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MATH CITY MAP: PROVIDE AND SHARE OUTDOOR MODELLING TASKS. AN EXPERIENCE WITH CHILDREN

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ABSTRACT. The MathCityMap-project (MCM-Project), carried out by the IDMI Goethe-University Frankfurt (Germany), aims to provide and share outdoor modelling tasks. A web portal and a mobile app for math-trail program were created and several math-trail tasks were designed around cities allover the world and uploaded into a system by the teachers. Then students walk the trail and its tasks by the help of mobile app or a .pdf file (generated by the web portal) to find and solve mathematical modelling tasks around the city. In this paper we present the results of an experimental teaching activity through MCM-Project for primary school students. Thanks to MCM-Project, we were able to design our math-trail in Catania (Italy) and involve some 10-11 aged students in its walking. Data were gathered by means of participatory observation, video-recording, interviews, and questionnaires. We will show how much this kind of activity can be effective to support primary school students in the process of modelling mathematics outside the classroom.

1. Introduction

Mathematical modelling is not easy to teach (Ludwig and Jablonski 2019): there is a lack of good beginner modelling tasks even for undergraduates students and a lot of so called modelling tasks are not authentic and not realistic (Vos 2011).

The MathCityMap-Project (MCM-Project, www.mathcitymap.eu), carried out by the IDMI Goethe - University Frankfurt (Germany), aims to motivate students to solve real world tasks by using expedient mathematical modelling ideas outside the classroom. In particular, MCM-Project, supported by the use of digital technology, helps both for preparing the tasks through a web portal and for doing math trail in the environment. The MCM-Project is based on the math trail idea introduced in Melbourne, Australia in 1984 (Blane and Clark 1984). To solve a typical task which is provided by the MCM-app (e,g,. to calculate the mass of a rock) you need to transfer the real model into a mathematical model. The MCM-app provides hints, checks the answer of the user and gives a direct feedback.

A math trail is a set of mathematical tasks or questions that are bound to objects of the real world. Usually they are located in walking distance. A mathematical trail is accompanied by a guiding map that displays interesting locations and descriptions of different tasks to discover mathematics in the outdoor environment. Shoaf, Pollak & Schneider (Shoaf *et al.* 2004) saw a potential in math trails to popularize mathematics since everyone (e.g. families) can walk them, the participants' work cooperatively and thus experiencing mathematics in a non-threatening environment. In addition, "walking a math trail is a good way to make experiences with the perceptual motor system, which is the base of all mathematical concepts"(Ludwig and Jablonski 2019).

In this paper we present some of the results of a teaching experiment carried out with primary school students, which consisted of five meetings, held alternately in classroom and in the city of Catania (Italy). The experimentation was, in particular, the subject of the Thesis of Master's Degree in Mathematics by A. Ariosto (Ariosto 2019), one of the authors of this paper.

Primary school students walked a math-trail, located in the city center of Catania, designed using MCM web portal. As mentioned, a math-trail could be walked by everyone, child or adult, citizen or tourist, just for fun. In the experimentation described in this paper, we show a didactical use of math-trail, allowing the merge of two traditionally separated environments: *in* (the classroom) and *out* (outside). Too many times the classroom environment is completely detached from what is happening outside. Here we used the math-trail idea to join *in* and *out* in the two senses of bringing the *in* work outside (*in-out*) and the *out* work inside (*out-in*):

- *in-out*: recognizing in the city's environment some objects or mathematical concepts already sawn and explained in class (in this experimentation this was done in meeting 2 and which will be the subject of study in this paper);
- *out-in*: starting from real city's objects or experiences and then bringing them in class, to introduce new objects or mathematical concepts (in this experimentation this was done in meeting 4).

Then, an *in-out-in* cycle is generated in the teaching methodology: the inside (the classroom) goes outside (city) and vice versa.

The aim of the outdoor meetings is to discover mathematics around the city and make students able to recognize in their city a lot of geometrical (and not only) objects, as well as to solve mathematical problems about them. In such a way the experimentation aims to make students able to apply their mathematical skills in tangible/real life. Moreover, the activity aims at helping students become familiar with a measurement tool, how to read a map, to juggle distances and orientation in the city. We underline also that the outdoor meetings were designed bearing in mind the idea of multidisciplinarity: in light of that, thanks to the *liaison* with the teacher of Italian and History, our field trip was structured so as to alternate mathematical and humanistic activities (the latter aimed at discovering the history, myths and legends of the city, as well as the discovery and admiration of cultural heritage sites that are considered assets of Catania's artistic heritage).

The research question guiding this paper is as follows: Is the math-trail idea able to support primary school students in the process of modelling mathematics outside the classroom, around the city?

2. Theoretical framework

To apply mathematics to reality, a model is needed, whether implicitly or explicitly recognised (Borromeo Ferri 2006) . Mathematical modelling is a cyclical process in which starting with a real problem, the student structures the problem according to mathematical concepts and solves it, and then translates the mathematical solution in relation to the real situation. It consists then of an extra-mathematical Domain (D), of some Mathematical domain (M) and a mapping between them resulting in outcomes that are translated back to the extra-mathematical domain (D), as shown in Figure 1.

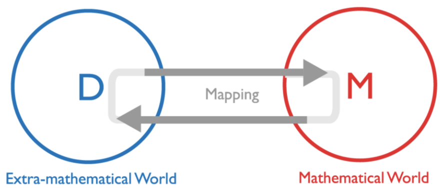

FIGURE 1. Mathematical modelling[1].

