

REGULARITY FOR NONLINEAR ELLIPTIC EQUATIONS AND SYSTEMS

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ABSTRACT. We study the regularity of *weak solutions* to the elliptic system in divergence form $\operatorname{div} A(x, Du) = 0$ in an open set Ω of \mathbb{R}^n , $n \geq 2$. The vector field $A(x, \xi)$, $A : \Omega \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$, has a variational nature in the sense that $A(x, \xi) = D_{\xi} f(x, \xi)$, where $f : \Omega \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is a convex *Carathéodory integrand*; i.e., $f = f(x, \xi)$ is measurable with respect to $x \in \mathbb{R}^n$ and it is a convex function with respect to $\xi \in \mathbb{R}^{m \times n}$. If $m = 1$ then the system reduces to a partial differential *equation*. In the context $m > 1$ of general vector-valued maps and *systems*, a classical assumption finalized to the *everywhere regularity* of the weak solutions is a modulus-dependence in the energy integrand; i.e., we require that $f(x, \xi) = g(x, |\xi|)$, where $g : \Omega \times [0, \infty) \rightarrow [0, \infty)$ is measurable with respect to $x \in \mathbb{R}^n$ and it is a convex and increasing function with respect to the gradient variable $t \in [0, \infty)$.

1. Introduction

I had the opportunity to write these notes in the occasion of the meeting "*Variational Analysis, PDE's and Mathematical Economics*", which took place on September 19th - 20th, 2019, in Messina, at the *Accademia Peloritana dei Pericolanti*, to celebrate Antonino Maugeri's 75th birthday. It was a pleasure for me to celebrate Antonino and here I like to summarize what I said in that occasion.

I recalled a paragraph written in the web site of this meeting: *In honour of Professor Antonino Maugeri: his scientific activity has been concerned with the existence and regularization theory for linear and nonlinear elliptic and parabolic equations and systems with discontinuous coefficients ... partial and global Hölder continuity for nonlinear systems, both in the variational and nonvariational case ... variational inequalities and their applications to equilibrium problems ... financial equilibrium problems ...* This was a motivation for me to give a conference in this meeting at the *Accademia Peloritana dei Pericolanti* about *regularity for nonlinear elliptic equations and systems*. In the following sections I will describe some recent results about these topics. Here, for more precision, I can refer to some of scientific publications by Antonino Maugeri in these fields, also joint with some of his coauthors. For instance I refer to Daniele *et al.* (2007), Marino and Maugeri (1989), Marino and Maugeri (1995), Maugeri *et al.* (2000).

Let me formulate again to Antonino my best wishes and let me also thank the Organizing Committee: Patrizia Daniele, *University of Catania*, Maria Bernadette Donato, *University of Messina*, Sofia Giuffrè, *Mediterranean University of Reggio Calabria*, Monica Milasi, *University of Messina*, Laura Scrimali, *University of Catania*, Carmela Vitanza, *Ordinary member of Accademia Peloritana dei Pericolanti*.

2. Elliptic systems with general growth conditions

We consider an elliptic system in *divergence form*, of the type

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i^\alpha(x, Du) = 0, \quad \alpha = 1, 2, \dots, m, \quad (1)$$

where Ω is an open set of \mathbb{R}^n , $n \geq 2$, u is a vector-valued map $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$. The vector field $A(x, \xi) = (a_i^\alpha(x, \xi))_{i=1,2,\dots,n}^{\alpha=1,2,\dots,m}$, $A : \Omega \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$, has a variational nature in the sense that $A(x, \xi) = D_\xi f(x, \xi)$, where $f : \Omega \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is a *convex* function with respect to $\xi \in \mathbb{R}^{m \times n}$ and it is a *Carathéodory integrand*, with $f_\xi(x, \xi)$, $f_{\xi\xi}(x, \xi)$, $f_{\xi x}(x, \xi)$ Carathéodory functions too; in particular they are measurable with respect to $x \in \mathbb{R}^n$ and continuous with respect to $\xi \in \mathbb{R}^{m \times n}$. We emphasize the variational condition

$$A(x, \xi) = D_\xi f(x, \xi) \quad (2)$$

which in terms of components reads

$$a_i^\alpha(x, \xi) = \frac{\partial f(x, \xi)}{\partial \xi_i^\alpha} = f_{\xi_i^\alpha}(x, \xi), \quad \forall \alpha = 1, 2, \dots, m; \forall i = 1, 2, \dots, n. \quad (3)$$

