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UNIQUE CONTINUATION FOR DEGENERATE QUASILINEAR EQUATIONS AND SUM OPERATORS

GIUSEPPE DI FAZIO [a], MARIA S. FANCIULLO [a] AND PIETRO ZAMBONI [a*]

ABSTRACT. We prove unique continuation property for positive solutions of some quasilinear degenerate elliptic equations.

1. Introduction

This paper is a contribution to the study of qualitative properties of solutions of degenerate elliptic equations. The project we are focused to consists with two main kind of properties: the smoothness of weak solutions and the unique continuation property. Some of our previous contributions are contained in the paper of Zamboni (2002) and Di Fazio *et al.* (2008, 2011, 2015a, 2020b) while our contribution here is answering the second issue. We prove a unique continuation result for a quasilinear degenerate elliptic equation that are degenerate according to a suitable geometry introduced by Franchi *et al.* (2003). The resulting setting is so general that we cannot ensure neither the validity of a (1-1) Poincaré nor the validity of the usual sub representation formula as in the papers of Franchi *et al.* (1994) and Lu and Wheeden (1998). We overcome this difficulty by using a different sub representation formula in term of a so called sum operator and, accordingly, the assumptions on the lower order terms are shaped on the geometry of the space.

The equation we are going to study is the following

$$\sum_{i=1}^{m} X_i^* A_i(x,u,Xu) + B(x,u,Xu) = 0$$

where

$$\begin{cases} |A(x,u,\xi)| \le aw(x)|\xi|^{p-1} + b|u|^{p-1} \\ |B(x,u,\xi)| \le c|\xi|^{p-1} + d|u|^{p-1} \\ \xi \cdot A(x,u,\xi) \ge w(x)|\xi|^p - d|u|^p \end{cases}$$

for a suitable doubling weight w satisfying condition (A3) in section 2 and

$$a \in \mathbb{R}, \ \left(\frac{b}{w}\right)^{p/p-1}, \left(\frac{c}{w}\right)^p, \ \frac{d}{w} \ \in \tilde{S}_p(\Omega)$$

(see section **2** for definitions and also Sawano *et al.* (2020a,b)).

We prove that any weak solution of the above equation which is nonnegative in a given domain Ω cannot have infinite order zeroes unless it is identically zero in Ω. The assumption on lower order terms are minimal for the smoothness of the weak solutions.

2. Stummel-Kato type classes and Fefferman - Phong inequality

Let $X = (X_1, X_2, \ldots, X_m)$ be a system of locally Lipschitz vector fields in $\mathbb{R}^n$ and d be the associated Carnot-Carathéodory distance. We assume that $d(x,y) < \infty$ for any $x, y \in \mathbb{R}^n$.

We denote by $B = B_r = B(x,r)$ the Carnot-Carathéodory (C-C) ball centered at $x \in \mathbb{R}^n$ of radius r.

Throughout the paper we shall assume the following

(A1) The distance d is continuous with respect to the Euclidean distance in $\mathbb{R}^n$.

(A2) Let w be a finite Borel measure, absolutely continuous with respect to the Lebesgue measure. We assume the following doubling condition, i.e. there exists a positive constant C_D such that $\forall x \in \mathbb{R}^n, r > 0$,

$$w(B(x,2r)) \leq C_D w(B(x,r))$$

where $w(B(x,r)) = \int_{B(x,r)} w\,dy$.

(A3) If B_0 is a given ball in $\mathbb{R}^n$, $p > 1$, there exists a positive constant C_P such that

$$\frac{1}{w(B)} \int_B |u - u_B| w dy \leq C_P\, r \left(\frac{1}{w(B)} \int_B |Xu|^p w dy \right)^{1/p}$$

for all $B \subset B_0$ and all $u \in C^\infty(\overline{B}_0)$. Here $u_B = \frac{1}{w(B)} \int_B u w\,dy$ and r is the radius of B.