Showing examples of modelling, and even more teaching to model, can enrich and amplify in the learner the idea of mathematics and its role in the world and in society, illustrate and give meaning to mathematical entities and processes and, consequently, motivate the study of mathematics itself (Blum *et al.* 2007).
One of the main components of the theory of teaching and learning mathematical modelling is the important idea, accepted by all, that a general mathematical modelling process can exist.
One of the most comprehensive schematic descriptions of the modelling cycle was introduced by Blum and Leiß (Blum and Leiß 2005) and is shown in Figure 2.

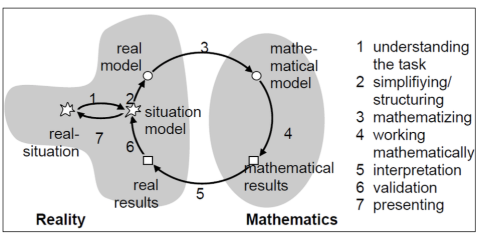

FIGURE 2. Modelling cycle according to Blum and Leiß (Blum and Leiß 2005).

As can be seen, the modelling process is divided into various phases and consequently into various sub-processes. There is always the pole of “reality" (left) and the one of “mathematics" (right), but within each of these are identified various moments and steps.

[1]retrieved from https://www.immchallenge.org.au/supporting-resources/what-is-mathematical-modelling

The process foresees that each individual making modelling can pass through six phases which we will describe below.

- **Real situation** (RS). It is the situation given in the problem. It can be a drawing, a text or both.
- **Situation model** or **Mental representation of the situation** (MRS).It is the moment when the individual begins to generate a mental representation of the real situation, at a rather implicit and mainly unconscious level. MRS varies from individual to individual. For example, it depends on the individual's mathematical thinking style; on the visual imagination linked to their experiences; or on the attention to numbers and data, which the individual wants to combine and relate. During the MRS the individual filters the information of the problem. Depending on the type of problem, demands for extra-mathematical knowledge may increase.
- **Real Model** (RM). This phase is very connected to MRS. In fact, during the transition from MRS to RM there is an idealization and simplification of the problem, at a more conscious level for the individual. The MRS is mostly built at the internal level of an individual. However, it can also be accompanied by a level of external representation (figures or formulas), depending on the individual's verbal statements while making an external representation.
- **Mathematical Model** (MM). At this stage, the individual proceeds in mathematization. He elaborates external representations in terms of figures or formulas. His verbal statements are more on a mathematical level, less on a level related to reality. The passage into mathematics is accomplished.
- **Mathematical results**. Here the individual uses his mathematical skills. He derives the results he has obtained on the basis of the MM he has chosen.
- **Real results**. In this last phase the interpretation of the results is carried out, i.e. the transition from mathematical results to real results. During the validation, the individual considers the correspondence between the real results and his MRS. This can be corrected for the results or not. Based on the data, there are two types of validation modes:
  - Intuitive validation (more unconscious): the individual realizes that the results could be wrong for reasons he cannot explain. Or he feels that the results are wrong because they do not correspond to the picture of his experiences and associations.
  - Knowledge-based validation (more conscious): the individual believes or does not believe that the results obtained on the basis of his extra-mathematical knowledge are valid.

  Both validations are related to the subject's previous reflections. The reason why the subjects mostly do not validate is due to the fact that they perform an internal validation in mathematics. Validating for them means "calculating" the mathematical model, not connecting the results with the real situation.

Of course, it is not always possible to identify all these phases, and many authors describe modelling cycles in a simplified way compared to the one just presented. Differentiation arises from a diversity of approach and conception of modelling; in some cases, it also depends on the type of problem used, whether it is complex or not. Borromeo Ferri (Borromeo Ferri 2006) indicates as an important point the separation between modelling cycles used for research and those for school purposes. In any case, according to various authors (e.g. (Lesh and Doerr 2003; Kaiser 2005; Borromeo Ferri 2006)) there is not necessarily linearity in the cyclic path, i.e. it is normal for the student to pass from one phase to another in a non-consequential way. Therefore, the diagrams reported are in any case to be understood as a synthetic/normative reference of what generally happens in practice.

3. Methodology

The teaching experimentation was carried out in the primary school "Francesco Ventorino", in Catania (Italy), by one of the authors of this paper. It involved a class of 17 pupils aged 10-11 years old. The whole experimentation aimed at helping students grasp mathematical concepts such as surface (the space included within a closed curve) and area (the number representing the measure of a surface), not through frontal lecture, but through discovery activities based on mathematical laboratory activities (Anichini *et al.* 2004). The experimentation began on February 2019 and finished on April 2019, and consisted of 14 hours divided in five meetings. Three of the five meetings were lessons in classroom, based on mathematical laboratory, with the involvement of the body, and two of them were outdoor activities, based on math trails designed by MCM around the city center of Catania. The structure of the five meetings follows:

1) Meeting 1 (21 February 2019): classroom activity carried out as a mathematics laboratory. It began with the Dido's legend on ox hide, and the Dido's problem on maximal surface figure among isoperimetric figures and it finished playing with a Tangram. The Tangram game had an important role to build a way to decompose polygons and, as we will see in the analysis section, students will apply this skill in the resolutions of some MCM-tasks of Meeting 2 (2 hours);
2) Meeting 2 (21 March 2019): outdoor activity based on a math trail on the recognition of polygons and calculation of perimeters and areas (4 hours);
3) Meeting 3 (22 March 2019): classroom activity. Some of the MCM-tasks that the children had encountered in Meeting 2 were commented on together, regarding solution strategies and data found on site. After that, the focus was on equidecompasability with a Tangram and other mathematical concepts that students would encounter in the second outdoor activity. Moreover, an appendix was devoted to errors and measurements. The lesson ended with a summary test allowing to verify the acquired skills (2 hours);
4) Meeting 4 (11 April 2019): outdoor activity based on a math trail on circumference and circle and construction of a "human compass" (4 hours);

5) Meeting 5 (17 April 2019): classroom activity that included a focus on the use of compasses as drawing tools, an appendix on the acceptability range of a given measurement, the acceptability range of a given measurement. The lesson ended with a summary test allowing to verify the acquired skills (2 hours).

