DOI: 10.1478/AAPP.98S2A2

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 98, No. S2, A2 (2020)

SET-VALUED ORTHOGONALITY AND NEARNESS

ANNAMARIA BARBAGALLO [a], OCTAVIAN-EMIL ERNST [b]
AND MICHEL THÉRA [c*]

ABSTRACT. The theory of set-valued mappings has grown with the development of modern variational analysis. It is a key in convex and non-smooth analysis, in game theory, in mathematical economics and in control theory. The concepts of nearness and orthogonality have been known for functions since the pioneering works of Campanato, Birkhoff and James. In a recent paper Barbagallo *et al.* [J. Math. Anal. Appl., 484 (1), (2020)] a connection between these two concepts has been made. This note is mainly devoted to introduce nearness and orthogonality between set-valued mappings with the goal to study the solvability of generalized equations involving set-valued mappings.

In seminal papers published at the end of the eighties, Campanato (1989, 1993, 1994) introduced and studied the notion of *nearness* between two functions defined on a set S and taking values in a real normed vector space $(X, \|\cdot\|)$. More precisely, given two functions f and g defined on S with values in X, we say that f is near g in the sense of Campanato, iff there exist two real constants $\alpha > 0$ and $\kappa \in [0,1)$ such that

$$\|(f(s_1) - \alpha g(s_1)) - (f(s_2) - \alpha g(s_2))\| \leq \kappa \|f(s_1) - f(s_2)\|, \tag{1}$$

for every $s_1, s_2 \in S$.

When X is a Banach space, Campanato (1989, Theorem 1 & 2) proved that if f is bijective and f is near g with constants α and κ, then g is necessarily a bijection. Moreover, the Lipschitz modulus of the bijective function $g \circ f^{-1} : X \to X$ is less than or equal to $\frac{\alpha}{1-\kappa}$, namely

$$\|g(f^{-1}(u)) - g(f^{-1}(v))\| \leq \frac{\alpha}{1-\kappa}\|u - v\|, \qquad \forall u, v \in X.$$

When $S = X$, f is the identity operator and g is a linear function, Campanato results boil down to the well-known Neumann's Lemma (Hansen 2016, Lemma 5.1.6), see also (Barbagallo *et al.* 2020) for more details and extensions.

Lemma 0.1 (Neumann's lemma). *Let X be a real Banach space, and A be a bounded linear operator on X, such that*

$$\exists\ \alpha > 0,\ 0 \leq \kappa < 1 : \quad \|Id - \alpha A\| \leq \kappa \|Id\|, \tag{2}$$

Dedicated to Professor Antonino Maugeri *where* $\|\cdot\|$ *is the operator norm defined by*

$$\|A\| := \inf\{c \geq 0 : \|A(x)\| \leq c\,\|x\|,\ \forall x \in X\}.$$

Then A *is invertible, and* $\|A^{-1}\| \leq \frac{\alpha}{1-\kappa}$.

In a locally convex space, a notion strongly related to Campanato nearness is Birkhoff-James orthogonality (see, Mazaheri and Kazemi (2008, Definition 1.1)). Given a vector space Y equipped with a family of semi-norms

$$\mathscr{P} := \{p_\iota : Y \to \mathbb{R} :\ \iota \in I\},$$

we say that the vector $u \in Y$ is *Birkhoff-James orthogonal* to $v \in Y$ on $(Y, \mathscr{P})$ iff

$$p_\iota(u) \leq p_\iota(u - t\,v), \qquad \forall t \in \mathbb{R},\ \iota \in I. \tag{3}$$

Of course, when Y is a normed space, $\mathscr{P}$ contains only one element, namely the norm of Y, and we retrieve the original Birkhoff's definition (see Birkhoff (1935) and James (1945); the reader is also referred to the very complete survey by Alonso *et al.* (2012)).

