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PROPERTIES OF $\gamma\mathscr{H}$-COMPACT SPACES WITH HEREDITARY CLASSES

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ABSTRACT. By using γ-operations on an m-structure and a hereditary class $\mathscr{H}$, we define the notion of γ-compactness modulo hereditary classes (or ideals) called $\gamma\mathscr{H}$-compact. We obtain several properties of $\gamma\mathscr{H}$-compact spaces and $\gamma\mathscr{H}$-compact sets relative to m-structures.

1. Introduction

Let (X,τ) be a topological space and $\mathscr{P}(X)$ the power set of X. Ogata (1991) introduced the notions of γ-operations and γ-open sets and investigated the associated topology τ_γ and weak separation axioms γ-T_i (i = 0, 1/2, 1, 2). More recently Noiri (2011) defined an operation on an m-structure with property $\mathscr{B}$. The operation is defined as a function $m\gamma : m \to \mathscr{P}(X)$ such that $U \subset m\gamma(U)$ for each $U \in m$ and is called an operation $m\gamma$ on m. Then it turns out that the operation is an unified form of several operations (for example, semi-γ-operation: Sai Sundara Krishnan *et al.* 2007; pre-γ-operation: An *et al.* 2008) defined on the family of generalized open sets. Moreover, Noiri obtained some characterizations of $m\gamma$-compactness.

In this paper, by using hereditary classes (Császár 2007) and ideals (Janković and Hamlett 1990), we define the notion of γ-compactness modulo hereditary classes (or ideals) called $\gamma\mathscr{H}$-compact. In Section 3 we obtain several properties of $\gamma\mathscr{H}$-compact spaces and $\gamma\mathscr{H}$-compact sets. In Section 4 we deal with functions between m-spaces with operations and hereditary classes and obtain several properties of such functions and some preservation theorems of $\gamma\mathscr{H}$-compact sets. Recent papers have introduced some new classes of sets via hereditary classes (Al-Omari and Noiri 2016, 2019).

This paper is dedicated to Professor Filippo Cammaroto (University of Messina) on the occasion of his retirement.

2. Preliminaries

Definition 2.1. Let X be a nonempty set and $\mathscr{P}(X)$ the power set of X. A subfamily m of $\mathscr{P}(X)$ is called a *minimal structure* (briefly *m-structure*) on X if m satisfies the following conditions:
(1) $\emptyset \in m$ and $X \in m$,
(2) The union of any family of subsets belonging to m belongs to m.

A set X with an m-structure is called an *m-space* and is denoted by (X,m). Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*.

Definition 2.2. Let (X,m) be an m-space. For a subset A of X, the *m-closure* of A is defined by Maki *et al.* (1999) as follows:

$$mCl(A) = \cap\{F : A \subset F, X \setminus F \in m\}.$$

Lemma 2.3. (Maki *et al.* 1999). *Let (X,m) be an m-space. For the m-closure, the following properties hold, where A and B are subsets of X:*
(1) $A \subset \mathrm{mCl}(A)$,
(2) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$,
(3) If $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$,
(4) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$.

Lemma 2.4. (Popa and Noiri 2000). *Let (X,m) be an m-space and A a subset of X. Then* $x \in \mathrm{mCl}(A)$ *if and only if* $U \cap A \neq \emptyset$ *for every* $U \in m$ *containing x.*

Lemma 2.5. (Popa and Noiri 2002). *Let (X,m) be an m-space and A a subset of X. Then, the following properties hold:*
(1) A is m-closed if and only if $\mathrm{mCl}(A) = A$,
(2) $\mathrm{mCl}(A)$ *is m-closed.*

Remark 2.6. Lemmas 2.3 and 2.4 hold without the condition (2) (property $\mathscr{B}$) in Definition 2.1.

Definition 2.7. (Noiri 2011). Let (X,m) be an m-space. Let $m\gamma : m \to \mathscr{P}(X)$ be a function from m into $\mathscr{P}(X)$ such that $U \subset m\gamma(U)$ for each $U \in m$. The function $m\gamma$ is called an $m\gamma$-operation on m. Hereafter, an $m\gamma$-operation is called a γ-operation and denoted by $\gamma : m \to \mathscr{P}(X)$.

Definition 2.8. (Noiri 2011). Let (X,m) be an m-space and γ an operation on m. A subset A of X is said to be γ-*open* if for each $x \in A$ there exists $U \in m$ such that $x \in U \subset \gamma(U) \subset A$. The complement of a γ-open set is said to be γ-*closed*. The family of all γ-open sets of (X,m) is denoted by $\gamma(X)$.

Remark 2.9. We assume that the empty set $\emptyset$ is a γ-open set, that is, $\emptyset \in \gamma(X)$.

Lemma 2.10. *Let (X,m) be an m-space. For $\gamma(X)$, the following properties hold:*
(1) $\emptyset, X \in \gamma(X)$,
(2) If $A_\alpha \in \gamma(X)$ *for each* $\alpha \in \Lambda$, *then* $\cup_{\alpha\in\Lambda} A_\alpha \in \gamma(X)$,
(3) $\gamma(X) \subset m$.

