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A CLASS OF SETS WHERE CONVERGENCE IN HAUSDORFF SENSE AND IN MEASURE COINCIDE

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ABSTRACT. We introduce a class of uniformly bounded closed sets such that, inside the class, convergence in Hausdorff sense and in measure do agree. We also show that the class is rich enough for applications to potential theory.

1. Introduction

A certain inverse problem in potential theory could find an easy solution if, given a fixed closed ball B in $\mathbb{R}^n$, it would be possible to exhibit a family of subsets which is relatively compact with respect to a topology compatible with convergence in measure. Such is for instance the generalization to L^∞ of the results of Sansò (2014). It is well known that the whole class of the closed sets inside a ball is compact for the Hausdorff metric topology, but in general the last and convergence in measure are not comparable. Aim of this short note is to single out an interesting subclass of closed subsets of B such that the two topologies on the restricted class agree. This class is also closed, and thus the compactness result holds as well. Similar results could be obtained by restricting the attention to closed sets with finite perimeter. Inside this class, it is possible to get compactness results exploiting the compact embedding of BV into L^1 (see Theorem 1.19 of Giusti (1984), Theorem 3.39 of Ambrosio *et al.* (2000)). Our result is however of different nature, it includes sets whose characteristic function is not of bounded variation.

2. Preliminaries

We shall consider a fixed closed ball K in some Euclidean space, and we shall denote by $c(K)$ the family of the nonempty closed (hence compact) sets contained in K. We start by giving some notions about set topologies.

Definition 1. *Given two closed sets C, D, we define the excess of C over D as $e(C,D) = \sup\{d(c,D) : c \in C\}$, and similarly the excess of D over C. We also define the following*

This paper is dedicated to Antonino Maugeri *distance on $c(K)$:*

$$H(C,D) = \max\{e(C,D), e(D,C)\}$$

It is usual to adopt the notation C^ε to denote the (closed) ε-expansion of a set C:

$$C^\varepsilon = \{x \in K : d(x,C) \le \varepsilon\}.$$

Thus Hausdorff convergence of a sequence $(C_n)_n$ to C can be proved by showing that for every $\varepsilon > 0$ eventually the two following inclusions hold:

$$C \subset C_n^\varepsilon \qquad \wedge \qquad C_n \subset C^\varepsilon.$$

It is well known that H defines a complete metric on $c(K)$ (for more on set convergence, we refer the reader to the books of Beer (1993) and Lucchetti (2006)). Furthermore, the space is also compact. To see this, we can for example make the following remarks: Hausdorff convergence of C_n to C is equivalent to uniform convergence of the family of functions $(f_n)_x = d(x,C_n)$ to the limit function $f_x = d(x,C)$ (see Lucchetti (2006), Theorem 8.2.12), while Kuratowski convergence is equivalent to the pointwise convergence of the same sequence of functions (see Lucchetti (2006), Proposition 8.2.9). Since K is a compact set, and the functions are equilipschitz and equibounded then pointwise and uniform convergence do coincide and thus the Hausdorff metric is equivalent to the topology of Kuratowski convergence. Moreover Kuratowski convergence is compatible with the Fell topology (see Lucchetti (2006), Proposition 8.2.4), which is compact (Theorem 8.4.4 of Lucchetti (2006), taking into account that compactness of K guarantees that a Kuratowski limit of a sequence is always a nonempty set). For the same proof, see also Falconer (1986). It follows in particular that every sequence $(C_n)_n$ of sets inside $c(K)$ has a subsequence converging to some $C \in c(K)$.

