollaries are of fundamental importance in Mathematical Analysis.

Corollary 6.4. *Let X and Y be classes and $f : X \to Y$ be a function. Assume that X is a set, then $Gr(f)$ is a set.*

Proof. By Axiom X the class $f(X)$ is a set. Consequently, by Proposition **6.3** the class $X \times f(X)$ is a set. Since $Gr(f) \subseteq X \times f(X)$ the conclusion follows by Axiom III. ☐

A careful analysis of Proposition **6.3** as well as of Axioms IX and III yield.

Corollary 6.5. *Let X and Y be sets. The class Y^X of all the functions $f : X \to Y$ is a set.*

Proof. Since X and Y are sets, by Proposition **6.3** the class $X \times Y$ is a set. Moreover, by Axiom IX, the class $\wp(X \times Y)$ is a set. Let $f \in Y^X$. By definition $Gr(f) \subseteq X \times Y$, that is $Gr(f) \in \wp(X \times Y)$. Consequently, $Y^X \subseteq \wp(X \times Y)$ and, on account of Axiom III, Y^X is a set as claimed. ☐

We refer the interested reader to

A. BERARDUCCI, Elementi di Teoria degli Insiemi 2012-13, available at the web–page of Alessandro Berarducci (University of Pisa), for a brilliant introduction on this subject and to the classical reference Jech 2003 for a detailed exposition of Set Theory. We also cite Nagel and Newmann 2001 for a very interesting trip in the Gödel universe.

From now on we restrict ourself to sets. Given the sets X and J, a **family** of elements of X, parameterized by J, is a function $x : J \to X$. The image $x(\alpha)$ is denoted by x_α, where α is considered as an **index**. Consequently, the image $x(J) = \{x_\alpha : \alpha \in J\} \subseteq X$ is denoted by $(x_\alpha)_{\alpha \in J}$. If $J = \mathbb{N}$ the family $x : \mathbb{N} \to X$, also denoted, by $(x_n)_n$ is said to be a **sequence** on X.

Let X be a set. If $(X_\alpha)_{\alpha \in J}$ is a family of $\wp(X)$, we set

$$\bigcup_{\alpha \in J} X_\alpha := \{x \in X \ : \ x \in X_\alpha \text{ for some } \alpha \in J\}$$

and

$$\bigcap_{\alpha \in J} X_\alpha := \{x \in X \ : \ x \in X_\alpha \text{ for any } \alpha \in J\}.[45]$$

The above notions of union and intersection are well definite on account of the **NGB** Axioms. In particular $\bigcup_{\alpha \in J} X_\alpha$ and $\bigcap_{\alpha \in J} X_\alpha$ are sets. A family $(X_\alpha)_{\alpha \in J}$ of $\wp(X)$ is said to be (pairwise) **disjoint** if $X_\alpha \cap X_\beta = \emptyset$ for every $\alpha, \beta \in J$ with $\alpha \neq \beta$.

[45] If $J = \emptyset$, then $\bigcup_{\alpha \in J} X_\alpha := \emptyset$ and $\bigcap_{\alpha \in J} X_\alpha := X$. Note that if $\Sigma \subseteq \wp(X)$ (*i.e.*, if Σ is a *collection* of subsets of X) we set $\bigcup_{X \in \Sigma} X := \{x : x \in X \text{ for some } X \in \Sigma\}$ (also denoted by $\bigcup \Sigma$) and $\bigcap_{X \in \Sigma} X := \{x : x \in X \text{ for every } X \in \Sigma\}$ (also denoted by $\bigcap \Sigma$).