An abstract notion of Campanato's nearness can be defined as follows: given two vectors u and v in a locally convex space $(Y, \mathscr{P})$, we say that u is Campanato near v iff there exist two real constants $\alpha > 0$ and $\kappa \in [0,1)$ such that

$$p_\iota(u - \alpha\,v) \leq \kappa\,p_\iota(u), \qquad \forall \iota \in I. \tag{4}$$

The original Campanato's nearness is obtained for the particular case of the locally convex space $(X^S, \mathscr{P})$, where X^S denotes the set of all functions from S to X, and the set $\mathscr{P}$ contains all semi-norms of the form

$$p_{s_1,s_2} : X^S \to \mathbb{R},\quad p_{s_1,s_2}(f) := \|f(s_1) - f(s_2)\|,\quad \forall f \in X^S, \tag{5}$$

for all the points $s_1, s_2 \in S$ such that $s_1 \neq s_2$. As no confusion risks to occur, we will drop, in the remaining part of this note, the wording “Birkhoff-James" and “Campanato", and simply speak of orthogonality and nearness.

The main object of this note is to extend the definitions of orthogonality and nearness from the original case of single-valued functions, to the case of set-valued mappings. As many operators of interest in non-smooth optimisation are set-valued - like the subdifferential of a convex function, to pick an example out of many - a correct definition of nearness covering the set-valued case should be a valuable tool in establishing set-valued generalizations of the Neumann and Campanato results.

This note is organized as follows. Section **1** addresses several very simple attempts to define set-valued orthogonality and nearness. Their coherence and possible uses are discussed in the light of several examples. As a consequence of this analysis, we propose in Section **2** a new definition of orthogonality and nearness in the set-valued setting.

1. Three attempts of defining set-valued orthogonality and nearness

This section is devoted to the study of three attempts to define orthogonality and nearness for set-valued mappings. The first one is based on the notion of Pompeiu-Hausdorff's distance in a metric space X, while the two others are stated in terms of the selections of the two involved set-valued mappings. The limitations of these three attempted definitions are highlighted by some elementary examples.

1.1. Distance-based nearness : first attempt of a definition. Let us consider the Pompeiu-Hausdorff distance, defined between two subsets of X by the following formula:

$$\mathbf{d}(A,B) := \max(\sup_{v\in B}\inf_{u\in A}\|u-v\|,\ \sup_{u\in A}\inf_{v\in B}\|u-v\|), \quad \forall A,B\subset X.$$

Let us recall that by a set-valued mapping (multifunction, correspondence, point-to-set, in some other terminologies) F, we mean a function between S and the set $\mathscr{P}(X)$ of all the subsets (possibly empty) of X. Throughout the paper, we will use the notation $F : S \rightrightarrows X$.

A simple transposition to the set-valued setting of the original definition of orthogonality and nearness may lead to the following tentative definition.

Definition 1. *Given two set-valued mappings $F,G : S \rightrightarrows X$, we say that F is orthogonal to G iff*

$$\boldsymbol{d}(F(s_1)-tG(s_1),F(s_1)-tG(s_2))) \geq \boldsymbol{d}(F(s_1),F(s_2)) \qquad \forall t\in\mathbb{R},\ s_1,s_2\in S,$$

and F is near G iff there are two real constants $\alpha>0$ and $\kappa\in[0,1)$ such that

$$\boldsymbol{d}(F(s_1)-\alpha G(s_1),\ F(s_1)-\alpha G(s_2))) \leq \kappa\, \boldsymbol{d}(F(s_1),\ F(s_2)), \qquad \forall s_1,s_2\in S.$$

The next proposition reveals that using this definition, there are set-valued mappings which are not near themselves.

Proposition 1.1. *Let S be a set containing at least two points, and X be a real normed vector space containing at least one non-null vector. Then, there is a set-valued mapping $F : S \rightrightarrows X$ which is not near itself, according to Definition 1.*

Proof. Let s_1 and s_2 two different elements of S, and consider the set-valued mapping $F : S \rightrightarrows X$ given by

$$F(s) = \begin{cases} \mathscr{B}_X & \text{if} \quad s = s_1 \\ \{0\} & \text{if} \quad s \neq s_1 \end{cases},$$

where $\mathscr{B}_X$ is the closed unit ball in X.