Together with the above set convergence notion, we want to consider also the convergence in measure, for sets in $c(K)$. Denote by μ the Lebesgue measure. A sequence $(C_n)_n$ (of Lebesgue measurable sets) converges in (Lebesgue) measure to C if

$$\mu[(C \setminus C_k) \cup (C \setminus C_k)] \to 0.$$

Denoting by I_C the characteristic function of the set C, i.e.

$$I_C(x) = \begin{cases} 1, & \text{if } x \in C \\ 0, & \text{otherwise} \end{cases}$$

it is easy to see that this convergence is equivalent to $L^1(K)$ convergence of the characteristic functions (see §II.2 in Yosida (1995)).
From now on, we shall use the notation $C \div D$ to denote the symmetric difference of two sets C and D:

$$C \div D = (C \setminus D) \cup (D \setminus C).$$

It is well known that in general convergence in measure and in Hausdorff sense are unrelated: as an easy example, consider $K = [0,1]$, denote by $N = \{x_n : n \in \mathbb{N}\}$ a countable dense subset of K and set $C_n = \bigcup_{k \le n} x_k$. Then $C_n \to K$ in Hausdorff sense but it converges to any null measure (closed) set: thus the same sequence cannot have the same limit in the two senses. Moreover, it is easy to see that, even if we restrict convergence in measure to the family of the closed sets contained in a compact set, convergence in measure does not enjoy the relative compactness properties of the Hausdorff metric. The following is a simple example showing this.

Example 1. *Let $X = [0,1]$ and consider the following sequence of sets:*

$$C_1 = \left[0, \frac{1}{2}\right],\ C_2 = \left[0, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right]\ \dots\ C_n = \left[0, \frac{1}{2^n}\right] \cup \left[\frac{2}{2^n}, \frac{3}{2^n}\right] \cup \dots \cup \left[\frac{2^n-2}{2^n}, \frac{2^n-1}{2^n}\right].$$

It is standard to see that the sequence of the characteristic functions of the sets weakly converges in $L^1[0,1]$ to the function valued $\frac{1}{2}$ everywhere in $[0,1]$, thus no subsequence of C_n can converge in measure to a set.

However observe that if a sequence converges both in Hausdorff and in measure sense, this does not imply that the limit set has boundary of null measure (take for instance a nowhere dense compact set C of positive measure on the interval $[0,1]$ and $C_n = C$ for every n).

Remark 1. In the sequel, we shall repeatedly use the following simple facts, concerning the two convergence modes, for closed sets C:

(1) For every $a, b > 0$ and (nonempty) set C; $(C^a)^b = C^{a+b}$;
(2) $C_n \to C$ implies $C_n^a \to C^a$ for every $a > 0$, where convergence is in Hausdorff sense;
(3) $\mu(\bigcap_{\varepsilon>0} C^\varepsilon) = \mu(C)$.

As mentioned in the introduction, we are interested in finding conditions under which convergence in measure and Hausdorff senses agree. To this end, we are led to define a subfamily of $c(K)$ in the following way: take a continuous increasing real valued function $h : [0,\infty) \to [0,\infty)$ such that $h(0) = 0$, and consider the set family $\mathscr{F}_h$ so defined:

$$\mathscr{F}_h = \{C \in c(K) : \mu(C^\varepsilon) - \mu(C) < h(\varepsilon), \varepsilon > 0\}.$$

Our main result wants to show that if we restrict our attention to the family $\mathscr{F}_h$, the convergence in Hausdorff sense and in measure do agree.

Theorem 1. *Let $(C_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}_h$ converging in Hausdorff sense to C. Then C_n converges to C in measure.*

Proof Given $\varepsilon > 0$ since it holds that for all large n

$$C_n \subset C^\varepsilon \subset C_n^{2\varepsilon}$$

it also follows that for sufficiently large n, m

$$C_n \subset (C_m)^{2\varepsilon} \quad \wedge \quad C_m \subset (C_n)^{2\varepsilon}$$

implying that

$$\mu(C_n \div C_m) \le 2h(2\varepsilon)$$

for all large n, m and this implies that $(C_n)_n$ is Cauchy sequence in measure. It follows that the sequence of the indicator functions is Cauchy in $L^1(K)$, thus it converges to some function which is the indicator function of some set. We can then conclude that there exists a set D such that $C_n \to D$ in measure. We now prove that $\mu(C \div D) = 0$, so that we can assume that $D = C$, which will prove the theorem. First of all, fix $a > 0$ and observe that

$$D \setminus C = \bigcup_{\varepsilon>0} (D \setminus C^\varepsilon).$$

Thus there are $b > 0$ and sufficiently large n such that

$$\mu(D \setminus C_n) \leq a \quad \wedge \quad D \setminus C^b \subset D \setminus C_n \quad \wedge \quad \mu(D \setminus C) \leq \mu(D \setminus C^b) + a.$$