Generalized De Morgan's formulas:

$$X \setminus \bigcup_{\alpha\in J} X_\alpha = \bigcap_{\alpha\in J}(X\setminus X_\alpha)$$

and

$$X \setminus \bigcap_{\alpha\in J} X_\alpha = \bigcup_{\alpha\in J}(X\setminus X_\alpha).$$

Let X and Y be sets. If $f : X \to Y$ is a function, we define for every $B \in \wp(Y)$ the set

$$f^{\leftarrow}(B) := \{x \in X : f(x) \in B\},$$

namely the **preimage** of B. Following De Marco 1986, Section 0.8, for any family $(Y_\alpha)_{\alpha\in J}$ of $\wp(Y)$, one has

$$f^{\leftarrow}\left(\bigcup_{\alpha\in J} Y_\alpha\right) = \bigcup_{\alpha\in J} f^{\leftarrow}(Y_\alpha) \quad \text{and} \quad f^{\leftarrow}\left(\bigcap_{\alpha\in J} Y_\alpha\right) = \bigcap_{\alpha\in J} f^{\leftarrow}(Y_\alpha).$$

Moreover, for every $A \in \wp(X)$, we set

$$f(A) := \{y \in Y : \exists x \in A \wedge y = f(x)\},$$

namely the **image** of A. For any family $(X_\alpha)_{\alpha\in J}$ of $\wp(X)$ then

$$f\left(\bigcup_{\alpha\in J} X_\alpha\right) = \bigcup_{\alpha\in J} f(X_\alpha) \qquad \text{and} \qquad f\left(\bigcap_{\alpha\in J} X_\alpha\right) \subseteq \bigcap_{\alpha\in J} f(X_\alpha).$$

The following facts also hold:

(*i*) $A \subseteq f^{\leftarrow}(f(A))$ for every $A \in \wp(X)$ and $f(f^{\leftarrow}(B)) \subseteq B$ for every $B \in \wp(Y)$;

(*ii*) Are equivalent:
 f is injective
 $A = f^{\leftarrow}(f(A))$ for every $A \in \wp(X)$
 $\{x\} = f^{\leftarrow}(f(\{x\}))$ for every $\{x\} \in \wp(X)$;

(*iii*) If $B \in \wp(Y)$ then $B = f(f^{\leftarrow}(B)) \Leftrightarrow B \subseteq f(X)$. Consequently, the next statements are equivalent:
 f is surjective
 $B = f(f^{\leftarrow}(B))$ for every $B \in \wp(Y)$
 $\{y\} = f(f^{\leftarrow}(\{y\}))$ for every $\{y\} \in \wp(Y)$.

Let X be a set. If $(X_\alpha)_{\alpha\in J}$ is a family of $\wp(X)$, the **direct product of the family** is defined as follows:

$$\prod_{\alpha\in J} X_\alpha := \left\{x \ : \ x : J \to \bigcup_{\alpha\in J} X_\alpha \text{ such that } x_\alpha := x(\alpha) \in X_\alpha \text{ for any } \alpha \in J\right\}.$$

If J is finite, for instance $J := \{1, ..., m\}$ for some $m \in \mathbb{N}$, the direct product $\prod_{k=1}^m X_k$ can be identified with the set of m–tples $(x_1, \ldots, x_m)$ such that $x_k \in X_k$ for every $k \in J$. In such a case the product $\prod_{k=1}^m X_k$ is also called the **cartesian product** of the (finite) family $(X_k)_{k\in J}$.

Assume that $X_k = X$ for every $k \in J$. An element $(x_1, ..., x_m) \in X^m$ is also called a **finite sequence** on X. If $(x_1, ..., x_m), (y_1, ..., y_m) \in X^m$, then $(x_1, ..., x_m) =$

$(y_1, ..., y_m)$ iff $x_k = y_k$ for every $k = 1, ..., m$. With abuse of notation, a finite sequence $(x_1, ..., x_m) \in X^m$ is also denoted by the symbol $x_1x_2...x_m$ (namely **string**), frequently used in Logic and Computer Science. In such a contest the simbol () denotes the **empty sequence**.

6.2. Axiom of Choice: equivalent versions. In the next result we present some[46] equivalent versions of Axiom XI; see De Marco 1986, Teorema A.0.6.