At the end of the outdoor activities, students were given a questionnaire. This allowed, on the one hand, to get feedback on the satisfaction of the activity and on the other hand to ask them to report some reasoning they had done to solve the tasks they had met.
For the purposes of this contribution, in the following we will focus in detail only on the first outdoor activity (Meeting 2), the one where it is involved the use of MCM.
The Meeting 2 was planned together with the collaboration of the Math and Italian teachers of the class, as a multidisciplinary outdoor activity. MCM was introduced to the students at the beginning of the Meeting 2 by the experimenter. Since the school does not allow their student to use smartphones during the lessons, students could not use the MCM-app. Therefore, the experimenter provided the .pdf file of the math-trail, generated by the MCM web portal, to each students. He explained to the students how to approach to the MCM-tasks contained in the math-trail. In particular, since the students could not use MCM-app, the experimenter specified that if they needed a hint, they could ask him. In total it was possible to request up to 3 hints (as many as you can find in the MCM-app). However, the experimenter specified that the students should try to respect a time between one hint and the other one, in order to ensure that the students reasoned on the given suggestion and to avoid helping them too much. The math-trail used was edit specifically for this outdoor meeting. The math-trail title is "Areas and perimeters: tribute to music and local myths"[2] (Figure 3).

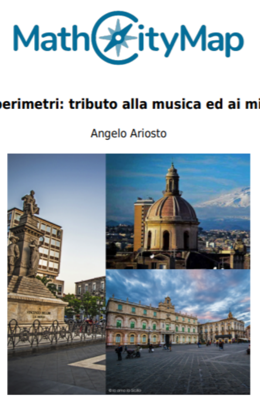

FIGURE 3. Locations of Math-trail designed with MCM.

[2]It is possible to find it on MCM-Portal or MCM-app with the Italian title: "Area e perimetri: tributo alla musica ed ai miti locali".

It contains 7 tasks located between two close squares of the city center (Piazza Stesicoro and Piazza Università, 480 m apart from each other – see Figure 4). Teachers let students to localize the tasks by themselves reading the MCM-map in their .pdf file.

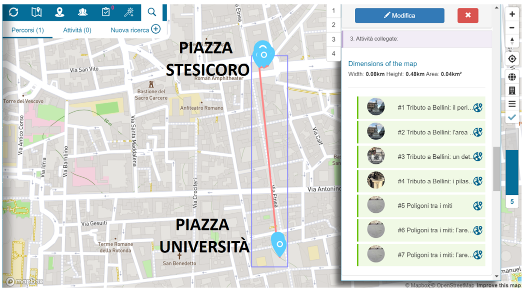

FIGURE 4. Math-trail designed with MCM.

All the students were divided in 4 groups (heterogeneous groups according students' skills and attitudes), with about 4 pupils each. Each group had the name of a legendary local myth (Colapesce, Fratelli Pii, Uzeda and Gammazita). The experimenter asked students to organized roles inside each groups (some of them read the task aloud, some of them took measures, others took notes, others thought how to solve the MCM-task, others computed). For the first MCM-tasks students respected the roles, but later on (as we can read in the final test filled by students) some of them redefined spontaneously the roles inside the group, in particular during the last tasks because of weather condition (that day it was raining). All the students had his/her personal block notes, calculator and measuring tape, as request by the experimenter, together with the math trail in .pdf. As you can see from the Meeting list, each outdoor activity was preceded by an indoor meeting, through which the experimenter could give those prerequisites required to approach outdoor problems. In this way, pupils could manage the MCM-tasks independently, arranged in collaborative groups.

All the groups solved successfully 3 of the 4 tasks located in Piazza Stesicoro, while not all the groups could try all the tasks located in Piazza Università. In fact, these last were all part of a block of tasks called "Polgygons among myths" and show several difficulties, both from the textual point of view and the mathematical point of view.
The difficulty of the tasks text results from several reasons:

- They are longer with respect to the previous ones;
- Together with mathematical information, they contain references to epic-literaly tales on the square;
- Some mathematical information are not necessary to solve the task;
- To locate the polygon, subject of the tasks, you have to follow specific geographic indications.

Moreover, from the mathematical point of view, some difficulties arose because the polygon to be measured (for perimeter and area):

- is very big;

- is a not regular one, it is an irregular heptagon.

Finally, as mentioned, the day students went around the city to solve these tasks, it was raining.
In the analysis section we will focus on two of the MCM-task located in Piazza Università. During the activity, the teachers and the experimenter recorded some videos and took some photos. Some dialogues have been extracted from the videos and will be reported in italic style in the following analysis. At the end of Meeting 2 the students were given a homework assignment: they had to rearrange the notes taken (measurements, counts, calculations, ...) during the activity in the notebook. The transcriptions on their notebooks were commented on together the following day, during Meeting 3. As already mentioned, in this meeting students could compare their solutions and debated about them. This was an important way to involve students to argue and do math discussion. They tried to explain how they have solved some tasks and debating different ways to solve a task. The experimenter supervised the debate and carried out students to think about different tasks solving. In the analysis we will also show some photos taken from the students' notebooks.

4. Data analysis

We analyse here the MCM-task "Polygons among myths: the perimeter of the boundary" (Figure 5).