Remark 2.11. (Noiri 2011). By (1) and (2) of Lemma 2.10, it turns out that $\gamma(X)$ is an m-structure. However, in general, $\gamma(X)$ is not a topology. It was shown by Ogata (1991, Example 2.8) that the intersection of two γ-open sets is not always γ-open.

Definition 2.12. (Noiri 2011). An m-space (X,m) is said to be *γ-regular* if for each $x \in X$ and each $U \in m$ containing x, there exists $V \in m$ such that $x \in V \subset \gamma(V) \subset U$.

Lemma 2.13. (Noiri 2011). *For an m-space (X,m), the following properties are equivalent:*
(1) $m = \gamma(X)$;
(2) (X,m) is γ-regular;
(3) For each $x \in X$ and each $U \in m$ containing x, there exists $W \in \gamma(X)$ such that $x \in W \subset \gamma W \subset U$.

Definition 2.14. (Noiri 2011). Let (X,m) be an m-space. For a subset A of X, the *γ-closure* of A (γCl(A)) and *γ-interior* of A (γInt(A)), are defined as follows:
(1) $\gamma\mathrm{Cl}(A) = \cap\{F : A \subset F, X \setminus F \in \gamma(X)\}$,
(2) $\gamma\mathrm{Int}(A) = \cup\{U : U \subset A, U \in \gamma(X)\}$.

Lemma 2.15. (Noiri 2011). *Let (X,m) be an m-space on X. For the γ-closure and the γ-interior, the following properties hold, where A and B are subsets of X:*
(1) $\gamma\mathrm{Int}(A) \subset A \subset \gamma\mathrm{Cl}(A)$,
(2) $\gamma\mathrm{Cl}(\emptyset) = \emptyset = \gamma\mathrm{Int}(\emptyset)$, $\gamma\mathrm{Cl}(X) = X = \gamma\mathrm{Int}(X)$,
(3) If $A \subset B$, then $\gamma\mathrm{Cl}(A) \subset \gamma\mathrm{Cl}(B)$ and $\gamma\mathrm{Int}(A) \subset \gamma\mathrm{Int}(B)$,
(4) $\gamma\mathrm{Cl}(\gamma\mathrm{Cl}(A)) = \gamma\mathrm{Cl}(A)$ and $\gamma\mathrm{Int}(\gamma\mathrm{Int}(A)) = \gamma\mathrm{Int}(A)$,
(5) A is γ-closed if and only if $\gamma\mathrm{Cl}(A) = A$ and A is γ-open if and only if $\gamma\mathrm{Int}(A) = A$,
(6) $\gamma\mathrm{Cl}(A)$ is γ-closed and $\gamma\mathrm{Int}(A)$ is γ-open,
(7) (i) $x \in \gamma\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every γ-open set U containing x,
(ii) $x \in \gamma\mathrm{Int}(A)$ if and only if for each $x \in A$ there exists a γ-open set U containing x such that $U \subset A$.

Proof. The proof follows easily from Lemmas 2.3, 2.4 and 2.5.

3. $\gamma\mathscr{H}$-compact spaces

First, we recall the definitions of a hereditary class and an ideal used in the sequel. A subfamily $\mathscr{H}$ of the power set $\mathscr{P}(X)$ is called a *hereditary class* on X (Császár 2007) if it satisfies the following property: $A \in \mathscr{H}$ and $B \subset A$ implies $B \in \mathscr{H}$. A hereditary class $\mathscr{H}$ is called an *ideal* (Janković and Hamlett 1990) if it satisfies the additional condition: $A \in \mathscr{H}$ and $B \in \mathscr{H}$ implies $A \cup B \in \mathscr{H}$. An m-space (X,m) with a hereditary class $\mathscr{H}$ on X is called a hereditary m-space and is denoted by $(X,m,\mathscr{H})$.

Definition 3.1. Let $(X,m,\mathscr{H})$ be a hereditary m-space and γ an operation on m, where $\mathscr{H}$ a hereditary class on X. A subset A of X is said to be *$\gamma\mathscr{H}$-compact (resp. $\mathscr{H}$-compact) relative to m* if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by m-open sets of X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathscr{H}$ (resp. $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathscr{H}$).

Definition 3.2. Let $(X,m,\mathscr{H})$ be a hereditary m-space and γ an operation on m. The m-space (X,m) is said to be *$\gamma\mathscr{H}$-compact (resp. $\mathscr{H}$-compact)* if X is $\gamma\mathscr{H}$-compact (resp. $\mathscr{H}$-compact) relative to m.

Definition 3.3. Let (X,m) be an m-space and γ an operation on m. Then (X,m) is said to be *γ-compact* (Noiri 2011) if for each cover $\{U_\alpha : \alpha \in \Delta\}$ of X by m-open sets of X, there exists a finite subset Δ_0 of Δ such that $\cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} = X$.

Definition 3.4. Let (X,m) be an m-space. A subset A of X is said to be *m-compact* (Popa and Noiri 2002) (resp. *m-closed,* Popa and Noiri 2002) relative to m if for each cover $\{U_\alpha : \alpha \in \Delta\}$ of A by m-open sets of X, there exists a finite subset Δ_0 of Δ such that $A \subset \cup\{U_\alpha : \alpha \in \Delta_0\}$ (resp. $A \subset \cup\{\mathrm{mCl}(U_\alpha) : \alpha \in \Delta_0\}$).