Theorem 6.6. *The following facts are equivalent:*

(*i*) *If $(X_\alpha)_{\alpha\in J}$ is a disjoint family of nonempty sets, then there exists a set S such that $S \cap X_\alpha = \{x_\alpha\}$ for every $\alpha \in J$ (Axiom of Choice);*

(*ii*) *Every surjective fuction admits a right inverse, i.e., if $f : X \to Y$ is a surjective function, there exists a function $g : Y \to X$ such that $f \circ g = id_Y$ (Axiom of the right inverse);*

(*iii*) *If $(X_\alpha)_{\alpha\in J}$ is a family of noempty sets then the product $\prod_{\alpha\in J} X_\alpha$ is notempty, i.e., there exists a function $x : J \to \bigcup_{\alpha\in J} X_\alpha$ such that $x(\alpha) \in X_\alpha$ for every $\alpha \in J$ (Axiom of product);*

(*iv*) *For every set X there exists a function $s : \wp(X) \setminus \{\emptyset\} \to X$ such that $s(E) \in E$ for every $E \in \wp(X) \setminus \{\emptyset\}$ (Axiom of the Choice function);*

(*v*) *Let X be a poset and assume that any chain[47] C in X is bounded above. Then X admits a maximal element (The Zorn's lemma);*

(*vi*) *Every set admits a well order[48] (Principle of well–ordering).*

Proof. $(i) \Rightarrow (ii)$ Let X and Y be two nonempty sets. Let us consider the disjoint family $(X_y)_{y\in Y}$ of $\wp(X) \setminus \{\emptyset\}$, where $X_y := f^{\leftarrow}(y)$, with $y \in Y$. By (i) we can fix $S \in \wp(X)$ such that $S \cap X_y = \{x_y\}$ for every $y \in Y$ so that we can define the function $g : Y \to X$ which maps any y into $x_y \in S \cap X_y$. By construction, it follows that $f \circ g = id_Y$, *i.e.*, condition (ii) holds.

$(ii) \Rightarrow (iii)$ Let us consider the function $f : \bigcup_{\alpha\in J} X_\alpha \times \{\alpha\} \to J$, defined by $f(z, \alpha) := \alpha$ for every $(z, \alpha) \in X_\alpha \times \{\alpha\}$. Hence f is surjective and by (ii) admits a right inverse $g : J \to \bigcup_{\alpha\in J} X_\alpha \times \{\alpha\}$ such that $g(\alpha) = (x_g, \alpha) \in X_\alpha \times \{\alpha\}$. Now, since the function $x : J \to \bigcup_{\alpha\in J} X_\alpha$ which maps any $\alpha \in J$ into the first component x_g of the pair $g(\alpha) = (x_g, \alpha)$ belongs to $\prod_{\alpha\in J} X_\alpha$ we get the thesis.

$(iii) \Rightarrow (iv)$. Assume that (iii) holds and let $J := \wp(X) \setminus \{\emptyset\}$. Hence, by (iii) there exists a function $x : \wp(X) \setminus \{\emptyset\} \to \bigcup_{E\in\wp(X)\setminus\{\emptyset\}} E$ such that $x(E) \in E$ for every $E \in \wp(X) \setminus \{\emptyset\}$. Since $\bigcup_{E\in\wp(X)\setminus\{\emptyset\}} E = X$, the conclusion is achieved.

[46] We emphasize that there are other statements which are logically equivalent to the Axiom of Choice. For instance: every vector space has a basis; every nontrivial ring has a maximal ideal; the Cartesian product of (arbitrarily many) compact topological spaces is compact (Tychonoff's Theorem); any two sets either have the same cardinality, or one of them has a smaller cardinality than the other (Hartogs' Theorem). See Curzio *et al.* 2014 for related topics.

[47] Let $(X, \leq)$ be a poset. A totally ordered subset $C \subseteq X$, with respect to the induced order, is said to be a **chain** in X.

[48] A **well order** (or well ordering or well order relation) on a set X is an order $\leq$ on X with the property that every nonempty subset E of X admits minimum with respect to the induced order. A set X together with a well order relation on X is then called a **well ordered set**.

$(iv) \Rightarrow (v)$ Thanks to (iv) it is possible, by using the argument in De Marco 1986, A.0.7, p. 651, to prove that any poset in which all well ordered sets are bounded above admits a maximal element. Then (v) follows since any ordered set is also totally ordered.