In the context of vector-valued maps and systems a classical assumption finalized to the *everywhere regularity* of the solutions is a modulus-dependence in the energy integrand; i.e., we require that

$$f(x, \xi) = g(x, |\xi|), \quad (4)$$

where $g(x, t)$, $g : \Omega \times [0, \infty) \rightarrow [0, \infty)$, is a Carathéodory function, convex and increasing with respect to the gradient variable $t \in [0, \infty)$, with derivatives $g_t(x, t)$, $g_{tt}(x, t)$, $g_{tx}(x, t)$ Carathéodory functions and $g_t(x, 0) = 0$. The system (1) is *elliptic* if the following condition holds

$$\sum_{i,j=1}^n \sum_{\alpha,\beta=1}^m \frac{\partial a_i^\alpha(x, \xi)}{\partial \xi_j^\beta} \lambda_i^\alpha \lambda_j^\beta > 0, \quad \forall \lambda, \xi \in \mathbb{R}^{m \times n} : \lambda \neq 0.$$

Under the variational condition (3), the previous ellipticity condition can be equivalently written in the form

$$\sum_{i,j,\alpha,\beta} \frac{\partial^2 f(x, \xi)}{\partial \xi_i^\alpha \partial \xi_j^\beta} \lambda_i^\alpha \lambda_j^\beta > 0, \quad \forall \lambda, \xi \in \mathbb{R}^{m \times n} : \lambda \neq 0.$$

Thus the ellipticity condition of the system is equivalent to the positivity on $\mathbb{R}^{m \times n}$ of the quadratic form $D_\xi^2 f(\xi)$

$$\left(D_\xi^2 f(x, \xi) \lambda, \lambda \right) > 0, \quad \forall \lambda, \xi \in \mathbb{R}^{m \times n}, \lambda \neq 0,$$

which implies the (*strict*) *convexity* of the function $f(x, \xi)$ with respect to $\xi \in \mathbb{R}^{m \times n}$ for almost every $x \in \Omega$.

In this context of convexity of the function $f(x, \xi)$ with respect to $\xi \in \mathbb{R}^{m \times n}$, any *weak solution* (in a class of maps u to be defined) to the differential elliptic system (1) is a *minimizer* (also here, we need to define the class of maps which compete with u in the minimization process) of the energy integral

$$F(u) = \int_{\Omega} f(x, Du) dx. \quad (5)$$

In general, the vice-versa does not hold; i.e., not necessarily a minimizer satisfies a weak form of the system (1) unless we assume some special growth conditions (for instance the so called *natural growth conditions*), which however can rule out several energy integrands $f(x, \xi)$.

In the following we consider some classes of elliptic equations and systems where we can obtain some smoothness of the solutions by assuming *general growth conditions* of the energy integrand $f(x, \xi)$.

3. Some examples with general growth

Let us first mention examples for some $p > 1$ and $q > p$

$$\int_{\Omega} |Du(x)|^{p(x)} dx, \quad (6)$$

$$\int_{\Omega} |Du(x)|^p \log(1 + |Du(x)|) dx, \quad (7)$$

$$\int_{\Omega} \exp\left(a(x) |Du(x)|^2\right) dx, \quad (8)$$

$$\int_{\Omega} \{a(x) |Du(x)|^p + b(x) |Du(x)|^q\} dx, \quad (9)$$

the last one in (9) with possibly zero coefficients, by assuming however that $a(x), b(x)$ are not equal to zero at the same time; i.e.,

$$a(x), b(x) \geq 0, \quad a(x) + b(x) > 0, \quad \text{a.e. } x \in \Omega. \quad (10)$$