Definition 3.5. An m-space (X,m) is said to be *m-compact* (Popa and Noiri 2002) (resp. *m-closed*, Popa and Noiri 2002) if X is m-compact (resp. m-closed) relative to m.

Remark 3.6. Let $(X,m,\mathscr{H})$ be a hereditary m-space and γ an operation on m.

(1) If γ is the identity (resp. m-closure) operation, then "$\gamma\mathscr{H}$-compact relative to m" coincides with "$\mathscr{H}$-compact (resp. m-closed) relative to m".

(2) If $\mathscr{H} = \{\emptyset\}$, then "$\gamma\mathscr{H}$-compact relative to m" coincides "γ-compact relative to m". Moreover if γ is the identity operation, then "$\gamma\mathscr{H}$-compact relative to m" coincides with "m-compact relative to m".

Theorem 3.7. *Let $(X,m,\mathscr{H})$ be a hereditary m-space and γ an operation on m. Then the following properties are equivalent:*

(1) (X,m) is $\mathscr{H}$-compact;

(2) For every family $\{F_\alpha : \alpha \in \Delta\}$ of m-closed sets satisfying $\cap\{F_\alpha : \alpha \in \Delta_0\} \notin \mathscr{H}$ for every finite subfamily Δ_0 of Δ, $\cap\{F_\alpha : \alpha \in \Delta\} \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2): Let (X,m) be $\mathscr{H}$-compact. Suppose that $\cap\{F_\alpha : \alpha \in \Delta\} = \emptyset$. Then $X \setminus F_\alpha$ is m-open for each $\alpha \in \Delta$ and $\cup_{\alpha\in\Delta} X \setminus F_\alpha = X \setminus \cap_{\alpha\in\Delta} F_\alpha = X$. By (1), there exists a finite subfamily Δ_0 of Δ such that $X \setminus \cup_{\alpha\in\Delta_0}(X \setminus F_\alpha) = \cap\{F_\alpha : \alpha \in \Delta_0\} \in \mathscr{H}$. This is a contradiction.

(2) $\Rightarrow$ (1): Suppose that (X,m) is not $\mathscr{H}$-compact. There exists a cover $\{U_\alpha : \alpha \in \Delta\}$ of X by m-open sets of X such that $X \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \notin \mathscr{H}$ for every finite subset Δ_0 of Δ. Since $X \setminus U_\alpha$ is m-closed for each $\alpha \in \Delta$ and $\cap\{(X \setminus U_\alpha) : \alpha \in \Delta_0\} \notin \mathscr{H}$ for every finite subset Δ_0 of Δ. By (2), we have $\cap\{(X \setminus U_\alpha) : \alpha \in \Delta\} \neq \emptyset$. Therefore, $X \setminus \cup\{U_\alpha : \alpha \in \Delta\} \neq \emptyset$. This is contrary that $\{U_\alpha : \alpha \in \Delta\}$ is an m-open cover of X.

Theorem 3.8. *Let $(X,m,\mathscr{H})$ be a hereditary m-space, γ an operation on m and A a subset of X. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ hold. If (X,m) is γ-regular, then the following properties are equivalent:*

(1) *A is $\mathscr{H}$-compact relative to m;*
(2) *A is $\gamma\mathscr{H}$-compact relative to m;*
(3) *A is $\mathscr{H}$-compact relative to $\gamma(X)$;*
(4) *A is $\gamma\mathscr{H}$-compact relative to $\gamma(X)$.*

Proof. (1) $\Rightarrow$ (2): For any cover $\{U_\alpha : \alpha \in \Delta\}$ of A by m-open sets of X, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathscr{H}$; hence $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathscr{H}$. Therefore, A is $\gamma\mathscr{H}$-compact relative to m.

(2) $\Rightarrow$ (3): Let A be $\gamma\mathscr{H}$-compact relative to m and $\{U_\alpha : \alpha \in \Delta\}$ a cover of A by γ-open sets of X. For each $x \in A$ there exists $\alpha(x) \in \Delta$ such that $x \in U_{\alpha(x)}$. Since $U_{\alpha(x)}$ is γ-open,

there exists $V_{\alpha(x)} \in m$ such that $x \in V_{\alpha(x)} \subseteq \gamma(V_{\alpha(x)}) \subseteq U_{\alpha(x)}$. Since the family $\{V_{\alpha(x)} : x \in A\}$ is an m-open cover of A and A is $\gamma\mathscr{H}$-compact relative to m, there exists a finite subset A_0 of A such that $A \setminus \cup\{\gamma(V_{\alpha(x)}) : x \in A_0\} \in \mathscr{H}$ and hence $A \setminus \cup\{U_{\alpha(x)} : x \in A_0\} \in \mathscr{H}$. This shows that A is $\mathscr{H}$-compact relative to $\gamma(X)$.

(3) $\Rightarrow$ (4): By Lemma **2.10**, $\gamma(X)$ is an m-structure and it follows from the same argument as (1) $\Rightarrow$ (2) that A is $\gamma\mathscr{H}$-compact relative to $\gamma(X)$.