Poligoni tra i miti: il perimetro del bordo	**Polygons among myths: the perimeter of the boundary**
La giovane Gammazita, il marinaio ColaPesce dalle doti subacquee, i fratelli Anfinomo e Anapia e il leggendario paladino catanese Uzeta. I quattro miti leggendari catanesi, ai quattro angoli della piazza, contemplano i poligoni sulla pavimentazione che li separano. Aiutali a capire di che poligoni si trattano! La pavimentazione della piazza presenta un motivo decorativo centrale romboidale con lo stemma della città, circondato da quattro poligoni uguali, il cui perimetro è contornato da blocchi di pietra lavica che costruiscono un "bordo" non trascurabile. Concentrati sul primo poligono che trovi alla tua destra guardando porta Uzeta e calcolane il perimetro esterno in metri.	The young Gammazita, the sailor Colapesce with underwater skills, the brothers Amphinomo and Anapia and the legendary paladin from Catania Uzeta. The four legendary myths of Catania, at the four corners of the square, contemplate the polygons on the floor that separate them. Help them understand what polygons they're talking about! The floor of the square has a rhomboidal central decorative motif with the coat of arms of the city, surrounded by four equal polygons, whose perimeter is surrounded by blocks of lava stone that build a not negligible "boundary". Concentrate on the first polygon on your right looking at Porta Uzeta and calculate the outer perimeter in meters.

Keywords	Plane geometry, polygons, irregular heptagon, perimeter.
Tools	Calculator, measuring tape.
Grade	6
Hints	1. Count the sides: it's just a heptagon! 2. Note the tiles of the boundary of the heptagon, what are their sizes? Are all the tiles the same? 3. Use the number of tiles and their length to find the measurements of the sides.

FIGURE 5. MCM-task "Polygons among myths: the perimeter of the boundary".

This task is one of the most difficult tasks, as explained in the methodology, but also the most appreciated by students, according to the analysis of the received feedbacks. The task asked to calculate the perimeter of the polygon sketched in Figure 6, so this is the real situation (RS) that the children have come across.

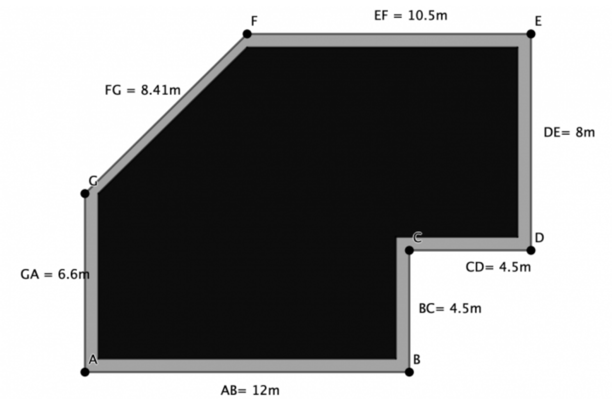

FIGURE 6. The heptagon whose perimeter is requested by the task.

The described polygon is a heptagon and its perimeter measures about 54.51 *m* and the settled solution MCM web portal is an interval[3]. Taking into account measurement errors, the expected solution is a value between 54 *m* and 56 *m* (green zone in Figure 7). Acceptable solutions are also those between 52 *m* and 58 *m* (orange zone in Figure 7). In the appendix A, we report a possible solution of the task.

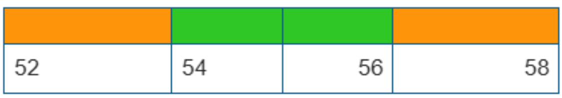

FIGURE 7. Perimeter interval of solution.

The difficulty to calculate the perimeter is related to the long sides of the polygon (Figure 6), and children had measuring tapes one-meter long. They had to face the problem of real world (in maths world you always have ready data) and each of them began to generate their own mental representation of the situation (MRS). By filtering the information about the RS, each child initiated the transition from MRS to real model (RM) by implementing problem-solving strategies. Some strategies adopted by children in different groups have been:

[3]The measures taken by the person who designs the task and/or the student who carries it out are subject to error. MCM, where it does not provide exact value as answers, suggests considering an interval set by the designer that identifies two zones: green if the answer is correct; orange if the answer is acceptable.

- S. decided to count how many steps she used to cover the whole perimeter;
- W., similarly, walked on the perimeter, but he used his longest step, saying that he was able to do 1 meter long steps, in such a way he will have been able to measure the perimeter more quickly than S.;
- V., I. and L. put together their measuring tapes in such a way to measure a whole side;
- D. tried to understand if there were equal sides, by comparing tiles.

Here we follow in detail the reasoning of a group. We analyse, in particular, the behaviour of A., G. and S. A. started jump on the edge of the heptagon. Then he decided to step and he counted the steps (Figure 8).

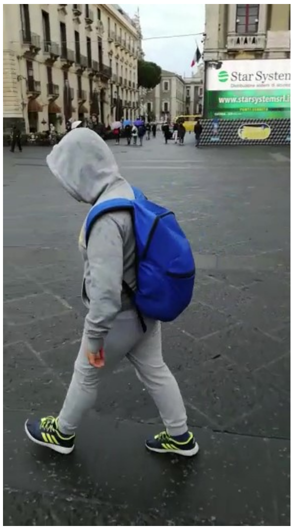

FIGURE 8. A. counting the tiles by steps, to get a rough measure of the boundary.

