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COMPUTER VIRUS PROPAGATION MODELLED AS A STOCHASTIC DIFFERENTIAL GAME

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ABSTRACT. The propagation of a computer virus is expressed as a stochastic differential game based on the two-dimensional Kermack-McKendrick model for the spread of epidemics. One optimizer tries to maximize the expected value of a cost function with quadratic control costs, while the other one wants to minimize this expected value. A particular problem is solved explicitly by making use of the method of similarity solutions to obtain the solution to the partial differential equation satisfied by the value function, subject to the appropriate conditions.

1. Introduction

Let $X(t)$ be the number of individuals, at time t, in a certain population who are susceptible to a given disease, and let $Y(t)$ be the number of those who are infected. Instead of the number of individuals, $X(t)$ and $Y(t)$ could represent the percentage of individuals. A classic model for the spread of epidemics is the following one proposed by Kermack and McKendrick (1927):

$$\left.\begin{aligned} dX(t) &= -k_1 X(t)Y(t)\,dt, \\ dY(t) &= k_1 X(t)Y(t)\,dt - k_2 Y(t)\,dt, \end{aligned}\right\} \tag{1}$$

where k_1 and k_2 are positive constants. That is, they assumed that the rate at which the susceptible individuals become infected is proportional to the product $X(t)Y(t)$.

There is also a three-dimensional version of this model:

$$\left.\begin{aligned} dX(t) &= -k_1 X(t)Y(t)\,dt, \\ dY(t) &= k_1 X(t)Y(t)\,dt - k_2 Y(t)\,dt, \\ dZ(t) &= k_2 Y(t)\,dt \end{aligned}\right\} \tag{2}$$

in which $Z(t)$ is the number (or percentage) of individuals who are removed from the population, because they are either recovered and immune, or quarantined, or dead. This is a particular SIR model (for Susceptible, Infected, Removed). These models have been used successfully to explain the spread of various diseases; Rachah and Torres (2015), for instance, proposed such a model to explain the 2014 outbreak of the Ebola virus in West

Africa. Lefebvre (2018b) considered the problem of optimally ending an epidemic (see also Ionescu *et al.* 2017).

Although the system in (2) was proposed to model the spread of diseases, in recent years many authors have used systems of the same type to model computer virus propagation (see, in particular, Mishra and Saini 2007; Mishra and Pandey 2011; Gan *et al.* 2012; Peng *et al.* 2013; Song *et al.* 2014; Qin 2015; Xu and Ren 2016, as well as the references therein).

In this paper, we consider a controlled version of the two-dimensional model (1):

$$\left.\begin{array}{rcl} dX(t) & = & -k_1 X(t) Y(t) u_1(t)\, dt, \\ dY(t) & = & k_1 X(t) Y(t) u_1(t)\, dt - k_2 Y(t) u_2(t)\, dt, \end{array}\right\} \tag{3}$$

where $u_i(t)$ is a control variable, for $i = 1,2$. Moreover, $X(t)$ is now the number or percentage of computers in a certain institution or region that are susceptible to a given virus, and $Y(t)$ is the number or percentage of those that are infected with the virus. Notice that Tong *et al.* (2016) also considered an optimal control problem based on the Kermack-McKendrick virus propagation model.

Next, we introduce some *noise* into the system:

$$\left.\begin{array}{rcl} dX(t) & = & -k_1 X(t) Y(t) u_1(t)\, dt, \\ dY(t) & = & k_1 X(t) Y(t) u_1(t)\, dt - k_2 Y(t) u_2(t)\, dt + \{v[Y(t)]\}^{1/2}\, dB(t), \end{array}\right\} \tag{4}$$

where $v(\cdot)$ is a non-negative function (the infinitesimal variance of the controlled stochastic process $\{Y(t), t \geq 0\}$) and $\{B(t), t \geq 0\}$ is a standard Brownian motion starting at $B(0) = 0$. Thus, there are two optimizers (or players). The first one (respectively second one), using $u_1(t)$ (resp. $u_2(t)$), wants to maximize (resp. minimize) the expected value of the following cost criterion:

$$J(x,y) := \int_0^{T(x,y)} \frac{1}{2} \left\{ q_2[X(t),Y(t)]\, u_2^2(t) - q_1[X(t),Y(t)]\, u_1^2(t) \right\} dt + c\, \frac{Y(T)}{X(T)}, \tag{5}$$

where $q_1(\cdot,\cdot)$ and $q_2(\cdot,\cdot)$ are positive functions, $c > 0$ is a constant and $T(x,y)$ is a *first-passage time* defined by

$$T(x,y) = \inf\left\{ t > 0 : \frac{Y(t)}{X(t)} = d_1 \text{ or } d_2 \mid X(0) = x > 0, Y(0) = y > 0 \right\}. \tag{6}$$

We assume that $0 \leq d_1 < y/x < d_2$.

The problem set up above defines a stochastic differential game (see Friedman 1972). However, while in Friedman's paper the final time T was fixed, here it is a random variable: the game ends the *first time* the ratio $Y(t)/X(t)$ becomes small or large enough. If $d_1 = 0$, there are no more infected computers, so that the second player should choose $u_2(t) \equiv 0$. Similarly, if $X(t)$ decreases to zero, so that the ratio $Y(t)/X(t)$ tends to infinity, the first player should take $u_1(t) \equiv 0$. Both optimizers must of course take the quadratic control costs into account.

Remarks

(i) The player using $u_1(t)$ is the person (the *hacker* or *pirate*) who wants to infect the computers, while the one who uses $u_2(t)$ could be a technician whose task is to disinfect the machines.

(ii) We could add a term in the cost function (5) that involves the value of $Y(t)$. Moreover, there could be two Brownian motions in the system. However, having only one such process is probably sufficient to take the randomness of the virus propagation into account.

(iii) In the three-dimensional version of the above problem, a third player could work to immunize the computers, so that they move from $X(t)$ to $Z(t)$, instead of waiting for them to become infected and then try to disinfect or *cure* them.

(iv) The stochastic differential game that we consider is related to the so-called LQG homing problems in optimal control (see Whittle 1982, 1990). The author has written several papers on these problems (see, for instance, Lefebvre 2018a; Lefebvre and Moutassim 2019, for recent papers).

In the next section, we will define the function from which the optimal values of $u_1(t)$ and $u_2(t)$ can be deduced, and we will derive the partial differential equation (p.d.e.) that this function satisfies. Then, in Section 3, a particular problem will be solved explicitly by making use of the method of similarity solutions to reduce the p.d.e. to an ordinary differential equation (o.d.e.). Finally, we will make a few concluding remarks in Section 4.

2. Optimal controls

To solve the problem set up in the previous section, we can make use of *dynamic programming*. First, we define the *value function*:

$$F(x,y) = \sup_{u_1(t), 0 \leq t \leq T(x,y)} \inf_{u_2(t), 0 \leq t \leq T(x,y)} E\left[J(x,y)\right]. \tag{7}$$

Next, we will derive the p.d.e. that $F(x,y)$ satisfies. From the solution of this p.d.e., subject to the appropriate boundary conditions, the optimal controls can be obtained.

Let $u_i := u_i(0)$, for $i = 1,2$. According to Bellman's principle of optimality, whatever the values of $u_i(t)$ in the interval $[0,\Delta t]$ and the resulting values $X(\Delta t)$ and $Y(\Delta t)$ of $X(t)$ and $Y(t)$, the players must choose the optimal values $u_i^*(t)$ of $u_i(t)$ in the interval $(\Delta t, T(x,y)]$, from $X(\Delta t)$ and $Y(\Delta t)$, in order to obtain the optimal policy. It follows that

$$\begin{aligned}
F(x,y) &= \sup_{u_1(t), 0 \leq t \leq \Delta t} \inf_{u_2(t), 0 \leq t \leq \Delta t} E\left[\int_0^{\Delta t} \frac{1}{2}\left[q_2 u_2^2(t) - q_1 u_1^2(t)\right] dt \right. \\
&\quad + F\big(x - k_1 x y u_1 \Delta t, y + (k_1 x y u_1 - k_2 y u_2)\Delta t \\
&\quad \left. + v^{1/2} B(\Delta t)\big) + o(\Delta t)\right] \\
&= \sup_{u_1(t), 0 \leq t \leq \Delta t} \inf_{u_2(t), 0 \leq t \leq \Delta t} \left\{\frac{1}{2}\left(q_2 u_2^2 - q_1 u_1^2\right)\Delta t \right. \\
&\quad + E\Big[F\big(x - k_1 x y u_1 \Delta t, y + (k_1 x y u_1 - k_2 y u_2)\Delta t \\
&\quad \left. + v^{1/2} B(\Delta t)\big)\Big] + o(\Delta t)\right\}.
\end{aligned}$$