The class of energy functionals modeled by the example (9) enters in the context of general p, q -growth conditions; it is also named *double phase integrals* and has been recently - starting from 2015 - explored in a series of interesting papers by Colombo and Mingione (2015b), Colombo and Mingione (2015a), Baroni *et al.* (2015), Baroni *et al.* (2016), Baroni *et al.* (2018). From different points of view Eleuteri *et al.* (2016b), Eleuteri *et al.* (2016a), Eleuteri *et al.* (2019), Eleuteri *et al.* (2020) and De Filippis (2018), De Filippis and Oh (2019). For related recent references we also quote Düzgün *et al.* (2014b), Düzgün *et al.* (2014a), Rădulescu and Zhang (2018), Cencelj *et al.* (2018), Chlebicka (2018), Chlebicka and De Filippis (2020), Cupini *et al.* (2017a), Carozza *et al.* (2018), Cupini *et al.* (2009, 2012, 2013, 2014, 2015, 2017b, 2018), Harjulehto *et al.* (2017), Hästö and Ok (2019), Bousquet and Brasco (2020), De Filippis and Palatucci (2019), Nguyen and Tran (2020). General growth conditions even for the one-dimensional case $n = 1$ have been studied in the papers of Botteron and Marcellini (1991), Fusco *et al.* (1998). For the general case $n > 1$ and $m > 1$ under *quasiconvexity conditions* see Marcellini (1984) and the *integral convexity condition* Bögelein *et al.* (2020); see also Bögelein *et al.* (2013, 2015a,b, 2018a,b). Further

references can be found in the papers of Marcellini (2020c), Marcellini (2020a), Marcellini (2020b), Marcellini (2021). We devote the next section to describe some results related to the class of energy integrals as in the example (9).

4. Double phase integrals

We consider the energy integral (9), with $q \neq p$, let us say $q > p > 1, p < n$, and

$$\begin{cases} a(x), b(x) \geq 0 \\ a(x) + b(x) > 0 \end{cases}, \quad \text{a.e. } x \in \Omega.$$

Independently of the continuity of the coefficients $a(x), b(x)$, we first state a local boundedness result for minimizers of the energy integral as in (9), obtained by Cupini *et al.* (2018) in the spirit of related results by Boccardo *et al.* (1990).

Theorem 1 (Cupini-Marcellini-Mascolo). *Let $q \geq p > 1, a^{-1} \in L^r_{loc}(\Omega)$ and $b \in L^s_{loc}(\Omega)$ for some exponents $r \in (\frac{1}{p-1}, +\infty], s \in (1, +\infty]$, with*

$$\frac{1}{pr} + \frac{1}{qs} + \frac{1}{p} - \frac{1}{q} < \frac{1}{n}. \tag{11}$$

Then every local minimizer of the energy integral

$$F(u) = \int_{\Omega} \{a(x) |Du(x)|^p + b(x) |Du(x)|^q\} dx, \tag{12}$$

is locally bounded in Ω .

Note that in the special relevant case $r = s = +\infty$ then the above condition reduces to $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$, that is

$$\frac{q}{p} < 1 + \frac{q}{n} \iff q < \frac{np}{n-p} =: p^*. \tag{13}$$

More regularity of minimizers, in fact the local Hölder continuity of their gradients, has been obtained in the quoted papers by Colombo and Mingione (2015b), Colombo and Mingione (2015a), Baroni *et al.* (2015), Baroni *et al.* (2016), Baroni *et al.* (2018), Eleuteri *et al.* (2016b), Eleuteri *et al.* (2016a), Eleuteri *et al.* (2019), Eleuteri *et al.* (2020) and De Filippis (2018). The following results have been proved by M.Colombo-Mingione; of course in the first one we need a more strict assumption than either (11) or (13).

Theorem 2 (Colombo-Mingione). *Let $q \geq p > 1, a^{-1} \in L^{\infty}_{loc}(\Omega)$ and $a, b \in C^{\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1]$, with*

$$\frac{q}{p} < 1 + \frac{\alpha}{n}. \tag{14}$$

Then every local minimizer of the energy integral (12) is of class $C^{1,\beta}_{loc}(\Omega)$ for some $\beta \in (0, 1)$.

Theorem 3 (Colombo-Mingione). *Let $q \geq p$ and $1 < p \leq n, a^{-1} \in L^{\infty}_{loc}(\Omega)$ and $a, b \in C^{\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1]$, with*

$$\frac{q}{p} < 1 + \frac{\alpha}{p}. \tag{15}$$

Any locally bounded minimizer of the energy integral (12) is also of class $C_{loc}^{1,\beta}(\Omega)$ for some $\beta \in (0, 1)$.