(4) $\Rightarrow$ (1): Suppose that (X,m) is γ-regular. Let A be $\gamma\mathscr{H}$-compact relative to $\gamma(X)$. By Lemma **2.13**, $m = \gamma(X)$ and A is $\gamma\mathscr{H}$-compact relative to $\gamma(X)$. Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of A by m-open sets of X. For each $x \in A$, there exists $\alpha(x) \in \Delta$ such that $x \in U_{\alpha(x)}$. Since (X,m) is γ-regular, there exists $V_{\alpha(x)} \in \gamma(X)$ such that $x \in V_{\alpha(x)} \subseteq \gamma(V_{\alpha(x)}) \subseteq U_{\alpha(x)}$. Since $\{V_{\alpha(x)} : x \in A\}$ is a cover of A by γ-open sets of X and A is $\gamma\mathscr{H}$-compact relative to $\gamma(X)$, there exists a finite subset A_0 of A such that $A \setminus \cup\{\gamma(V_{\alpha(x)}) : x \in A_0\} \in \mathscr{H}$ and hence $A \setminus \cup\{U_{\alpha(x)} : x \in A_0\} \in \mathscr{H}$. This shows that A is $\mathscr{H}$-compact relative to m.

Corollary 3.9. *For any γ-regular m-space (X,m), the following properties are equivalent. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ hold without the assumption γ-regular on (X,m).*

(1) *(X,m) is $\mathscr{H}$-compact;*
(2) *(X,m) is $\gamma\mathscr{H}$-compact;*
(3) *$(X,\gamma(X))$ is $\mathscr{H}$-compact;*
(4) *$(X,\gamma(X))$ is $\gamma\mathscr{H}$-compact.*

Remark 3.10. In Corollary 3.9, if we put $\mathscr{H} = \{\emptyset\}$, then we obtain Theorem 5.1 of Noiri (2011).

Theorem 3.11. *Let $(X,m,\mathscr{H})$ be a hereditary m-space, γ an operation on m and A,B be subsets of X. If A is $\gamma\mathscr{H}$-compact relative to m and B is γ-closed, then $A \cap B$ is $\gamma\mathscr{H}$-compact relative to m.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a cover of $A \cap B$ by m-open subsets of X. Then $A \setminus B \subseteq X \setminus B$ and $X \setminus B$ is γ-open. For each $x \in A \setminus B$, there exists an m-open set U_x containing x such that $x \in U_x \subseteq \gamma(U_x) \subseteq X \setminus B$. Then $\{U_\alpha : \alpha \in \Delta\} \cup \{U_x : x \in A \setminus B\}$ is a cover of A by m-open sets of X. Since A is $\gamma\mathscr{H}$-compact relative to m, there exist finite subsets Δ_0 of Δ and A_0 of A such that $A \subseteq [\cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}] \cup [\cup\{\gamma(U_x) : x \in A_0\}] \cup H_0$, where $H_0 \in \mathscr{H}$. Then we have

$$\begin{aligned} A \cap B &\subseteq [\cup\{\gamma(U_\alpha) \cap B : \alpha \in \Delta_0\}] \cup [\cup\{\gamma(U_x) \cap B : x \in A_0\}] \cup (H_0 \cap B) \\ &\subseteq [\cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\}] \cup H_0. \end{aligned}$$

Therefore, $(A \cap B) \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \subseteq H_0 \in \mathscr{H}$. This shows that $A \cap B$ is $\gamma\mathscr{H}$-compact relative to m.

Corollary 3.12. *Let $(X,m,\mathscr{H})$ be a hereditary m-space and γ an operation.*

(1) *If $(X,m,\mathscr{H})$ is $\gamma\mathscr{H}$-compact and B is γ-closed, then B is $\gamma\mathscr{H}$-compact relative to m.*
(2) *If (X,m) is γ-compact and B is γ-closed, then B is γ-compact relative to m.*

Definition 3.13. Let $(X,m,\mathscr{H})$ be a hereditary m-space. A subset A of X is said to be

(1) *$m\mathscr{H}g$-closed* (Al-Omari and Noiri 2020) if $mcl(A) \subseteq U$ whenever $A \setminus U \in \mathscr{H}$ and $U \in m$,
(2) *mg-closed* (Noiri 2007) if $mcl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in m$.

Remark 3.14. We note that:
(1) If $\mathscr{H} = \{\emptyset\}$, then $m\{\emptyset\}g$-closed and mg-closed coincide.
(2) If A is $m\mathscr{H}g$-closed, then A is mg-closed. The converse is not always true as shown by the following example due to Qahis *et al.* (2021).

Example 3.15. *Let $X = \mathbb{R}$, $m = \{\emptyset, \mathbb{R}\} \cup \{(r, \infty) : r \in \mathbb{R}\}$ and $\mathscr{H} = \{H : H \subseteq \mathbb{Q} \cap [0, \infty)$ or $H \subseteq \mathbb{Q} \cap (-\infty, 0]\}$. If $A = \mathbb{Q}$, then*
(1) *A is mg-closed because if $A \subseteq U$ and $U \in m$, then $U = \mathbb{R}$ and $mcl(A) = \mathbb{R} \subseteq U$.*
(2) *A is not $m\mathscr{H}g$-closed because if $U = (0, \infty)$, then $U \in m$ and $A \setminus U = \mathbb{Q} \setminus (0, \infty) = \mathbb{Q} \cap (-\infty, 0] \in \mathscr{H}$ but $mcl(A) = \mathbb{R} \nsubseteq (0, \infty)$.*