He had fun and found that you could use your body to measure. Here we report an excerpt of the dialogue between the experimenter, indicated by E., and A..
E.: *Can you say us what did you do? We saw you were jumping.*
A.: *Since it is 50* (meaning the measure of a single tile).
E.: *50... what?*
A.: *Centimetres. I went on: 100, 150, 200, 250, 300, 350, 400, 450, 500, 550, 600 ...*
E.: *But, they are all equal to each other, isn't it?*
A.: *Mmm... I think so ...* (Looking around) *Nooo, maybe this one is smaller than this one.*
E.: *Eh, we should measure them (Pause) Tatatatà!*
The experimenter used the famous incipit of Beethoven 5-th symphony, that, according to Beethoven himself, represents "the fate knocking at the door" (Figure 9).

FIGURE 9. Vocal sound used by the experimenter to provoke a doubt in the student.

A. had a rest for a few seconds and then he replied with "*taratataaaaaaam*", letting the fate entering the door.
With this musical stratagem, the experimenter is implicitly inviting A. to adjust the RM he has idealized. In fact, A. believes that all the tiles have the same length (50*cm* each). When A. repeats the sound tune, he understands that there is something in his RM that needs to be modified and agrees to consider the experimenter's suggestion.
E.: *Let's measure them! One of the two dimensions is equal, which one?* [...]
G.: *The height.*
E.: *We should see whether they also have the same base.*
A.: *According to me, they are the same.*
E.: *Let's measure. Do you have a measuring tape?*
A., along with teammate G., started measuring the tiles.
G.: *Then we can compute a rectangle*(meaning a tile).
E.: *and then?* G.:*and then multiplied for all.*
E.: *... if they* (the tiles) *are all equal to each other. If so, G.'s idea will let us save much time.*
Children are measuring regularly with a measuring tape with a regular ruler. A girl, S., is thinking
S.: *Wait, could you see the measure of a square* (a tile)*?*
Students: *30, 30.*
Student.: *I can count them all!*
And, running, he started counting the tiles. In the same time, the girl S. was thinking while looking and slowly walking.
S.: *I discovered something...*
E.: *What?*
S.: *These two* (two sides of the heptagon) *are the same* (sides *BC* and *CD* in figure 6)
E.: *Good, say it to your classmates who are noting the measures, in such a way they can use this fact. Boys! S. founds out a fact, maybe useful. Say it!*
S.: *These two* (sides *BC* and *CD* in figure 6) *are equal* [...]
S., in her actions of slow walking and evident state of reflection, is in the transition from RM to mathematical model (MM). She intuits that two sides seem to be equal. She measures them and then confirms her RM. At that point she shares her discovery with the experimenter who invites her to communicate it also to the other group mates.
At this time all the children of the group decided to use the strategy to use tiles, measuring one of them and counting them, but they had to face the difficulty of some tiles (the ones in the corners – in the vertex *F* and *G* in Figure 6) different from all the other ones.
V.: *But those half* [tiles]*, can't we... like...?* (She used the two hands to represent a cut as

you can see in Figure 10).

FIGURE 10. V. shows with gestures her idea to count two far half tiles as one.

V.: *...because over there, there is one, and you can only do like this ... this one is another one and then you sum this one* – she measured the tile in the corner (vertex F in Figure 6) and then she runs on the other corner (vertex G in Figure 6) – *with this one.*

FIGURE 11. V. measures one the two half tiles with her feet.

She compared the length of the two tiles. (Figure 11).
E.: *You are saying that if we sum this one and that one, we find out a whole* [tile].
We observe that the two tiles that the girl wants to measure and then add up are distant from each other (for the whole length of the FG side). V. is idealizing her own RM in which she believes that there are regularity links between the tiles. The experimenter, with his statement, repeats more clearly the verbal statement that V. shared with him with her gestures (cutting with her hands, running from one point to another and measuring the tiles with her feet).
V.: *Yes!*
E.: *This is a good idea, but this is a hypothesis that we should verify. We should take the ruler and see if it is true that this piece plus that piece of tile you show me over there, give the whole side of a tile. If it is true, you are right, good!*
Proceeding experimentally, V. discovers that her conjecture is wrong. The MM assumed, i.e.

assuming that the two tiles placed at the F and G vertexes have the same length, leads her to mathematical results that contradict the RM she had created. The experimenter, who did not immediately dismantle her conjecture, favoured her modelling process. This allowed V. not only to verify herself that the MM she had chosen was not the appropriate one, but it also prompted her to think about changing her resolution strategy by choosing a new RM. After measuring time, pupils went ahead, putting down on paper the data, and finally facing the problem from a strictly mathematical point of view (Figure 12), namely moving from the RM to the MM.

FIGURE 12. Students write on paper the get measures.

The mathematical results obtained by the students came very close to the real results of the problem. In fact, we can say that students solved very well this tasks, and the most of them found an acceptable solution (as the reader can sees in the picture taken from a student's notebook in Figure 13).

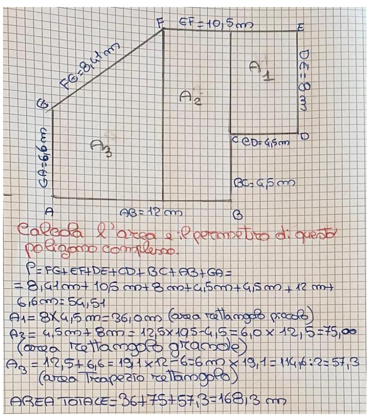

FIGURE 13. MM of the heptagon put on paper by a student.

We now move on to the analysis of another MCM-task: “Polygons among myths: the inner area” (Figure 14).