The following is a related regularity result by Eleuteri *et al.* (2020), valid for a generalized class of double (or *multi*) phase energy integrands, whose prototype is given by

$$f(x, \xi) = a(x)|\xi|^p + b(x)|\xi|^s + |\xi_n|^q, \tag{16}$$

ξ_n being the last (or any other) component of the vector $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ and $s \leq \frac{p+q}{2}$. Note however that we do not assume a structure representation of the integrand, for instance of the type (16), which is only a model example.

In fact we can also consider more general energy integrands $f = f(x, \xi)$ without a structure, i.e. not necessarily depending on the modulus of ξ . We assume that $f : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty)$ is a convex function with respect to the gradient variable and it is strictly convex only at infinity; more precisely, $f_{\xi\xi}, f_{\xi x}$ are Carathéodory functions satisfying

$$\begin{cases} M_1 |\xi|^{p-2} |\lambda|^2 \leq \sum_{i,j} f_{\xi_i \xi_j}(x, \xi) \lambda_i \lambda_j \\ |f_{\xi_i \xi_j}(x, \xi)| \leq M_2 |\xi|^{q-2} \\ |f_{\xi x}(x, \xi)| \leq h(x) |\xi|^{\frac{p+q-2}{2}} \quad \text{or, respectively} \quad |f_{\xi x}(x, \xi)| \leq h(x) |\xi|^{q-1} \end{cases} \tag{17}$$

for some constants $M_0, M_1, M_2 > 0$, for almost every $x \in \Omega$ and for all $\lambda, \xi \in \mathbb{R}^n$ with $|\xi| \geq M_0$. Here $1 < p \leq q$ and $h \in L^r(\Omega)$ for some $r > n$.

Theorem 4 (Eleuteri-Marcellini-Mascolo). *Under the growth assumptions (17) with exponents p, q satisfying*

$$\frac{q}{p} < 1 + 2 \left(\frac{1}{n} - \frac{1}{r} \right) \quad \text{or, respectively} \quad \frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}, \tag{18}$$

any local minimizer of the energy integral $\int_{\Omega} f(x, Du(x)) dx$ is locally Lipschitz continuous in Ω .

If we specialize the above theorem with integrand $f(x, \xi)$ as in (16), with

$$a(x) = 1, \quad b(x) = |x|^\alpha,$$

for some $\alpha \in (0, 1)$ and $0 \in \Omega$, then $b \in C^{0,\alpha} \cap W^{1,r}$ with $\frac{1}{r} = \frac{1-\alpha}{n}$. The function h belongs to L^r for the same $r = \frac{n}{1-\alpha}$ and the assumption on the exponents p, q can be written in terms of the parameter α in the equivalent form

$$\frac{q}{p} < 1 + \frac{2\alpha}{n}. \tag{19}$$

Differently, if we take under consideration the double phase integral (12) with the same coefficients $a(x) = 1$ and $b(x) = |x|^\alpha$, then a computation gives $\frac{q}{p} < 1 + \frac{\alpha}{n}$, as in the Colombo-Mingione Theorem.

5. Minimizers and weak solutions

Let us emphasize a well known problem which arises in the process to go from a minimizer of an energy integral to a weak solution to the corresponding first variation equation or system. There are aspects to make precise even in the definition of weak solution. For instance, in the context of the double phase energy integral, if we assume the

continuity of the coefficients $a(x), b(x)$ in the compact set $\overline{\Omega}$, since $q \geq p$ then, for some positive constant m the following coercivity condition holds

$$a(x)|Du(x)|^p + b(x)|Du(x)|^q \geq m|Du(x)|^p, \quad \text{if } |Du(x)| \geq 1.$$

If u is a local minimizers of the energy integral (9) then $u \in W_{\text{loc}}^{1,p}(\Omega)$. We may ask if u is also a weak solution to the corresponding Euler's first variation

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i(x, Du) = 0, \quad x \in \Omega.$$

The vector-field $a(x, Du) = (a_i(x, Du))_{i=1,2,\dots,n}$ is given by

$$a(x, \xi) =: \left\{ p a(x) |Du(x)|^{p-2} + q b(x) |Du(x)|^{q-2} \right\} Du(x)$$

and it satisfies the growth condition $|a(x, \xi)| \leq M(1 + |\xi|^{q-1})$ for some constant $M > 0$ and for every $\xi \in \mathbb{R}^n$.