$(v) \Rightarrow (vi)$ Let X be a set and let $\mathscr{X} \subseteq \wp(X)$ be the set of well ordered subsets $(W, \leq_W)$ of X. The structure $(\mathscr{X}, \preceq)$ is a poset when the binary relation $\preceq$ is defined by setting for every $(W_1, \leq_{W_1})$ and $(W_2, \leq_{W_2}) \in \mathscr{X}$

$$(W_1, \leq_{W_1}) \preceq (W_2, \leq_{W_2})$$

iff

either $W_1 = W_2$ or $W_1 = (-\infty, m)_{W_2} := \{w \in W_2 : w <_{W_2} m\}$ for some $m \in W_2$;

or $\leq_{W_1}$ and $\leq_{W_2}$ coincide on W_1.

In $\mathscr{X}$ let us consider a chain $C := ((W_\alpha, \leq_\alpha))_{\alpha \in J}$ with respect to $\preceq$ and set $\mathscr{W} := \bigcup_{\alpha \in J} W_\alpha$. On $\mathscr{W}$ define the total order $\leq_{\mathscr{W}}$ as follows:

$$\forall x, y \in \mathscr{W}\ (x \in W_\lambda, y \in W_\mu)\ x \leq_{\mathscr{W}} y \Leftrightarrow x \leq_\beta y, \text{ where } \beta := \begin{cases} \lambda & \text{if } W_\mu \subseteq W_\lambda \\ \mu & \text{if } W_\lambda \subseteq W_\mu. \end{cases}$$

By De Marco 1986, Lemma A.0.5, the following facts hold:

$(\mathscr{W}, \leq_{\mathscr{W}})$ is well ordered;

$(W_\alpha, \leq_\alpha) \preceq (\mathscr{W}, \leq_{\mathscr{W}})$ for every $\alpha \in J$.

Hence $(\mathscr{X}, \preceq)$ is a poset in which every chain C is bounded above in $(\mathscr{X}, \preceq)$. Then, the Zorn's lemma (v) ensures that there exists a maximal element $(M, \leq_M)$ in $(\mathscr{X}, \preceq)$. We claim that $M = X$ (and, as a consequence, that X is a well ordered set). Indeed, arguing by contradiction, assume that $M \neq X$, let $x \in X \setminus M$ and extend $\leq_M$ to $M \cup \{x\}$ as follows: $m <_{M\cup\{x\}} x$ for every $m \in M$. Clearly, $(M, \leq) \preceq (M \cup \{x\}, \leq_{M\cup\{x\}})$ in $\mathscr{X}$ against the maximality of $(M, \leq)$ in $\mathscr{X}$.

$(vi) \Rightarrow (i)$ Let $(X_\alpha)_{\alpha \in J}$ be a disjoint family of nonempty sets. Consider the set $X := \bigcup_{\alpha \in J} X_\alpha$. By (vi) there exists a well order $\leq$ on X. Now, denote by $\min_X X_\alpha$ the minumum of X_α with respect to $\leq$ for every $\alpha \in J$. Then (i) follows by taking $S := \{\min X_\alpha : \alpha \in J\}$ since $S \cap X_\alpha = \{x_\alpha\}$ for every $\alpha \in J$. The proof is now complete. □

Among several and deep results on Set Theory in Jech 2003, Abstract Algebra in Curzio *et al.* 2014, as well as Differential and Algebraic Geometry in Hartshorne 1977; Do Carmo 1992, some meaningful consequences of the Axiom of Choice are implicitly used in several text books of Functional Analysis; see, for instance, Brézis 1983.

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[a] Politecnico di Bari
Dipartimento di Meccanica, Matematica e Management
Via Orabona, 4, 70126 Bari, Italy

[b] Università degli Studi di Urbino, Carlo Bo
Dipartimento di Scienze Pure e Applicate (DiSPeA),
Piazza della Repubblica 13, 61029 Urbino (Pesaro e Urbino), Italy

* To whom correspondence should be addressed | email: giovanni.molicabisci@uniurb.it

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