Poligoni tra i miti: l’area interna	**Polygons among myths: the inner area**
La pavimentazione della piazza presenta un motivo decorativo centrale romboidale con lo stemma della città, circondato da quattro poligoni uguali, il cui perimetro è contornato da blocchi di pietra lavica che costruiscono un “bordo” non trascurabile. Concentrati sul primo poligono che trovi alla tua destra guardando porta Uzeta. Aiuta i fratelli Pii a trovare l’area interna del poligono considerato (esprimi il risultato in m^2).	The floor of the square has a rhomboidal central decorative motif with the coat of arms of the city, surrounded by four equal polygons, whose perimeter is surrounded by blocks of lava stone that form a not negligible “boundary”. Concentrate on the first polygon on your right looking at Porta Uzeta . Help the Pii brothers to find the inner area of the polygon (express the result in m^2).

Keywords	Geometry, Trapezoid, Rectangle, Areas, Equivalence of plane surfaces.
Tools	Calculator, measuring tape.
Grade	6
Hints	1. The boundary has a constant thickness on the entire perimeter. 2. The thickness of the boundary is 50cm. 3. Decompose the polygon in question into the sum of a trapezoid and two rectangles.

FIGURE 14. MCM-task “Polygons among myths: the inner area”.

The task asked to calculate the internal area of the polygon (the black part in Figure 6), so this is the real situation (RS) that the children had come across.
The internal area is about 143.15 m^2 and the settled solution in MCM web portal is an interval. Taking into account measurement errors, the expected solution is a value between 141 m^2 and 145 m^2 (green zone in figure 15). They are also acceptable solutions those ones between 139 m^2 and 147 m^2 (orange zone in figure 15). Note that this heptagon is not the same of the previous task (Figure 6) but it is smaller and subsets of the previous one. We report a solution of the task in the appendix B.

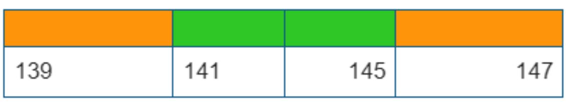

FIGURE 15. Area interval of solution.

This MCM-task presents a mathematical difficulty for the students involved in the experimentation. In fact, related to the calculus of area, students never worked on heptagons, they did not know any formula for this polygon and they were not familiar with irregular polygons. However, in the Meeting 1 the children had seen and played with the Tangram. The experimenter had insisted that the Tangram was a way to decompose polygons.
We report an excerpt from a dialogue among students and the experimenter, in particular, with the student M.
E.: *How can we do to find the area? First of all, is it a regular heptagon? What does regular mean?*
Students: *it is not regular, it hasn't all the sides equal to each other.*
E.: *So, children, how can we do to find this area, since it is not regular? We don't have a formula. Remember the Tangram.*
M.: *We should divide it into rectangles, squares, ...*
The experimenter accompanies the children in the MRS, that is he tries to recall a previous experience (the one made in the previous meeting with Tangram) to activate their visual imagination. M. remembers the experience immediately.
E.: *Oh! Very good! I.e. we use equidecomposability, so we divide this figure in polygons that are familiar to us. Let's see if we find rectangles, trapezoids, squares ... in such a way we can compute the area of each piece. Then, if we sum all the ...*
M.: (interrupting): *I already found a figure!*

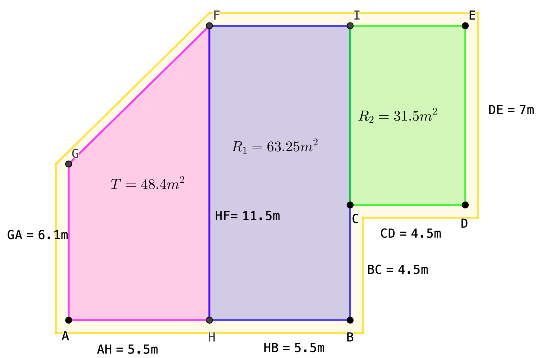

FIGURE 16. The heptagon decomposed in known figures in students' minds.

E.: *Which one?*
M.:*A trapeozoid.*
E.:*A trapezoid. And do we have a formula for a trapezoid?*
M.: *Yes! Long base plus short base, between brackets, times height, divided by two.*
Having played with the Tangram has allowed M. to "mentally" cut the heptagon to achieve figures, whose formulas he knows. By identifying a trapezoid (the pink one in Figure 16),

M. idealized and simplified the RS in a RM. Subsequently, the recall of the formula brings him into the MM phase. We observe that while explaining the formula, M. is processing its external representation. In fact, saying "*between brackets*" highlights how he is actually visualizing the formula in his mind.
E.: *Good, so as for the trapezoid, we can do it. Can you see other figures?*
The experimenter wanted to be sure that all the students caught the trapezoid, and asked, several times .
E.: *Have you seen the trapezoid?*
The experimenter is always working with the children in the MRS. He makes sure that everyone "sees" in their minds the figure, decomposed in known figures. He wants everyone to be able to model the same RM.
M.: *The trapezoid is this one. That one is the short base* (pointing at side *GA* in figure 16)*, the long base* (*HF* in figure 16) ... *we have to mark it by ourselves.*
M. is located between the RM and the MM. He has already understood that in order to apply the formula he called up, it is necessary to identify the major base. It is not visible in the RS, in fact it is necessary to "*mark it*" (mentally), while the shorter base is real and visible. The experimenter asked for the other polygons to be used, and they all found two rectangles (the violet one and the green one in Figure 16).
E.: *Ok, so two rectangles and a trapezoid. Remember this, then we can draw the figure. Even if it is not regular, we succeded to find a method to compute the area. We will do it in class.*
The work proceeded in class, where the experimenter realized that different groups found different decompositions of the heptagon, that were compared. Here we report, as an example, two different strategies (Figure 17).