The number $Q = \log_2 C_D$ will be called the homogeneous dimension of $\mathbb{R}^n$.

Now we prove an useful lemma that has been proved by Di Fazio *et al.* 2020a in a more general case.

Lemma 2.1. *For all $x_0 \in \mathbb{R}^n$ and $r > 0$*

$$w(B(x_0,r)) \leq \gamma w(B(x_0,2r)),$$

where $\gamma = \frac{C_D^3}{C_D^3 + 1}$.

Proof. Let $x_0 \in \mathbb{R}^n$, $r > 0$ and $x \in \mathbb{R}^n$ such that $d(x_0,x) = \frac{3}{2} r$. We have $B(x_0,r) \cap B(x,r/2) = \emptyset$ and $B(x,r/2) \subseteq B(x_0,2r)$, from which $B(x,r/2) \subseteq B(x_0,2r) \setminus B(x_0,r)$ and

$$w(B(x,r/2)) \leq w(B(x_0,2r)) - w(B(x_0,r)).$$

Moreover $B(x_0,r) \subseteq B(x, 2^3 r/2)$, then, from $(A2)$

$$\begin{aligned} w(B(x_0,r)) \leq w(B(x_0,2^3 r/2)) \leq C_D^3 w(B(x,r/2)) \leq \\ \leq C_D^3 [w(B(x_0,2r)) - w(B(x_0,r))], \end{aligned}$$

from which the thesis follows. □

In the sequel we will need Sobolev spaces and Stummel-Kato type spaces with respect to the doubling measure $w\,dx$.

Definition 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $1 \leq p \leq \infty$. We say that u belongs to $W^{1,p}(\Omega,w)$ if u, $X_i u \in L^p(\Omega,w)$ for $i = 1,\dots m$. We denote by $W_0^{1,p}(\Omega,w)$ the closure of the smooth and compactly supported functions in $W^{1,p}(\Omega,w)$ with respect to the norm*

$$\|u\|_{W^{1,p}(\Omega,w)} = \|u\|_{L^p(\Omega,w)} + \sum_{i=1}^{m} \|X_i u\|_{L^p(\Omega,w)}.$$

We say that u belongs to $W^{1,p}_{loc}(\Omega,w)$ if $u \in W^{1,p}(\Omega',w)$ for any $\Omega' \Subset \Omega$.

In order to state the sub representation formula we are going to use, we recall some definitions.

Definition 2.2. *(see Franchi* et al. *2003) Given $B_0 = B(x_0,r)$ and $x \in B_0$, let us denote by $\{B_i\} = \{B_i(x)\}_{i=1}^{\infty}$ a chain of balls, of radius $r(B_i)$, such that*
- (H1) *$B_i \subset B_0$ for all $i \geq 0$*
- (H2) *$r(B_i) \sim 2^{-i} r(B_0)$ for all $i \geq 0$*
- (H3) *$d(B_i,x) \leq C r(B_i)$ for all $i \geq 0$*
- (H4) *for all $i \geq 0$, $B_i \cap B_{i+1}$ contains a ball S_i such that $r(S_i) \sim r(B_i)$.*

Remark 2.1. Carnot-Carathéodory metric satisfies the segment property, i.e. given $x, y \in \mathbb{R}^n$ there exists a continuous curve $\gamma : [0,T] \to \mathbb{R}^n$ connecting x and y such that $d(\gamma(t),\gamma(s)) = |t-s|$ for all s, $t \in [0,T]$ (see Franchi *et al.* 1998, Remark 2.6 and Franchi *et al.* 2003, Remark 2). Then, in a Carnot-Carathéodory metric space $\forall x \in B_0 = B(x_0,r)$ there exists a chain of balls satisfying (H1), (H2), (H3) and (H4). An example is given by the family of balls given by $\{B(x, 2^{-i} d(x,\partial B_0))\}_{j=1}^{\infty}$.