We have now to require that $u \in W_{\text{loc}}^{1,q}(\Omega)$. Let us recall here the reason: if $u \in W_{\text{loc}}^{1,q}(\Omega)$ we would obtain

$$|a(x, Du)| \leq M(1 + |Du|^{q-1}) \in L_{\text{loc}}^{\frac{q}{q-1}}(\Omega) = L_{\text{loc}}^{q'}(\Omega), \quad \frac{1}{q} + \frac{1}{q'} = 1,$$

and $u \in W_{\text{loc}}^{1,q}(\Omega)$ would satisfy the integral form of the equation

$$\int_{\Omega} \sum_{i=1}^n a_i(x, Du) \frac{\partial \varphi}{\partial x_i} dx = 0, \quad (20)$$

for every $\varphi \in W_0^{1,q}(\Omega)$ with $\text{supp } \varphi \subset \Omega$. However a minimizer is only a function of class $W_{\text{loc}}^{1,p}(\Omega)$, while, for the validity of the weak equation we have to impose a priori that $u \in W_{\text{loc}}^{1,q}(\Omega)$. This is a difference with respect to the case of the so-called natural growth conditions with $q = p$. We refer to Carozza *et al.* (2015) for general conditions for the validity of the Euler-Lagrange equation in the weak sense.

6. Vector-valued maps and elliptic systems

In this section we consider elliptic systems in divergence form as in (1)

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i^\alpha(x, Du) = 0, \quad \alpha = 1, 2, \dots, m,$$

with the variational structure as in (2), i.e. $a_i^\alpha(x, \xi) = f_{\xi_i^\alpha}(x, \xi)$ for some convex function $f: \Omega \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ of class $C^2(\mathbb{R}^{m \times n})$ with respect to $\xi \in \mathbb{R}^{m \times n}$. First, let us mention again the celebrated example by De Giorgi (1968), who considered the energy integral in (5)

$$F(u) = \int_{\Omega} f(x, Du(x)) dx$$

and he proved that his techniques, valid for the scalar case $m = 1$, cannot be extended to the vector-valued case $m > 2$ with systems. Later (but published in the same year: 1968) Giusti

and Miranda (1968) proposed a similar example for a minimizer of the energy-integral

$$F(u) = \int_{\Omega} f(u, Du(x)) dx.$$

The first example of a singular minimizer of an energy-integral without x and u explicit dependence appeared in 1977 and it is due to Nečas (1977), for an energy integral of the type

$$F(u) = \int_{\Omega} f(Du(x)) dx. \quad (21)$$

The minimizer found by Nečas is a map $u : \mathbb{R}^n \rightarrow \mathbb{R}^{n^2}$ with n large. Later, in 2000, Šverák and Yan (2000) found an example of a singular minimizer in 3 dimensions; precisely for a map $u : \mathbb{R}^3 \rightarrow \mathbb{R}^5$. More recently, in 2016, Mooney and Savin (2016) constructed a singular minimizing map $u : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ of a smooth uniformly convex energy-integral. We also refer to a recent study by Mooney (2019) for a singular minimizer map u defined in $\Omega \subset \mathbb{R}^4$.

As already said, in the study of the *everywhere regularity* in the vector-valued case $m > 1$ we consider a structure condition of the type

$$f(\xi) = g(|\xi|), \quad \forall \xi \in \mathbb{R}^{m \times n}. \quad (22)$$

In this context (22) a main reference is the p -Laplacian energy-integral studied in 1977 by Uhlenbeck (1977), with $f(\xi) = |\xi|^p$ and $p \geq 2$. The energy-integral has the form

$$F(u) = \int_{\Omega} |Du(x)|^p dx$$

and the p -Laplace operator, as well known, either in the scalar context $m = 1$ for equations or for vector-valued maps and systems $m > 1$, is

$$\begin{aligned} \operatorname{div} A(Du) &= \operatorname{div} |Du|^{p-2} Du \\ &= \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(|Du|^{p-2} u_{x_i}^\alpha \right), \quad \alpha = 1, 2, \dots, m. \end{aligned}$$