Assume that F is differentiable with respect to x and twice differentiable with respect to y. Then, Taylor's formula enables us to write that

$$
\begin{aligned}
& E\left[F\left(x-k_1\,x y u_1\,\Delta t, y+(k_1\,x y u_1-k_2\,y u_2)\,\Delta t+v^{1/2}\,B(\Delta t)\right)\right] \\
& \quad = F(x,y)-k_1\,x y u_1\,\Delta t\,\frac{\partial F(x,y)}{\partial x}+(k_1\,x y u_1-k_2\,y u_2)\,\Delta t\,\frac{\partial F(x,y)}{\partial y} \\
& \qquad +\frac{1}{2}\,v\,\Delta t\,\frac{\partial^2 F(x,y)}{\partial y^2}+o(\Delta t),
\end{aligned}
$$

where we used the facts that $E[B(\Delta t)]=0$ and that $E[B^2(\Delta t)]=V[B(\Delta t)] = \Delta t$. Hence, we have

$$
\begin{aligned}
0 \quad = \quad & \sup_{u_1(t),0\leq t\leq \Delta t}\ \inf_{u_2(t),0\leq t\leq \Delta t}\left\{\frac{1}{2}\left(q_2\,u_2^2-q_1\,u_1^2\right)\Delta t-k_1\,x y u_1\,\Delta t\,\frac{\partial F(x,y)}{\partial x}\right. \\
& \left.+(k_1\,x y u_1-k_2\,y u_2)\,\Delta t\,\frac{\partial F(x,y)}{\partial y}+\frac{1}{2}\,v\,\Delta t\,\frac{\partial^2 F(x,y)}{\partial y^2}+o(\Delta t)\right\}.
\end{aligned}
$$

Finally, if we divide both sides of the above equation by Δt, and if we let Δt decrease to 0, we obtain the following *dynamic programming equation* (d.p.e.):

$$
\begin{aligned}
& \sup_{u_1}\inf_{u_2}\left\{\frac{1}{2}\left(q_2\,u_2^2-q_1\,u_1^2\right)-k_1\,x y u_1\,F_x+(k_1\,x y u_1-k_2\,y u_2)\,F_y\right. \\
& \qquad \left.+\frac{1}{2}\,v\,F_{yy}\right\}=0. \qquad (8)
\end{aligned}
$$

From the d.p.e., we deduce at once that, in terms of $F(x,y)$, the optimal values of u_1 and u_2 are given by

$$
u_1^*=\frac{k_1}{q_1}\,x y\,(F_y-F_x) \quad \text{and} \quad u_2^*=\frac{k_2}{q_2}\,y\,F_y. \qquad (9)
$$

Substituting these expressions for u_i^* into Eq. (8), we obtain (after simplification) the following proposition.

Proposition 2.1. *The value function satisfies the following second-order non-linear p.d.e.:*

$$
-\frac{k_2^2}{q_2}\,y^2\,F_y^2+\frac{k_1^2}{q_1}\,x^2\,y^2\,(F_y-F_x)^2+v\,F_{yy}=0. \qquad (10)
$$

Moreover, the boundary conditions are

$$
F(x,y)=c\,y/x \quad \text{if } y/x=d_1 \text{ or } d_2 \qquad (11)
$$

(since $T(x,y)=0$ if we start on either boundary).

In the next section, we will try to find the explicit solution of (10), (11) in particular cases. To do so, we will appeal to the method of similarity solutions.