Theorem 2.1. *(Franchi* et al. *2003) Given B_0 a ball, let $\{B_j(x)\}_{j=1}^{\infty}$ be a chain of balls as in Definition 2.2. Let $p > 1$ and $u \in W^{1,p}(B_0,w)$ be such that for any ball $B \subset B_0$ the following $(1-p)$ Poincaré inequality hold true*

$$\frac{1}{w(B)} \int_B |u - u_B| w dx \leq Cs \left(\frac{1}{w(B)} \int_B |Xu|^p w dy \right)^{1/p}$$

where s is the radius of the ball B.

Then, there exists $C' > 0$ such that

$$|u(x) - u_{B_0}| \leq C' \sum_{j=0}^{\infty} r(B_j(x)) \left(\frac{1}{w(B_j(x))} \int_{B_j(x)} |Xu|^p w(y) dy \right)^{1/p}$$

where C' is a geometric constant which also depends on C.

In order to handle lower order terms of the differential equation we now introduce the Stummel classes adapted to our setting. Our definition is a generalization of the classical one that was first introduced by Aizenman and Simon (1982).

Definition 2.3. *Let B_0 a ball and $\{B_j(x)\}_{j=1}^{\infty}$ be a chain of balls as in Definition 2.2. We say that $V \in L^1_{loc}(\mathbb{R}^n,w)$ belongs to the space $\tilde{S}_p(\mathbb{R}^n,w)$ if*

$$\eta_V(r) \equiv \sup_{x_0 \in \mathbb{R}^n} \sup_{y \in B_0} \int_{B_0} \sum_{j=0}^{\infty} \frac{r^p(B_j(x)) |V(x)|}{w(B_j(x))} \chi_{B_j(x)}(y) w(x) dx$$

is finite for all $r > 0$.

Now we can state and prove the weighted embedding result that will allow us to get our main results (see also Di Fazio *et al.* (2015b), Di Fazio and Zamboni (2002), Di Fazio and Zamboni (2006), Lu (1996), Zamboni (1987), Zamboni (2001), Di Fazio *et al.* (2013) Ragusa and Zamboni (2001) and Di Fazio *et al.* (2019)).

Theorem 2.2. *Let V be a function in $\tilde{S}_p(\mathbb{R}^n, w)$. Then, there exists a positive constant C such that*

$$\int_{B_0} |V|\,|u-u_{B_0}|^p w\,dx \leq C\,\eta_V(r) \int_{B_0} |Xu|^p w\,dx$$

*for any $u \in C^\infty(B_0)$. Here η_V and B_0 are as in the Definition **2.3**.*

Proof. If u is any smooth function in B_0 Theorem **2.1** yields the following sub representation formula

$$|u(x)-u_{B_0}| \leq C \sum_{j=0}^{\infty} r(B_j(x)) \left(\frac{1}{w(B_j(x))} \int_{B_j(x)} |Xu|^p w(y)dy \right)^{1/p} \tag{1}$$

for a.e. $x \in B_0$. Now from (1) and Hölder inequality

$$\begin{aligned}
&\int_{B_0} |V(x)||u(x)-u_{B_0}|^p w(x)dx \leq \\
&\qquad \leq C \int_{B_0} |V(x)||u(x)-u_{B_0}|^{p-1} \sum_{j=0}^{\infty} r(B_j(x)) \cdot \\
&\qquad \cdot \left[\frac{1}{w(B_j(x))} \int_{B_j(x)} |Xu(y)|^p w(y)dy \right]^{1/p} w(x)dx \leq \\
&\qquad \leq C \left[\int_{B_0} |V(x)||u(x)-u_{B_0}|^p w(x)dx \right]^{\frac{p-1}{p}} \cdot \\
&\qquad \cdot \left[\int_{B_0} \sum_{j=0}^{\infty} |V(x)| \frac{r^p(B_j(x))}{w(B_j(x))} \int_{B_j(x)} |Xu(y)|^p w(y)dy w(x)dx \right]^{1/p} \leq \\
&\qquad \leq C \left[\int_{B_0} |V(x)||u(x)-u_{B_0}|^p w(x)dx \right]^{\frac{p-1}{p}} \cdot \\
&\qquad \cdot \left[\int_{B_0} \sum_{j=0}^{\infty} |V(x)| \frac{r^p(B_j(x))}{w(B_j(x))} \int_{B_0} |Xu(y)|^p \chi_{B_j(x)}(y) w(y)dy\, w(x)dx \right]^{1/p} \leq
\end{aligned}$$