Let us pick a positive real number t. Since X contains at least a non-null element, the Pompeiu-Hausdorff distance between the sets $t\mathscr{B}_X$ and $\{0\}$ amounts to t. Accordingly, the Pompeiu-Hausdorff distance between the two sets

$$F(s_1)-\alpha F(s_1) = (1+\alpha)\mathscr{B}_X, \qquad F(s_2)-\alpha F(s_2) = \{0\}$$

equals to $1+\alpha$, and we deduce that

$$\mathbf{d}(F(s_1)-\alpha G(s_1),\ F(s_2)-\alpha G(s_2)) = 1+\alpha > \kappa = \kappa\,\mathbf{d}(F(s_1),\ F(s_2))$$

for any two real constants $\alpha>0$ and $\kappa\in[0,1)$. □

Let us remark that the existence of set-valued mappings which are not near themselves is still achieved even when the Pompeiu-Hausdorff distance is replaced by one of the numerous distances between sets available in the mathematical literature.

It is easy to verify that, if $\delta : 2^X \times 2^X \to \mathbb{R}_+$ is any of the distances described in the survey paper by Conci and Kubrusly (2017), then the function

$$e : \mathbb{R}_+ \to \mathbb{R}_+, \qquad e(s) := \delta(\{0\}, s\mathscr{B}_X), \ \forall s \geq 0$$

is increasing. Accordingly, Proposition **1.1** holds true even if the Pompeiu-Hausdorff distance is replaced in Definition 1 by another distance between sets. It appears thus that no valid definition of set-valued orthogonality and nearness can be achieved by using distance functions.

1.2. Selections-based nearness: second attempt of a definition. Given $F : S \rightrightarrows X$, we call *selection* of F any function $\sigma_F : S \to X$ such that $\sigma_F(s) \in F(s)$, for any $s \in S$.

Our second very simple attempt to define a correct notion of set-valued orthogonality and nearness is based on the analysis of all the selections of the two set-valued mappings involved.

Definition 2. *Let $F, G : S \rightrightarrows X$. Then F is orthogonal to (respectively near) G iff any selection of F is orthogonal to (respectively near) any selection of G.*

Once again, this definition leads to the existence of set-valued mappings which fails to be near themselves.

Proposition 1.2. *Let S be a set containing at least two points, and X be a real normed vector space containing at least one non-null vector. Then, there is a set-valued mapping $F : S \rightrightarrows X$ which is not near itself, according to Definition 2.*

Proof. Let us prove that the constant set-valued mapping $F : S \rightrightarrows X$,

$$F(s) := \mathscr{B}_X, \qquad \forall s \in S,$$

is not near itself according to Definition 2.

Since the vector space X contains non-null vectors, let us choose a vector u, such that $\|u\| = 1$. Picking now a pair (s_1, s_2) of two different elements of S, we observe that the functions

$$\sigma_1 : S \to X, \qquad \sigma_1(s) := \begin{cases} u, & \text{if} \quad s = s_1 \\ -u, & \text{if} \quad s \neq s_1 \end{cases}$$

and

$$\sigma_2 : S \to X, \qquad \sigma_2(s) := \begin{cases} u, & \text{if} \quad s \neq s_1 \\ -u, & \text{if} \quad s = s_1 \end{cases}$$

are two selections of the set-valued mapping F. But

$$\|(\sigma_1(s_1) - \alpha\,\sigma_2(s_1)) - (\sigma_1(s_2) - \alpha\,\sigma_2(s_2))\| = 2(1+\alpha) > 2\kappa = \kappa\|\sigma_1(s_1) - \sigma_1(s_2)\|$$

for any two real constants $\alpha > 0$ and $\kappa \in [0,1)$, completing the proof. □

1.3. Selections-based nearness: third attempt of a definition. In view of the above, two first attempts to define set-valued nearness, we have identified cases of set-valued mappings not being near themselves. This means that these attempts are too restrictive. Accordingly, in a further step, we will try a selection-based definition considerably broader than Definitions 1 and 2.