We can also consider the non-degenerate p -Laplace energy-integral and the corresponding p -Laplace operator with exponent $p > 1$, respectively given by

$$\begin{aligned} F(u) &= \int_{\Omega} \left(1 + |Du(x)|^2 \right)^{\frac{p}{2}} dx; \\ \operatorname{div} A(Du) &= \operatorname{div} \left(1 + |Du(x)|^2 \right)^{\frac{p-2}{2}} Du. \end{aligned}$$

We propose below a generalization of these examples by considering a general integrand of the form $f(\xi) = g(|\xi|)$, where $g : [0, +\infty) \rightarrow [0, +\infty)$ is an increasing convex function and $g'(0) = 0$. The energy-integral F takes the form

$$F(u) = \int_{\Omega} g(|Du|) dx.$$

A relevant difference with the p -Laplacian relies on the growth assumptions on $g = g(t)$. For instance, the local Lipschitz regularity result by Marcellini (1996) can be applied to the *exponential growth*, and also to any finite composition of exponential, such as (with $p_i \geq 1, \forall i = 1, 2, \dots, k$)

$$g(|\xi|) = (\exp(\dots (\exp(\exp|\xi|^2)^{p_1})^{p_2}) \dots)^{p_k}.$$

However, some other restrictions are imposed, such as, for instance, the fact that $t \in (0, +\infty) \rightarrow \frac{g'(t)}{t}$ is assumed to be an increasing function. To exemplify, the model case $g(t) = t^p$ gives the restriction $p \geq 2$. Also p, q -growth can be considered, with the energy-integrand $f(\xi) = g(|\xi|)$ which do not behave like a power when $|\xi| \rightarrow \infty$. For instance, for $|\xi|$ large (i.e., $|\xi| \geq e$), the integrand could be of the type

$$f(\xi) = g(|\xi|) = |\xi|^{a+b\sin\log\log|\xi|}, \quad g(t) = t^{a+b\sin\log\log t}. \tag{23}$$

In fact a computation shows that such an integrand is a convex function for $|\xi| \geq e$ ($t \geq e$). Therefore the function

$$g(|\xi|) = |\xi|^{a+b\sin\log\log|\xi|}$$

a-priori defined for $|\xi| \geq e$, can be extended to all $\xi \in \mathbb{R}^{m \times n}$ as a convex function on $\mathbb{R}^{m \times n}$ if a, b are positive real numbers such that $a > 1 + b\sqrt{2}$. In this case our integrand satisfies the p, q -growth conditions, with $p < q$, where $p = a - b$ and $q = a + b$,

$$|\xi|^p \leq f(\xi) \leq 1 + |\xi|^q, \quad \forall \xi \in \mathbb{R}^{m \times n}.$$

We notice that the " Δ_2 -condition" (well known in the mathematical literature; see i.e. Chlebicka 2018) is considered to be a *generalized uniformly elliptic case*. The function $f(\xi)$ in (23) satisfies the Δ_2 -condition. While we can construct (details by Bögelein *et al.* (2018b), Bögelein *et al.* (2018a)) some convex functions $f(\xi) = g(|\xi|)$, satisfying p, q -growth conditions, with $q > p$ and q arbitrarily close to p , which do not satisfy the Δ_2 -condition and which enter in the regularity theory presented here.

Some other references about this subject are: Fuchs and Mingione (2000), who concentrated on the case of nearly-linear growth; a typical examples is $f(\xi) = |\xi| \log(1 + |\xi|)$. Leonetti *et al.* (2003) considered the case of *subquadratic p, q -growth*, i.e. they assume $1 < p < q < 2$; their result includes energy densities f of the type (here $p < 2$) $f(\xi) = |\xi|^p \log^\alpha(1 + |\xi|)$. Bildhauer (2003) considered *nearly-linear growth*; he gave conditions that can keep " γ -elliptic linear growth" with $\gamma < 1 + \frac{2}{n}$, for instance

$$g_\gamma(t) = \int_0^t \int_0^s (1 + z^2)^{-\frac{\gamma}{2}} dz ds, \quad \forall t \geq 0;$$

for $\gamma = 1$, $g_\gamma(t)$ behaves like $t \log(1 + t)$ and in the (not included) limit case $\gamma = 3$, $g_\gamma(t)$ becomes $(1 + t^2)^{1/2}$. Note that the *minimal surface integrand* $g(t) = \sqrt{1 + t^2}$ does not enter in the assumptions of these quoted regularity results.