3. Explicit solution

Let us assume that the value function $F(x,y)$ is actually a function of $w := y/x$. We define

$$H(w) = F(x,y). \tag{12}$$

The variable w is called the *similarity variable*. Notice that for this method to work, it must be possible to express both the p.d.e. and the boundary conditions in terms of w. Here, the boundary conditions become

$$H(w) = c\,w \quad \text{if } w = d_1 \text{ or } d_2. \tag{13}$$

Moreover, we find that Eq. (10) reduces to the second-order non-linear differential equation

$$-\frac{k_2^2}{q_2}\,w^2\,[H'(w)]^2 + \frac{k_1^2}{q_1}\,y^2\,(1+w)^2\,[H'(w)]^2 + \frac{1}{x^2}\,v H''(w) = 0. \tag{14}$$

We can state the following result.

Proposition 3.1. *If $q_2(x,y)$, $y^2/q_1(x,y)$ and $v(y)/x^2$ can all be expressed in terms of the similarity variable w, then the optimal controls u_1^* and u_2^* in Eq. (9) can be obtained from Eq. (12) and the solution of (14), (13).*

Remarks
(i) Actually, we only need to find $G(w) := H'(w)$ to determine the value of the optimal controls. However, the boundary conditions (13) are in terms of $H(w)$.
(ii) To be more general, we could have written that we must have

$$-\frac{k_2^2}{q_2(x,y)}\,w^2 + \frac{k_1^2}{q_1(x,y)}\,y^2\,(1+w)^2 = \phi(w)$$

(together with $v(y)/x^2 = \psi(w)$).

Particular cases for which the above proposition holds include the one when

$$q_2(x,y) \equiv q_{2,0}, \quad q_1(x,y) = q_{1,0}x^2 \quad \text{and} \quad v(y) = v_0\,y^2, \tag{15}$$

where $q_{1,0}, q_{2,0}$ and v_0 are positive constants, and the one when $q_1(x,y)$ is replaced by $q_{1,0}\,y^2$. In the next subsection, we will consider one such particular case.

3.1. Particular case. Let us choose $k_1 = k_2 = c = 1$ in (4) and (5), respectively, and $q_{1,0} = q_{2,0} = v_0 = 1$ in (15). Then, Proposition **3.1** holds.

Remark. With the above values, the uncontrolled process $\{Y(t), t \geq 0\}$ is a *geometric Brownian motion*. This process can be expressed as the exponential of a Brownian motion, and is therefore always positive, which makes sense in the application considered.

We find that we must solve the second-order non-linear o.d.e.

$$w\,(2+w)\,[H'(w)]^2 + H''(w) = 0, \tag{16}$$

which is actually a first-order (Riccati) equation for $G(w)$. The general solution of the above equation can be expressed as follows:

$$H(w) = \int \frac{3}{w^3 + 3w^2 + 3c_1}\,dw + c_2, \tag{17}$$

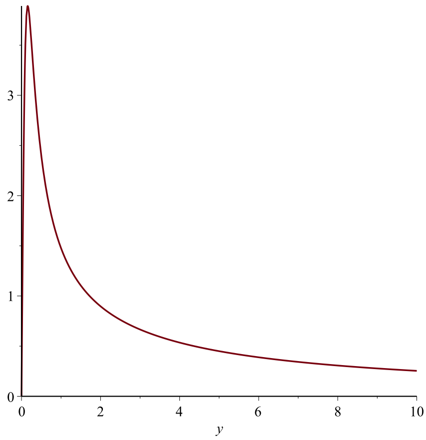

FIGURE 1. Optimal control u_1^* for $x = 1$ and $y \in [0,10]$ in the particular case considered in Subsection 3.1