$$\leq C\left[\int_{B_0}|V(x)||u(x)-u_{B_0}|^p w(x)dx\right]^{\frac{p-1}{p}}\cdot$$

$$\cdot\left[\int_{B_0}|Xu(y)|^p\int_{B_0}\sum_{j=0}^{\infty}|V(x)|\frac{r^p(B_j(x))}{w(B_j(x))}\chi_{B_j(x)}(y)w(x)dx\,w(y)dy\right]^{1/p}\leq$$

$$\leq C\left[\int_{B_0}|V(x)||u(x)-u_{B_0}|^p w(x)dx\right]^{\frac{p-1}{p}}\eta_V^{1/p}(r)\cdot$$

$$\cdot\left[\int_{B_0}|Xu(y)|^p w(y)dy\right]^{1/p}$$

from which

$$\int_{B_0}|V(x)||u(x)-u_{B_0}|^p w(x)dx\leq C\eta_V(r)\int_{B_0}|Xu(x)|^p w(x)dx.$$

□

Corollary 2.1. *Let V a function in $\tilde{S}_p(\mathbb{R}^n,w)$. Then, there exists a positive constant C such that*

$$\int_{B_0}|V|\,|u|^p w\,dx\leq C\,\eta_V(2r)\int_{B_0}|Xu|^p w\,dx$$

for any $u\in C_0^\infty(B_0)$.

Proof. Let x_0 and r the center and the radius of B_0 respectively. From Theorem **2.2**

$$\int_{B(x_0,r)}|V(x)|\,|u(x)|^p w\,dx\leq C\int_{B(x_0,r)}|V|\,|u-u_{B(x_0,r)}|^p w\,dx+$$
$$+C\int_{B(x_0,r)}|V|\,|u_{B(x_0,r)}|^p w\,dx\leq C\eta_V(r)\int_{B(x_0,r)}|Xu(x)|^p w\,dx+$$
$$+C\int_{B(x_0,r)}|V|\,|u_{B(x_0,r)}|^p w\,dx. \quad (2)$$

In order to estimate the second integral in the right hand side of (2) we observe that, since u has compact support in B_0,

$$u_{B(x_0,r)}=\frac{w(B(x_0,2r))}{w(B(x_0,2r))-w(B(x_0,r))}\left(u_{B(x_0,r)}-u_{B(x_0,2r)}\right)$$

and, from Lemma **2.1**,

$$\frac{w(B(x_0,2r))}{w(B(x_0,2r))-w(B(x_0,r))}\leq 1+C_D^3.$$

Then, since $\eta_v(r)\leq\eta_V(2r)$ and from Theorem **2.2**, we have

$$\int_{B(x_0,r)}|V|\,|u_{B_0}|^p w\,dx\leq C\int_{B(x_0,r)}|V|\,|u_{B(x_0,r)}-u_{B(x_0,2r)}|^p w\,dx\leq$$
$$\leq C\int_{B(x_0,r)}|V|\,|u-u_{B(x_0,r)}|^p w\,dx+C\int_{B(x_0,2r)}|V|\,|u-u_{B(x_0,2r)}|^p w\,dx\leq$$
$$\leq C\eta_V(2r)\int_{B(x_0,r)}|Xu|^p w\,dx. \quad (3)$$

The thesis follows from (2) and (3).