Definition 3. *A set-valued mapping F is orthogonal to (respectively near) G iff any selection of F is orthogonal (respectively near) to at least one selection of G, and for any selection of G, there is at least one selection of F which is orthogonal to (respectively near) it.*

It is now obvious that any set-valued mapping is near itself, in the sense of Definition 3. Thus, the main difficulty plaguing Definitions 1 and 2 is now removed. However, Definition 3 is not satisfactory since, unlike the single-valued case, basic algebraic properties are not inherited from F to G in case F is near G, as proved by the following result.

Proposition 1.3. *Let $S := \{s_1, s_2\}$ be a set containing two points, and X be a real normed vector space containing at least one non-null vector. Then, there is a set-valued mapping $F : S \rightrightarrows X$ which is injective, in the sense that $F(s_1) \neq F(s_2)$, and which is near a constant set-valued mapping G, according to Definition 3.*

Proof. Fix $u \in X$, $u \neq 0$, and define

$$F : S \rightrightarrows X, \qquad F(s_1) := \{-u, u\},\ F(s_2) := \{0, u\}$$

and

$$G : S \rightrightarrows X, \qquad G(s_1) = G(s_2) := \{0, u\}.$$

Each of the two set-valued mappings F and G has exactly four selections. More precisely:

$$\sigma_{F,1}(s_1) := -u,\ \sigma_{F,1}(s_2) := 0, \qquad \sigma_{F,2}(s_1) := -u,\ \sigma_{F,2}(s_2) := u$$

$$\sigma_{F,3}(s_1) := u,\ \sigma_{F,3}(s_2) := 0, \qquad \sigma_{F,4}(s_1) := u,\ \sigma_{F,4}(s_2) := u$$

are the selections of F, while the selections of G are

$$\sigma_{G,1}(s_1) := 0,\ \sigma_{G,1}(s_2) := 0, \qquad \sigma_{G,2}(s_1) := 0,\ \sigma_{G,2}(s_2) := u$$

$$\sigma_{G,3}(s_1) := u,\ \sigma_{G,3}(s_2) := 0, \qquad \sigma_{G,4}(s_1) := u,\ \sigma_{G,4}(s_2) := u.$$

It is straight-forward to prove that $\sigma_{F,1}$ is near $\sigma_{G,2}$, that $\sigma_{F,2}$ is near $\sigma_{G,2}$, that $\sigma_{F,3}$ is near $\sigma_{G,3}$, that $\sigma_{F,4}$ is near $\sigma_{G,1}$, and that $\sigma_{F,4}$ is near $\sigma_{G,4}$. So, the set-valued mappings F and G satisfy conditions of Definition 3. However, F is injective, while G is constant. □

2. Nearness for set-valued mappings

In view of the detailed analysis achieved in Section **1**, we conclude that the correct definition of set-valued orthogonality and nearness should be broader than Definition 2, but more restrictive than Definition 3. In order to attain this objective, let us introduce the following notations. Given $F, G : X \rightrightarrows X$, we say that a binary relation $\mathscr{R}$ on X is *(F,G)-compatible* if

$$\forall s \in S, \forall u \in F(s), \exists v \in G(s) \quad \text{s.t.} \quad u \mathscr{R} v,$$

and

$$\forall s \in S, \forall v \in G(s), \exists u \in F(s) \quad \text{s.t.} \quad u \mathscr{R} v.$$

Moreover, a selection σ_G of the set-valued mapping G is said *$\mathscr{R}$-compatible* with one selection σ_F de F, if $\sigma_F(s)\,\mathscr{R}\,\sigma_G(s)$ for any $s \in S$.

We are now ready to state the main notion of this note.