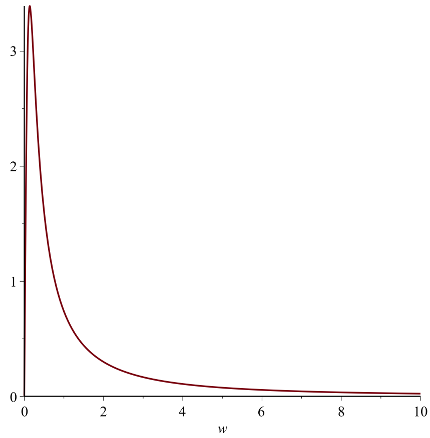

FIGURE 2. Optimal control u_2^* for $w := y/x \in [0,10]$ in the particular case considered in Subsection 3.1

where c_1 and c_2 are arbitrary constants. If we take $d_1 = 0$ in (6), we can write that

$$H(w) = \int_0^w \frac{3}{w^3 + 3w^2 + 3c_1} \, dw, \tag{18}$$

where c_1 must be such that

$$d_2 = \int_0^{d_2} \frac{3}{w^3 + 3w^2 + 3c_1} \, dw. \tag{19}$$

The value of c_1 can be computed numerically for any $d_2 > 0$. Let us choose $d_2 = 10$. We then find that $c_1 \simeq 0.02075$, so that

$$G(w) \simeq \frac{3}{w^3 + 3w^2 + 0.06225} \quad \text{for } 0 \leq w \leq 10. \tag{20}$$

Finally, from Eq. (9), we obtain that

$$u_1^* = \frac{y}{x^2}\left(1 + \frac{y}{x}\right) G(y/x) \quad \text{and} \quad u_2^* = wG(w). \tag{21}$$

Notice that u_1^* is not a function of w, contrary to u_2^*. The optimal controls are shown in Fig. 1 for $x = 1$ and $y \in [0,10]$, and in Fig. 2 for $w \in [0,10]$. We see that the two optimal control functions behave similarly. Indeed, they both move from 0 to a maximum for $y/x \simeq 0.15$, and then decrease rapidly toward 0. See Fig. 3 and Fig. 4, in which y or w belongs to $[0,1]$, for more clarity.

Therefore, both players make the maximum efforts to *win* the game when the ratio $x(t)/y(t)$ of remaining computers that are susceptible to the virus to the infected ones is around 6 or 7. When the ratio $y(t)/x(t)$ is either very small or relatively large, the players make little control efforts.

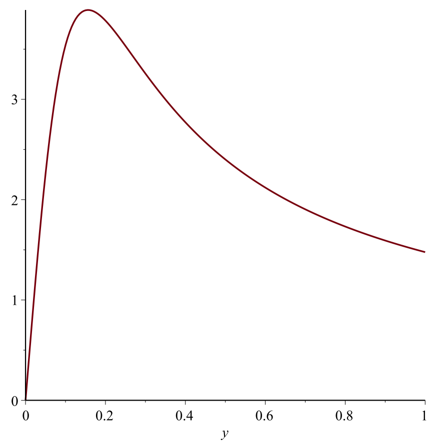

FIGURE 3. Optimal control u_1^* for $x = 1$ and $y \in [0, 1]$ in the particular case considered in Subsection 3.1

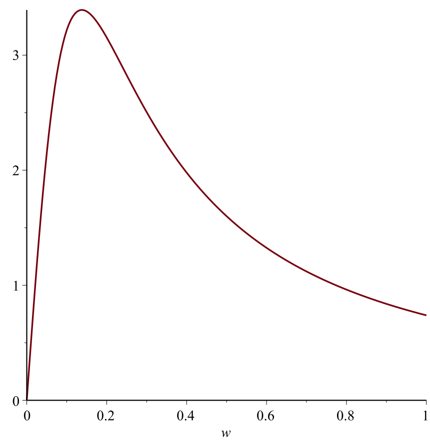

FIGURE 4. Optimal control u_2^* for $w := y/x \in [0, 1]$ in the particular case considered in Subsection 3.1

4. Conclusion

In this paper, we considered a stochastic differential game as a model for the propagation of a computer virus. The game ended at a random time. This type of problem is related to LQG homing problems and is difficult to solve explicitly, because it entails finding the solution of a non-linear partial differential equation. Here, we made use of the method of similarity solutions to obtain this solution. When this method does not apply, we could at least try to find approximate or numerical solutions.

As mentioned in the Introduction section, we could consider the three-dimensional version of the Kermack-McKendrick model, with three players. The game theoretical problem would obviously be even more difficult to solve, but it would also be more realistic.

Finally, both the two- and three-dimensional games could be considered in discrete time. Then, we would have to solve difference rather than differential equations.

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