□

We list some other useful definitions.

Definition 2.4. *Let f be a locally integrable function in $\mathbb{R}^n$ with respect to w. We say that f belongs to the class BMO_w if*

$$\sup_B \frac{1}{w(B)} \int_B |f(x) - f_B| w(x) dx < +\infty$$

where the supremum is taken over C-C balls.

Definition 2.5. *Let u be a locally integrable function such that $\frac{1}{u}$ is also locally integrable. We say the u belongs to the class $A_2(w)$ if*

$$\int_B u(x)w(x)dx \int_B \frac{1}{u(x)} w(x)dx \leq c(w(B))^2$$

for any C-C ball B.

Theorem 2.3. *Let u be a function in the class $A_2(w)$. Then*

$$\int_{B_{2r}} uwdx \leq C \int_{B_r} uwdx$$

for any C-C ball B_r.

Proof. The proof is similar to that of Theorem 4.5 proved by Di Fazio and Zamboni 2010. □

Definition 2.6. *Let Ω be a bounded domain of $\mathbb{R}^n$. Let μ be a locally integrable function such that $\frac{1}{\mu}$ is locally integrable and $u \in L^1_{loc}(\Omega)$. We say that u has a zero of infinite order at $x_0 \in \Omega$ if*

$$\lim_{\sigma \to 0} \frac{\int_{B(x_0,\sigma)} u(x)\mu(x)dx}{(\mu(B(x_0,\sigma)))^k} = 0, \quad \forall k > 0.$$

Now we can prove the following

Theorem 2.4. *Let $u \in L^1_{loc}(\Omega)$, $u \geq 0$ a.e. in Ω, $u \not\equiv 0$. If there exist positive numbers C and σ_0 such that*

$$\int_{B(x_0,2\sigma)} uwdx \leq C \int_{B(x_0,\sigma)} uwdx$$

for any $0 < \sigma < \sigma_0$ and any $x_0 \in \Omega$, then u cannot have zeros of infinite order in Ω.

Proof. We argue by contradiction. Let $x_0 \in \Omega$ be a zero of infinite order for u. Using Theorem **2.3**, Hölder inequality and Lemma **2.1** k times we obtain

$$\int_{B(x_0,r)} uwdz \le C^k \int_{B(x_0,\frac{r}{2^k})} uwdz =$$

$$= C^k w\left(B\left(x_0,\frac{r}{2^k}\right)\right)^\beta \frac{1}{w\left(B\left(x_0,\frac{r}{2^k}\right)\right)^\beta} \int_{B\left(x_0,\frac{r}{2^k}\right)} uw\,dx \le$$

$$\le C^k \gamma^{k\beta} \frac{w(B(x_0,r))^\beta}{w\left(B\left(x_0,\frac{r}{2^k}\right)\right)^\beta} \int_{B\left(x_0,\frac{r}{2^k}\right)} uw\,dx.$$

Now, if we choose $\beta = \log_\gamma C^{-1}$, we obtain

$$\int_{B(x_0,r)} uwdz \le \frac{w(B(x_0,r))^\beta}{w\left(B\left(x_0,\frac{r}{2^k}\right)\right)^\beta} \int_{B\left(x_0,\frac{r}{2^k}\right)} uw\,dx,$$

from which, as $k \to \infty$, we have that $uw \equiv 0$ in B_r and, since $w \not\equiv 0$, $u \equiv 0$. □

3. Unique continuation for variational quasilinear equations

Let Ω be a bounded domain in $\mathbb{R}^n$. Let $X = (X_1, X_2, \dots, X_m)$ be a system of locally Lipschitz vector fields in $\mathbb{R}^n$. For $i = 1, 2, \dots, m$ we denote by X_i^* the formal adjoint of the vector fields X_i. Let w be a weight satisfying assumption (A3).