Definition 4. *A set-valued mapping F is said to be orthogonal (respectively near) to G iff there exists a (F,G)-compatible binary relation $\mathscr{R}$ on X such that any selection σ_F of F is orthogonal to (respectively near) any $\mathscr{R}$-compatible selection σ_G of G.*

It is now easy to see that any set-valued mapping F is near itself (just take for $\mathscr{R}$ the identity relation, $x\mathscr{R}y \iff x = y$, remark that $\mathscr{R}$ is (F,F)-compatible, and notice that any selection σ_F of F is $\mathscr{R}$-compatible only with itself).

Given a set-valued mapping F, in a future article, we will investigate the properties which are inherited by any set-valued mapping G such that F is near G.

Acknowledgments

Research of A. Barbagallo is partially supported by PRIN 2017 Nonlinear Differential Problems via Variational, Topological and Set-valued Methods (Grant 2017AYM8XW).

Research of M. Théra is supported by the Australian Research Council (ARC) grant DP160100854 and benefited from the support of the FMJH Program PGMO and from the support of EDF.

References

Alonso, J., Martini, H., and Wu, S. (2012). "On Birkhoff orthogonality and isosceles orthogonality in normed linear spaces". *Aequationes Mathematicae* **83**(1-2), 153–189. DOI: 10.1007/s00010-011-0092-z.

Barbagallo, A., Ernst, O.-E., and Théra, M. (2020). "Orthogonality in locally convex spaces: Two nonlinear generalizations of Neumann's lemma". *Journal of Mathematical Analysis and Applications* **484**(1), 123663. DOI: 10.1016/j.jmaa.2019.123663.

Birkhoff, G. (1935). "Orthogonality in linear metric spaces". *Duke Mathematical Journal* **1**(2), 169–172. DOI: 10.1215/S0012-7094-35-00115-6.

Campanato, S. (1989). "A Cordes type condition for nonlinear nonvariational systems". *Accademia Nazionale delle Scienze detta dei XL. Rendiconti. Serie V. Memorie di Matematica e Applicazioni. Parte I* **13**(1), 307–321.

Campanato, S. (1993). "Further contribution to the theory of near mappings". *Le Matematiche* **48**(1), 183–187.

Campanato, S. (1994). "On the condition of nearness between operators". *Annali di Matematica Pura ed Applicata.* **167**(4), 243–256. DOI: 10.1007/BF01760335.

Conci, A. and Kubrusly, C. (2017). "Distances between set-a survey". *Advances in Mathematical Sciences and Applications* **26**(1), 1–18. arXiv: 1808.02574.

Hansen, V. L. (2016). *Functional analysis. Entering Hilbert space. Second edition*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ. DOI: 10.1142/9892.

James, R. C. (1945). "Orthogonality in normed linear spaces". *Duke Mathematical Journal* **12**(2), 291–302. DOI: 10.1215/S0012-7094-45-01223-3.

Mazaheri, H. and Kazemi, R. (2008). "The orthogonality in the locally convex spaces". *Taiwanese Journal of Mathematics* **12**(5), 1101–1106. DOI: 10.11650/twjm/1500574250.

[a] University of Naples Federico II
Department of Mathematics and Applications "R. Caccioppoli"
Via Cintia, Monte S.Angelo
80126 Napoli, Italy

[b] Aix-Marseille Université
Institut de Mathématiques de Marseille
I2M UMR 7373, 13453 Marseille Cedex, France

[c] Université de Limoges, Laboratoire XLIM
UMR-CNRS 6172, France
Federation University
Centre for Informatics and Applied Optimization
Australia

* To whom correspondence should be addressed | email: michel.thera@unilim.fr

Paper contributed to the meeting on "Variational Analysis, PDEs and Mathematical Economics",
held in Messina, Italy (19–20 September 2019), on the occasion of Prof. Antonino Maugeri's 75th birthday,
under the patronage of the *Accademia Peloritana dei Pericolanti*

Manuscript received 31 January 2020; published online 13 December 2020