It follows that

$$\mu(D \setminus C) \leq \mu(D \setminus C^b) + a \leq \mu(D \setminus C_n) + a \leq 2a$$

which means that $\mu(D \setminus C) = 0$. Let us now see that $\mu(C \setminus D) = 0$. Fix $a > 0$ and take $\varepsilon > 0$ and n large enough such that

$$C \subset C_n^{\varepsilon} \quad \wedge \quad h(\varepsilon) < a \quad \wedge \quad \mu(C_n \setminus D) \leq a.$$

Since

$$C \setminus D \subset C_n^{\varepsilon} \setminus D \subset (C_n^{\varepsilon} \setminus C_n) \cup (C_n \setminus D)$$

if follows that $\mu(C \setminus D) \leq 2a$ and from this we conclude. ■

Corollary 1. *The family $\mathscr{F}_h$ is closed in the Hausdorff topology.*

Proof Take a sequence $(C_n)_n$ in $\mathscr{F}_h$ converging to some set C and fix $\varepsilon > 0$. We need to prove that $\mu(C^{\varepsilon} \setminus C) < h(\varepsilon)$. Fix a arbitrary $a, b > 0$ and take n so large that

$$C^{\varepsilon} \subset C_n^{\varepsilon + a} \qquad \wedge \qquad C_n \subset C^b.$$

Then

$$C^{\varepsilon} \setminus C^b \subset C_n^{\varepsilon + a} \setminus C_n$$

and thus

$$\mu(C^{\varepsilon} \setminus C^b) \leq h(\varepsilon + a).$$

By continuity of h, it follows that $\mu(C^{\varepsilon} \setminus C^b) \leq h(\varepsilon)$. Since this is true for all b, this implies the thesis. ■

Corollary 2. *Let $(C_n)_{n \in \mathbb{N}}$ be a sequence of sets in $\mathscr{F}_h$. Then $(C_n)_{n \in \mathbb{N}}$ has a subsequence converging in measure.*

Proof The result follows from the compactness result mentioned in the preliminaries, relative to the Hausdorff convergence topology. ■

Corollary 3. *Let $(C_n)_n \subset \mathscr{F}_h$, and suppose $C_n \to C$ in Hausdorff sense. Then $\mu(C_n) \to \mu(C)$.*

Proof This is obvious from Theorem 1 since

$$|\mu(C_n) - \mu(C)| \leq \mu(C_n \div C). \quad ■$$

We now provide an example of a set belonging to the class $\mathscr{F}_h$, for a suitable choice of the function h, but whose perimeter is not finite. Thus our result can be applied to sets like that one in the example, but tools from the analysis of functions of bounded variation do not apply.

Example 2. Let C be the following subset of $\mathbb{R}^2$:

$$C = \bigcup_{n=0}^{\infty} C_n \; : \; C_0 = \{(0, y) : 0 \leq y \leq 1\},$$

$$C_n = \{(x, y) : \frac{1}{2^{n-2}} - \frac{1}{2^{n+2}} \leq x \leq \frac{1}{2^{n-2}} + \frac{1}{2^{n+2}}, 0 \leq y \leq 1\}, n \geq 1.$$

It is easy to show that the length of the boundary of C is infinite, on the other hand $\mu(C^\varepsilon \setminus C) \leq 2\varepsilon + 2\varepsilon a$, where $a = |\{n : C_n \nsubseteq C_0^\varepsilon\}|$. Since $a < 2 + \log_2(\frac{1}{\varepsilon})$, it is enough to take $h(\varepsilon) = k\varepsilon \log_2(\frac{1}{\varepsilon})$, for suitable $k > 0$, to conclude.

The next question of interest is: how rich is the family of sets $\mathscr{F}_h$, for suitable h? Here we propose a class, which is rich enough for the application to inverse problems in potential theory.