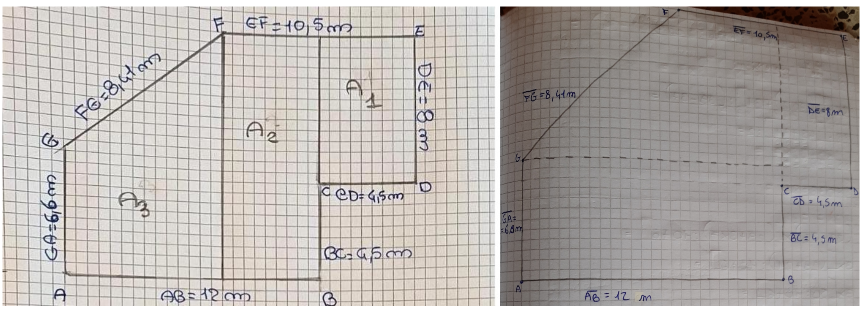

FIGURE 17. The heptagon decomposed in known figures in students' minds.

During Meeting 3, the whole class was able to retrace the shift from RM to MM, identifying a decomposition of the heptagon into known polygons. The measurements that had been taken on site were then shared with everyone in the class and the mathematical result was obtained. The children were able to see that, even if they considered different RMs and implemented different calculations related to the chosen MM, the final result was within the range of possible solutions.

5. Discussion and conclusion

First, let us say that students acquired those mathematical concepts, focused during the whole activity, as confirmed by final tests and by their teacher. We do not report here the details on mathematical skills, because the research question guiding the paper was on modelling, and precisely: Is the math-trail idea able to support primary school students in the process of modelling mathematics outside the classroom, around the city?
The carried out analyses, discussed in the previous section, show an acquired skill of modelling in analysed children. In the outdoor trip (meeting 2) they were able to start from a real situation, then generate a mental representation of that situation (for example, think about V. who is going to sum two far tiles, before being there to measure them). Then they passed to real models, when they measure, for instance, and also to mathematical model (we recall, for instance, M. who "see" the formula for the area of a trapezoid, even with brackets). The rest of the mathematization process was concluded in class, where strategies were compared, mathematical results were get, and an answer to the real situation was given.
We want to underline that in the experimentation, children were the main characters of their learning. The experimenter was there for hints (we recall that children were not allowed to use smartphones, and so they had not access to the hints in the app), but he never said more than what was useful to help students in the modelling phase. If some child showed an idea that could be used as a strategy, he did not say "right" or "wrong", but rather "if it is true, you are right". In such a way, he led them to think by themselves, favouring a process of mathematization. For example, when A. went on counting the tiles, without posing the problem if they are all equal, the experimenter did not just say that it is not true, but rather he posed the question (are they equal?) highlighting the question with a sound suggesting a doubt (from the Beethoven fifth symphony). Or when V. hypothesized that two half tiles could be counted as a whole, he took her to verify whether the conjecture is right or wrong.
At the end of the activity the experimenter submitted a questionnaire to the 17 students. In table 1 we report the answers' percentage to the following questions from this questionnaire:

(1) How much did you learn?
(2) How much were explanations clear?
(3) Were city's activities joint with classroom's activities?
(4) Have you seen mathematics around you?
(5) How much mathematics can you now notice around you?
(6) How much did you like MathCityMap?

Question	Not at all	Barely	A little	Sufficiently	A lot	Extremely
(1)	0%	0%	0%	24%	47%	29%
(2)	0%	0%	0%	41%	35%	24%
(3)	0%	0%	0%	18%	41%	41%
(4)	0%	0%	0%	0%	18%	82%
(5)	0%	0%	0%	12%	18%	71%
(6)	0%	0%	0%	18%	29%	53%

TABLE 1. Percentages of answers to the final questionnaire.

As we can read from the data, the 76% of pupils thought to have learned a lot or extremely. The 82% of them found (a lot or extremely) bonds between *in* and *out* activities, knocking over that wall between classroom's and city's environments, because they finally see mathematics not only in teacher's explanations and books, but also around them, and not only during the experimentation (100%), but 89% of them also after the experimentation. Children appreciated MCM (for a total of 82% of very satisfied pupils), as we can also deduce from the following free comments: "I want another trip equal to this one"; "We love MathCityMap"; "It was beautiful to know the teachers. We saw mathematics around us, everywhere, not only in class".

Last, we want to point out the effectiveness of the *in-out-in* methodology. When using the MCM-Project for teaching (whether you are using the app or not), it is very important to get to the mathematical point of the question with the students: in order to make this happen, it is useful to plan preparation meetings (like the first in the experimentation proposed in this paper) or discussion meetings (like the third or the fifth) in class. In this way we can make sure that all puplis have the knowledge to afford the trail and, if some student miss something, the teacher has a chance to assess the class.

We can finally say that children learned some mathematics, saw mathematics in the city, started modelling real situations with mathematical objects and, last but not least, perceived the outdoor lessons as trips, having fun with mathematics!