Theorem 3.16. *Let $(X, m, \mathscr{H})$ be an $\mathscr{H}$-compact space. If A is mg-closed, then A is $\mathscr{H}$-compact relative to m.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of A by m-open sets of X. Then $mcl(A) \subseteq \cup\{U_\alpha : \alpha \in \Delta\}$. Then we have $(X \setminus mcl(A)) \cup [\cup\{U_\alpha : \alpha \in \Delta\}]$ is an m-open cover of X. Since X is $\mathscr{H}$-compact, there exists a finite subset Δ_0 of Δ such that $X \setminus [(X \setminus mcl(A)) \cup [\cup\{U_\alpha : \alpha \in \Delta_0\}]] \in \mathscr{H}$. Then

$$\begin{aligned} &X \setminus [(X \setminus mcl(A)) \cup [\cup\{U_\alpha : \alpha \in \Delta_0\}]] \\ &= mcl(A) \cap (X \setminus [\cup\{U_\alpha : \alpha \in \Delta_0\}]) \\ &\supseteq A \cap (X \setminus [\cup\{U_\alpha : \alpha \in \Delta_0\}]) \\ &= A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\}. \end{aligned}$$

Therefore, $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathscr{H}$ and A is $\mathscr{H}$-compact relative to m.

Corollary 3.17. *If (X, m) is m-compact and A is mg-closed, then A is m-compact relative to m.*

Let (X, τ) be a topological space. A subset A of X is said to be g-closed (Levine 1970) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$.

Corollary 3.18. *(Levine 1970) If (X, τ) is a compact topological space and A is g-closed, then A is compact.*

Theorem 3.19. *Let $(X, m, \mathscr{H})$ be a hereditary m-space, A an $m\mathscr{H}g$-closed subset of X and $A \subseteq B \subseteq mcl(A)$, then the following properties are equivalent:*
(1) *A is $\mathscr{H}$-compact relative to m,*
(2) *B is $\mathscr{H}$-compact relative to m.*

Proof. (1) $\Rightarrow$ (2): Suppose that A is $\mathscr{H}$-compact relative to m. Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of B by m-open sets of X. Then $\{U_\alpha : \alpha \in \Delta\}$ is a cover of A by m-open sets of X. Since A is $\mathscr{H}$-compact relative to m, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathscr{H}$. Since A is $m\mathscr{H}g$-closed, $mcl(A) \subseteq \cup\{U_\alpha : \alpha \in \Delta_0\}$. Since $B \subseteq mcl(A)$, we have $B \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \subseteq mcl(A) \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} = \emptyset \in \mathscr{H}$. Therefore, B is $\mathscr{H}$-compact relative to m.

(2) $\Rightarrow$ (1): Suppose that B is $\mathscr{H}$-compact relative to m. Let $\{U_\alpha : \alpha \in \Delta\}$ is any cover of A by m-open sets of X. Since A is $m\mathscr{H}g$-closed, A is mg-closed and hence we have $B \subseteq mcl(A) \subseteq \cup\{U_\alpha : \alpha \in \Delta\}$. Since B is $\mathscr{H}$-compact relative to m, there exists a finite subset Δ_0 of Δ such that $B \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathscr{H}$. Since $A \subseteq B$, $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathscr{H}$. Therefore, A is $\mathscr{H}$-compact relative to m.

Theorem 3.20. *Let (X,m) be an m-space and $\mathscr{H}$ an ideal. If subsets A and B of X are $\gamma\mathscr{H}$-compact relative to m, then $A \cup B$ is $\gamma\mathscr{H}$-compact relative to m.*

Proof. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of $A \cup B$ by m-open sets of X. Then $\mathscr{U}$ is a cover of A and B by m-open sets of X. Since A and B are $\gamma\mathscr{H}$-compact relative to m, there exist finite subsets Δ_A and Δ_B of Δ and subsets H_A and H_B of $\mathscr{H}$ such that $A \subseteq \cup\{\gamma(U_\alpha) : \alpha \in \Delta_A\} \cup H_A$ and $B \subseteq \cup\{\gamma(U_\alpha) : \alpha \in \Delta_B\} \cup H_B$. Hence we have $A \cup B \subseteq \cup\{\gamma(U_\alpha) : \alpha \in \Delta_A \cup \Delta_B\} \cup (H_A \cup H_B)$. Since $\mathscr{H}$ is an ideal, we have $(A \cup B) \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_A \cup \Delta_B\} \in \mathscr{H}$. This shows that $A \cup B$ is $\gamma\mathscr{H}$-compact relative to m.

The following example is due to Qahis *et al.* (2021).