We consider the following quasilinear equation

$$\sum_{i=1}^m X_i^* A_i(x,u,Xu) + B(x,u,Xu) = 0 \tag{4}$$

in a given bounded domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} |A(x,u,\xi)| \le aw(x)|\xi|^{p-1} + b|u|^{p-1} \\ |B(x,u,\xi)| \le c|\xi|^{p-1} + d|u|^{p-1} \\ \xi \cdot A(x,u,\xi) \ge w(x)|\xi|^p - d|u|^p \end{cases} \tag{5}$$

where X_i^* is the formal adjoint of the vector fields X_i and

$$a \in \mathbb{R}\,,\ \left(\frac{b}{w}\right)^{p/p-1},\left(\frac{c}{w}\right)^p,\ \frac{d}{w},\ \in \tilde{S}_p(\Omega,w)\,. \tag{6}$$

Definition 3.1. *A function $u \in W^{1,p}_{loc}(\Omega,w)$ is said to be a weak solution of* (4) *in Ω if*

$$\sum_{j=1}^m \int_\Omega A_j(x,u(x),Xu(x))X_j\varphi(x)\,dx = \int_\Omega B(x,u(x),Xu(x))\varphi(x)\,dx \tag{7}$$

for every $\varphi \in W_0^{1,p}(\Omega,w)$.

We remark that the integrals appearing in (7) are all finite because of (5).

Theorem 3.1. *Let $u \in W^{1,p}(\Omega, w)$, $u \geq 0$, $u \not\equiv 0$, be a weak solution of* (4) *satisfying* (5) *and* (6). *Then u has no zero of infinite order in Ω.*

Proof. Let $x_0 \in \Omega$, $B(x_0, r)$ be a C-C ball such that $B(x_0, 2r)$ is contained in Ω. Consider any B_h contained in $B(x_0, r)$. Let $\eta \in W_0^{1,p}(\Omega, w)$ be a nonnegative function supported in B_{2h}. Using $\phi = \eta^p u^{1-p}$ as test function in (7) we get

$$\int_\Omega |Xu|^p \eta^p u^{-p} w\, dx \leq c \left\{ \int_\Omega |X\eta|^p w\, dx + \int_\Omega V \eta^p w\, dx \right\}$$

that is

$$\int_\Omega |X \log u|^p \eta^p w\, dx \leq c \left\{ \int_\Omega |X\eta|^p w\, dx + \int_\Omega V \eta^p w\, dx \right\} \tag{8}$$

where

$$V = \left(\frac{b}{w}\right)^{\frac{p}{p-1}} + \left(\frac{c}{w}\right)^p + \frac{d}{w} \in \tilde{S}_p(\Omega, w).$$

Now we want to estimate the second term in the right hand side of (8). Using Corollary **2.1**, we get

$$\int_\Omega V \eta^p w\, dx \leq c \int_\Omega |X\eta|^p w\, dx.$$

Inserting this inequality in (8), we obtain

$$\int_\Omega \eta^p |X \log u|^p w\, dx \leq c \int_\Omega |X\eta|^p w\, dx. \tag{9}$$

Choosing η so that $\eta = 1$ in B_h and $|X\eta| \leq \frac{c}{h}$, by (9) we have

$$\int_{B_h} |X \log u|^p w\, dx \leq c \frac{w(B_h)}{h^p}.$$

Therefore, by (A4) we have $\log u \in BMO_w$ and from John-Nirenberg lemma (see Buckley (1999)) it follows that there exists $\delta > 0$ such that is u^δ belongs to $(A_2)_w$. By Theorem **2.3** u^δ has the doubling property, i.e. the assumption of Theorem **2.4**. So the conclusion follows for u^δ and hence also for u. □

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[a] Università degli Studi di Catania
Dipartimento di Matematica e Informatica,
Viale Andrea Doria, 6, Catania, Italy

* To whom correspondence should be addressed | email: zamboni@dmi.unict.it

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