6. Acknowledgments

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Appendix A. Solution of "The perimeter of the boundary"

The polygon can be approximated to one irregular heptagon. To compute the measures of the sides let's observe that the border of the figure is constituted from rectangular tiles wide $0.50m$ long and of variable length between $0.63m$ or $0.75m$ [4]. More, in the calculation of the measures of the sides as the sum of the sides of the rectangles of the tiles, the account should be taken of the space between tiles (approximately $0.005m$ long). Following the letters of the attached Figure 18, the sides measures are:

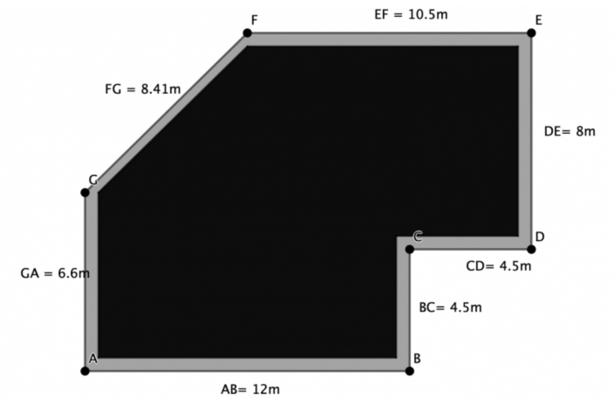

FIGURE 18. Sides measures of heptagon.

$$
\begin{aligned}
AB &= 12m \\
BC = CD &= 4.50m \\
DE &= 8.00m \\
EF &= 10.5m \\
FG &= 8.41m \\
GA &= 6.6m
\end{aligned}
$$

So the perimeter P of the $ABCDEFG$ heptagon is

$$P = AB + BC + CDDE + EF + FG + GA = 54.51m$$

[4]The first and last tile of the FG side are not exactly rectangular (but pentagonal), for the calculation of the outer perimeter this difference is negligible as it is sufficient to pay attention to the outer sides that are part of the perimeter, noting that these are exceptions in that: - the one that has as extreme G is about $0.73m$ long - the one that has as extreme F is about $0.42m$ long.

Appendix B. Solution of “The inner area"

This is a problem of equivalence of areas. To calculate the area of the inner heptagon $ABCDEFG$ one possible way is to think it composed by a rectangle trapezoid ($AHFG$) and two rectangles ($HBIF$ and $CDEI$) as in the figure.

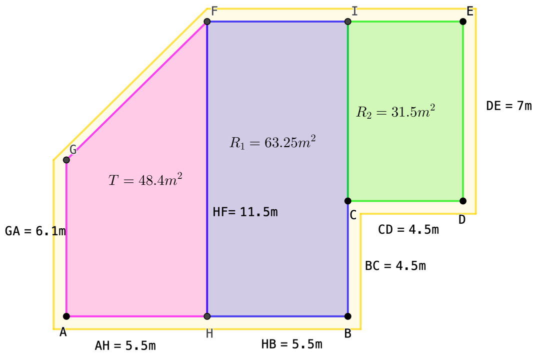

FIGURE 19. Sides measures of heptagon.

Called T the area of the trapezoid $AHFG$, we have that:

$$T = \frac{(HF + GA) \times AH}{2} = \frac{(11.5m + 6.1m) \times 5.5m}{2} = 48.4m^2$$

Called R_1 the area of the rectangle $HBIF$, we have that:

$$R_1 = HB \times HF = 5.5m \times 11.5m = 63.25m^2.$$

Called R_2 the area of the rectangle $CDEI$, we have that:

$$R_2 = CD \times DE = 4.5m \times 7m = 31.5m^2.$$

Therefore, called E_{int} the area of the heptagon $ABCDEFG$, we have that:

$$E_{int} = T + R_1 + R_2 = 48.4m^2 + 63.25m^2 + 31.5m^2 = 143.15m^2.$$

References

Anichini, G., Arzarello, F., Ciarrapico, L., and Robutti, O. (2004). *Matematica 2003*. Lucca: Matteoni stampatore.

Ariosto, A. (2019). “Le idee umane dietro e dentro la matematica: MathCityMap e il Laboratorio di Matematica”. Master thesis. Università degli Studi di Catania. (Visited on).

Blane, D. and Clark, D. (1984). *A Mathematics Trail Around the City of Melbourne*. Monash Mathematics Education Centre, Monash University.

Blum, W., Galbraith, P. L., Henn, H., W., and Niss, M. (2007). *Modelling and applications in mathematics education*. New York: Springer. DOI: 10.1007/9780387298221.

Blum, W. and Leiß, D. (2005). "How do Students and Teachers Deal with Modelling Problems? The example "Sugarloaf"". In: *ICTMA 12 proceedings*, pp. 222–231. DOI: 10.1533/9780857099419.5.221.

Borromeo Ferri, R. (2006). "Theoretical and empirical differentiations of phases in the modeling process." *ZDM* **38(2)**, 86–95. DOI: 10.1007/BF02655883.

Kaiser, G. (2005). "Mathematical Modelling in School – Examples and Experiences. In Kaiser, G. and Henn, H.-W. (Eds.)" In: *Mathematikunterricht im Spannungsfeld von Evaluation und Evolution*. Hildesheim: Franzbecker. DOI: 10.1007/9783834823892.

Lesh, R. and Doerr, H. (2003). *Beyond Constructivismen - Models and Modeling Perspectives on Mathematics Problem Solving, Learning and Teaching*. Mahwah: Lawrence Erlbaum. DOI: 10.1080/10986065.2003.9680000.

Ludwig, M. and Jablonski, S. (2019). "Doing Math Modelling Outdoors - A Special Math Class Activity designed with MathCityMap". In: *HEAD'19. 5th International Conference on Higher Education Advances*. Editorial Universitat Politècnica de València., pp. 901–909.

Shoaf, M., Pollak, H., and Schneider, J. (2004). *Math Trails*. Lexington: COMAP.

Vos, P. (2011). "What is 'authentic' in the teaching and learning of mathematical modelling? G. Kaiser, W. Blum, R. Borromeo Ferri, and G. Stillman (Eds.)" In: *Trends in teaching and learning of mathematical modelling: ICTMA 14*. New York: Springer.

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