Definition 2. *We say that a set D in $\mathbb{R}^n$ is* star shaped *if there exists $x_0 \in D$ such that for every $x \in D$, $tx + (1-t)x_0 \in D$, $0 \leq t \leq 1$.*

In other words the line segment connecting any $x \in D$ with x_0, the center of the star, belongs to D. We shall consider a special subclass of star shaped sets, that we now define.

Given a star shaped set D with x_0 as the center of the star, call $C(x, x_0, a)$ the set $\{y \in K : \langle y, x - x_0\rangle \geq |y||x - x_0|a\}$, let $\mathscr{S}$ be the family of the star shaped sets contained in K, and consider the following subset of $\mathscr{S}$:

$$\mathscr{S}_a = \{D \in \mathscr{S} : {x_0}^1 \in \operatorname{int} D \wedge [x + C(x, x_0, a)] \cap D = \{x\}\ \forall x \in \partial D\}$$

We want to prove the following theorem.

Theorem 2. *There exists $C > 0$ such that the class of the star shaped sets S_a belongs to $\mathscr{F}_h$, for $h(\varepsilon) = C\varepsilon$.*

Proof Given the star shaped set $D \in \mathscr{S}$, with center x_0, define the following set, for a fixed $\eta > 0$:

$$R_\eta = \{\bigcup_{x \in \partial D} t(x - x_0) : 1 \leq t \leq 1 + \eta\}.$$

The first step of the proof consists in showing that $D^\varepsilon \setminus D \subset R_\eta$, with $\eta = \frac{\varepsilon}{\sqrt{1-a^2}}$. To see this, fix $z \in D^\varepsilon \setminus D$. Let $\bar{x} \in \partial D$ be such that $x = bz + (1-b)x_0$, for some $b \in (0,1)$. Let $t = (1-b)||z - x_0||$ be the distance between z and x. Then

$$d(z, \partial[x + C((x, x_0, a)]) = t\sqrt{1-a^2} \leq d(z, D) = \varepsilon.$$

It follows that $t \leq \frac{\varepsilon}{\sqrt{1-a^2}} = \eta$, so that $z \in R_\eta$.
The above inclusion allows us to conclude by majorizing the measure of the set R_η: consider the spherical change of coordinates $x \rightsquigarrow (r, \sigma)$, denote by σ an arbitrary element in the boundary of the unit ball and by r_σ the (unique) element in the boundary of D whose ray contains σ. Thus

$$\mu(R_\eta) = \int_\Sigma d\sigma_{n-1} \int_{r_\sigma}^{r_\sigma+\eta} r^{n-1}\, dr = \frac{1}{n}\int_\Sigma d\sigma_{n-1}\left[(r_\sigma + \eta)^n - (r_\sigma)^n\right] \leq c\eta \leq C\varepsilon$$

for some suitable C. ■

Remark 2. Let us make the following observations:

[1] x_0 is the center of the star D.

(1) When we defined the special class of star shaped sets, we ask that any cone with vertex at a point x of the boundary of D and given angle $\theta = \arccos a$ does not intersect the set D at any other point but x: of course we only need a local condition of this type, thus actually we could ask that the cone, intersected with a ball of small radius $b > 0$, has the unique point x in common with D; of course the choice of b must be the same for every point of the boundary;

(2) Clearly, any union, up to at most a fixed integer N, of elements taken form classes $\mathscr{F}_h$ (for some h) still belongs to some class $\mathscr{F}_h$. Thus for instance sets which are union of a set in S_a and a set like that one in Example 2 belong to some suitable class $\mathscr{F}_h$;

(3) The whole family of the star shaped sets cannot be included in any $\mathscr{F}_h$. It is enough to consider the following: take a dense countable $\{x_n\}_n$ set on the boundary of the unit ball, let A_n be the line segment $[0, x_n]$ and set $C_n = \cup_{k \leq n} A_k$. A simple variant of the above example adding the condition that the set have nonempty interior still does not guarantee inclusion in any $\mathscr{F}_h$. However we do not know if the condition above that the center of the star be an interior point of the set is necessary to get the result.

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