Example 3.21. *Let $X = \mathbb{R}$ be the real number, $m = \tau$ the usual topology, $\gamma : m \to \mathscr{P}(X)$ such that $\gamma(U) = cl(U)$ for every $U \in m$ and $\mathscr{H} = \{H \subseteq \mathbb{R} : H \subseteq (0,1)$ or $H \subseteq (1,2)\}$. If $A = (0,1)$ and $B = (1,2)$, then*

(1) *A and B are $\gamma\mathscr{H}$-compact relative to m.*
(2) *$A \cup B$ is not $\gamma\mathscr{H}$-compact relative to m.*

Proof. (1) The proof is obvious.
(2) The family $\{(\frac{1}{n}, 2-\frac{1}{n}) : n \in \mathbb{Z}^+\}$, where $\mathbb{Z}^+$ is the family of positive integers, is a cover of $A \cup B$ by m-open sets of X. For any finite subsets $\{n_1, n_2, ..., n_k\}$ of $\mathbb{Z}^+$, put $N = max\{n_1, n_2, ..., n_k\}$. Then we have

$(A \cup B) \setminus \cup\{\gamma(\frac{1}{n_i}, 2-\frac{1}{n_i}) : 1 \le i \le k\} = (A \cup B) \setminus \cup\{[\frac{1}{n_i}, 2-\frac{1}{n_i}] : 1 \le i \le k\} = (A \cup B) \setminus [\frac{1}{N}, 2-\frac{1}{N}] = (0, \frac{1}{N}) \cup (2-\frac{1}{N}, 2) \notin \mathscr{H}$. Therefore, $A \cup B$ is not $\gamma\mathscr{H}$-compact relative to m.

Definition 3.22. (Noiri and Popa 2018) Let $(X, m, \mathscr{H})$ be a hereditary m-space and A a subset of X.

(1) The *minimal local function* $A^\star_{mH}(\mathscr{H}, m)$ of A is defined as follows: $A^\star_{mH}(\mathscr{H}, m) = \{x \in X : U \cap A \notin \mathscr{H}$ for every $U \in m(x)\}$, where $m(x) = \{U : x \in U \in m\}$. Hereafter, $A^\star_{mH}(\mathscr{H}, m)$ is denoted by $A^\star_{mH}$.
(2) The *minimal $\star$-closure* $mcl^\star_H(A)$ of A is defined as $mCl^\star_H(A) = A \cup A^\star_{mH}$. The *$m^\star_H$-structure* is defined as follows: $m^\star_H = \{U \subseteq X : mCl^\star_H(X \setminus U) = X \setminus U\}$. Each member of $m^\star_H$ is said to be *$m^\star_H$-open* and the complement of an $m^\star_H$-open set is said to be *$m^\star_H$-closed.*

Remark 3.23. (Noiri and Popa 2018) Let $(X, m, \mathscr{H})$ be a hereditary m-space and A a subset of X. If $\mathscr{H} = \{\emptyset\}$ (resp. $\mathscr{P}(X)$), then $A^\star_{mH} = \mathrm{mCl}(A)$ (resp. $A^\star_{mH} = \emptyset$).

Lemma 3.24. *(Noiri and Popa 2018) For a hereditary m-space $(X, m, \mathscr{H})$, the following properties hold:*

(1) *m^*_H is an m-structure on X such that m^*_H has property $\mathscr{B}$ and $m \subseteq m^*_H$.*
(2) *$\beta(m, \mathscr{H}) = \{U \setminus H : U \in m, H \in \mathscr{H}\}$ is a basis for m^*_H such that $m \subseteq \beta(m, \mathscr{H})$.*

Theorem 3.25. *Let $(X,m,\mathscr{H})$ be a hereditary m-space. If a subset A of X is $\mathscr{H}$-compact relative to $m^\star_H$, then A is $\mathscr{H}$-compact relative to m. The converse is true if $\mathscr{H}$ is an ideal.*

Proof. Let A be $\mathscr{H}$-compact relative to $m^\star_H$. By Lemma **3.24**, $m \subseteq m^*_H$ and hence A is $\mathscr{H}$-compact relative to m.

Conversely, suppose that $\mathscr{H}$ is an ideal and A is $\mathscr{H}$-compact relative to m. Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of A by $m^\star_H$-open sets of X. For each $x \in A$ there exists $\alpha(x) \in \Delta$ such that $x \in V_{\alpha(x)} \in m^*_H$. By Lemma **3.24**, there exists $U_{\alpha(x)} \in m$ and $H_{\alpha(x)} \in \mathscr{H}$ such that $x \in U_{\alpha(x)} \setminus H_{\alpha(x)} \subseteq V_{\alpha(x)}$; hence $x \in U_{\alpha(x)} \subseteq V_{\alpha(x)} \cup H_{\alpha(x)}$, where $H_{\alpha(x)} \in \mathscr{H}$. Then $\{U_{\alpha(x)} : x \in A\}$ is a cover of A by m-open sets of X. Since A of X is $\mathscr{H}$-compact relative to m, there exists a finite subset A_0 of A such that $A \setminus \cup\{U_{\alpha(x)} : x \in A_0\} \in \mathscr{H}$. Hence we have

$$\begin{aligned} A &\subseteq \cup\{U_{\alpha(x)} : x \in A_0\} \cup H_0, \text{ where } H_0 \in \mathscr{H} \\ &\subseteq \cup\{V_{\alpha(x)} \cup H_{\alpha(x)} : x \in A_0\} \cup H_0 \\ &= \cup\{V_{\alpha(x)} : x \in A_0\} \cup \big(\cup\{H_{\alpha(x)} : x \in A_0\}\big) \cup H_0. \end{aligned}$$

Therefore, we obtain $A \setminus \cup\{V_{\alpha(x)} : x \in A_0\} \subseteq \big(\cup\{H_{\alpha(x)} : x \in A_0\}\big) \cup H_0 \in \mathscr{H}$. This shows that A is $\mathscr{H}$-compact relative to $m^\star_H$.