Marcellini and Papi (2006) gave conditions which include different kind of growths: *more general conditions on the function g embracing growths moving between linear and exponential functions*. The conditions are the following (we consider explicitly the case $n \geq 3$, while if $n = 2$ then the exponent $\frac{n-2}{n}$ can be replaced by any real number): *let $t_0, H > 0$ and let $\beta \in (\frac{1}{n}, \frac{2}{n})$. For every $\alpha \in (1, \frac{n}{n-1}]$ there exist $K = K(\alpha)$ such that, for all $t \geq t_0$,*

$$Ht^{-2\beta} \left[\left(\frac{g'(t)}{t} \right)^{\frac{n-2}{n}} + \frac{g'(t)}{t} \right] \leq g''(t) \leq K \left[\frac{g'(t)}{t} + \left(\frac{g'(t)}{t} \right)^\alpha \right]. \tag{24}$$

The exponent α in the right hand side is a parameter to be used to test more functions g . The condition in the left-hand side allows us to achieve functions - for instance - with second derivative going to zero as a power $t^{-\gamma}$, with γ small, i.e. $\gamma < 1 + \frac{2}{n}$.

In the paper of Marcellini and Papi (2006) the following two results are proved, the first one valid under *general growth conditions*, the second one specific for the *linear case*.

Theorem 5 (general growth). *Let $g : [0, +\infty) \rightarrow [0, +\infty)$ be a convex function of class $W_{\text{loc}}^{2,\infty}$ with $g(0) = g'(0) = 0$, satisfying the general growth conditions (24). Let $u \in W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^m)$ be a minimizer. Then*

$$u \in W_{\text{loc}}^{1,\infty}(\Omega; \mathbb{R}^m).$$

Moreover, for every $\varepsilon > 0$ and $R > \rho > 0$ there exists a constant $C = C(\varepsilon, n, \rho, R)$ such that

$$\|Du\|_{L^\infty(B_\rho; \mathbb{R}^{m \times n})}^{2-\beta n} \leq C \left\{ \int_{B_R} (1 + g(|Du|)) dx \right\}^{\frac{1}{1-\beta} + \varepsilon}.$$

Theorem 6 (linear growth). *Let $g : [0, +\infty) \rightarrow [0, +\infty)$ be a convex function of class $W_{\text{loc}}^{2,\infty}$ with $g(0) = g'(0) = 0$. If g has the linear behavior at infinity, i.e.*

$$\lim_{t \rightarrow +\infty} \frac{g(t)}{t} = l \in (0, +\infty)$$

and if its second derivative satisfies the inequalities

$$H \frac{1}{t^\gamma} \leq g''(t) \leq K \frac{1}{t}, \quad \forall t \geq t_0,$$

for some positive constants H, K, t_0 and for some $\gamma \in [1, 1 + \frac{2}{n})$, then any minimizer $u \in W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^m)$ is of class $W_{\text{loc}}^{1,\infty}(\Omega; \mathbb{R}^m)$ and, for every $R > \rho > 0$, there exists a constant $C = C(n, \rho, R, l, H, K)$ such that

$$\|Du\|_{L^\infty(B_\rho; \mathbb{R}^{m \times n})}^{\frac{2-n(\gamma-1)}{2}} \leq C \int_{B_R} (1 + g(|Du|)) dx.$$

We end by quoting a related recent a-priori regularity result by Di Marco and Marcellini (2020), where some energy integrals the form

$$F(u) = \int_{\Omega} g(x, |Du|) dx$$

are considered, with $x \in \Omega \subset \mathbb{R}^n$, $\xi \in \mathbb{R}^{m \times n}$, and the integrand $g(x, |\xi|)$ may explicitly depend on x too.

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