Corollary 3.26. *Let $(X,m,\mathscr{H})$ be an ideal m-space, then for a subset A of X the following properties are equivalent:*

(1) *A is $\mathscr{H}$-compact relative to m,*
(2) *A is $\mathscr{H}$-compact relative to $m^\star_H$.*

4. (γ,δ)-continuous functions

In this section, let (X,m) and (Y,n) be minimal spaces and γ (resp. δ) be an operation on m (resp. n).

Definition 4.1. A function $f : (X,m) \to (Y,n)$ is said to be

(1) (γ,δ)-*continuous* if for each $x \in X$ and each $V \in n$ containing $f(x)$, there exists $U \in m$ containing x such that $f(\gamma(U)) \subseteq \delta(V)$.
(2) *weakly* (γ,δ)-*continuous* if $f^{-1}(V)$ is γ-open in X for every δ-open set V of Y.

Theorem 4.2. *If $f : (X,m) \to (Y,n)$ is (γ,δ)-continuous, then it is weakly (γ,δ)-continuous.*

Proof. Let V be any δ-open set of Y. For each $x \in f^{-1}(V)$, $f(x) \in V$ and there exists $V_0 \in n$ such that $f(x) \in V_0 \subseteq \delta(V_0) \subseteq V$. Since f is (γ,δ)-continuous, there exists $U \in m$ containing x such that $f(\gamma(U)) \subseteq \delta(V_0)$. Therefore, we have $x \in U \subseteq \gamma(U) \subseteq f^{-1}(f(\gamma(U))) \subseteq f^{-1}(\delta(V_0)) \subseteq f^{-1}(V)$. This shows that f is weakly (γ,δ)-continuous.

The converse of Theorem **4.2** is not always true as shown by the following example.

Example 4.3. *Let $X = Y = \{a,b,c\}$, $m = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$, $n = \{\emptyset, Y, \{a\}, \{c\}, \{a,c\}\}$. Then m (resp. n) is a topology on X (resp. Y). Let $\gamma(A) = mCl(A) = Cl(A)$ for every subset A of X and $\delta(B) = nCl(B) = Cl(B)$ for every subset B of Y. Then the identity function $f : (X,m) \to (Y,n)$ is weakly (γ,δ)-continuous because $\gamma(X) = \{\emptyset, X\}$ and $\delta(Y) = \{\emptyset, Y\}$. For $a \in X$ and $f(a) = a \in \{a\} = V \in n$, $\delta(V) = Cl(\{a\}) = \{a,b\}$ and $f(\gamma(U))$ is not contained in $\delta(V)$ for every $U \in m$ containing a. Therefore, f is not (γ,δ)-continuous.*

Theorem 4.4. *Let (Y,n) be δ-regular. Then for a function $f:(X,m)\to(Y,n)$ the following properties are equivalent:*

(1) *f is (γ,δ)-continuous;*
(2) *f is weakly (γ,δ)-continuous.*

Proof. (1) $\Rightarrow$ (2): The proof follows from Theorem **4.2**.

(2) $\Rightarrow$ (1): For any $x\in X$ and $V\in n$ containing $f(x)$, by Lemma **2.13** there exists $W\in\delta(Y)$ such that $f(x)\in W\subseteq\delta(W)\subseteq V$. By (2), $f^{-1}(W)$ is a γ-open set containing x and hence there exists $U\in m$ such that $x\in U\subseteq\gamma(U)\subseteq f^{-1}(W)$. Therefore, $f(\gamma(U))\subseteq W\subseteq\delta(W)\subseteq V\subseteq\delta(V)$. This shows that f is (γ,δ)-continuous.

Lemma 4.5. *(Al-Omari and Noiri 2020) Let $f:X\to Y$ be a function.*

(1) *If $\mathscr{H}$ is a hereditary class on X, then $f(\mathscr{H})$ is a hereditary class on Y.*
(2) *If $\mathscr{H}$ is a hereditary class on Y, then $f^{-1}(\mathscr{H})$ is a hereditary class on X.*

Theorem 4.6. *If $f:(X,m,\mathscr{H})\to(Y,n,f(\mathscr{H}))$ is (γ,δ)-continuous and A is $\gamma\mathscr{H}$-compact relative to m, then $f(A)$ is $\delta f(\mathscr{H})$-compact relative to n.*

Proof. Let $\{V_\alpha:\alpha\in\Delta\}$ be any cover of $f(A)$ by n-open sets of Y. For each $x\in A$, there exists $\alpha(x)\in\Delta$ such that $f(x)\in V_{\alpha(x)}$. Since f is (γ,δ)-continuous, there exists $U_{\alpha(x)}\in m$ containing x such that and $f(\gamma(U_{\alpha(x)})\subseteq\delta(V_{\alpha(x)})$. Since $\{U_{\alpha(x)}:x\in A\}$ is a cover of A by m-open sets of X, there exists a finite subset A_0 of A such that $A\setminus\cup\{\gamma(U_{\alpha(x)}):x\in A_0\}\subseteq H_0$, where $H_0\in\mathscr{H}$. Therefore, $f(A)\subseteq\cup\{f(\gamma(U_{\alpha(x)})):x\in A_0\}\cup f(H_0)\subseteq\cup\{\delta(V_{\alpha(x)}):x\in A_0\}\cup f(H_0)$. Hence we have $f(A)\setminus\{\delta(V_{\alpha(x)}):x\in A_0\}\in f(\mathscr{H})$ and hence $f(A)$ is $\delta f(\mathscr{H})$-compact relative to n.

Theorem 4.7. *If $f:(X,m,\mathscr{H})\to(Y,n,f(\mathscr{H}))$ is weakly (γ,δ)-continuous and A is $\mathscr{H}$-compact relative to $\gamma(X)$, then $f(A)$ is $f(\mathscr{H})$-compact relative to $\delta(Y)$.*

Proof. Let $\{V_\alpha:\alpha\in\Delta\}$ be any cover of $f(A)$ by δ-open sets of Y. For each $x\in A$, there exists $\alpha(x)\in\Delta$ such that $f(x)\in V_{\alpha(x)}$. Since f is weakly (γ,δ)-continuous, $x\in f^{-1}(V_{\alpha(x)})\in\gamma(X)$ and $\{f^{-1}(V_{\alpha(x)}):x\in A\}$ is a cover of A by γ-open sets of X. Since A is $\mathscr{H}$-compact relative to $\gamma(X)$, there exists a finite subset A_0 of A and $H_0\in\mathscr{H}$ such that $A\subseteq\cup\{f^{-1}(V_{\alpha(x)}):x\in A_0\}\cup H_0$; hence $f(A)\subseteq\cup\{V_{\alpha(x)}:x\in A_0\}\cup f(H_0)$. Therefore, $f(A)$ is $f(\mathscr{H})$-compact relative to $\delta(Y)$.

Corollary 4.8. *Let $f:(X,m)\to(Y,n)$ be a surjective function.*

(1) *If f is (γ,δ)-continuous and (X,m) is $\gamma\mathscr{H}$-compact, then (Y,n) is $\delta f(\mathscr{H})$-compact.*
(2) *If f is weakly (γ,δ)-continuous and $(X,\gamma(X))$ is $\mathscr{H}$-compact, then $(Y,\delta(Y))$ is $f(\mathscr{H})$-compact.*

Definition 4.9. (Noiri and Popa 2006) A function $f:(X,m)\to(Y,n)$ is said to be *M-closed* if $f(B)$ is n-closed in Y for every m-closed subset B of X.

Lemma 4.10. *(Noiri and Popa 2006) For a function $f:(X,m)\to(Y,n)$, the following properties are equivalent:*

(1) *f is M-closed;*
(2) *for each $y\in Y$ and each $U\in m$ containing $f^{-1}(y)$, there exists $V\in n$ containing y such that $f^{-1}(V)\subseteq U$.*

Theorem 4.11. *Let $f : (X,m) \to (Y,n,\mathscr{H})$ be an M-closed surjective function. If $f^{-1}(y)$ is m-compact relative to m for each $y \in Y$ and B is $\mathscr{H}$-compact relative to n, then $f^{-1}(B)$ is $f^{-1}(\mathscr{H})$-compact relative to m.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of $f^{-1}(B)$ by m-open sets of X. Then for each $y \in B$, since $f^{-1}(y)$ is m-compact relative to m, there exists a finite subset $\Delta(y)$ of Δ such that $f^{-1}(y) \subseteq \cup\{U_\alpha : \alpha \in \Delta(y)\} = U_y$. Since U_y is an m-open set of X containing $f^{-1}(y)$ and f is M-closed, by Lemma **4.10** there exists an n-open set V_y containing y such that $f^{-1}(V_y) \subseteq U_y$. Since $\{V_y : y \in B\}$ is a cover of B by n-open sets of Y and B is $\mathscr{H}$-compact relative to n, there exists a finite subset B_0 of B such that $B \setminus \cup\{V_y : y \in B_0\} \in \mathscr{H}$. Therefore, $B \subseteq \cup\{V_y : y \in B_0\} \cup H_0$, where $H_0 \in \mathscr{H}$. Hence we have

$$\begin{aligned} f^{-1}(B) &\subseteq \cup\{f^{-1}(V_y) : y \in B_0\} \cup f^{-1}(H_0) \\ &\subseteq \cup\{U_y : y \in B_0\} \cup f^{-1}(H_0) \\ &\subseteq \cup\{U_\alpha : \alpha \in \Delta(y), y \in B_0\} \cup f^{-1}(H_0). \end{aligned}$$

We obtain $f^{-1}(B) \setminus \cup\{U_\alpha : \alpha \in \Delta(y), y \in B_0\} \in f^{-1}(\mathscr{H})$. This shows that $f^{-1}(B)$ is $f^{-1}(\mathscr{H})$-compact relative to m.

Corollary 4.12. *Let $f : (X,m) \to (Y,n,\mathscr{H})$ be an M-closed surjective function. If $f^{-1}(y)$ is m-compact relative to m for each $y \in Y$ and Y is $\mathscr{H}$-compact, then X is $f^{-1}(\mathscr{H})$-compact.*

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