DOI: 10.1478/AAPP.97S2A9

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 97, No. S2, A9 (2019)

A VARIANT OF THE ENERGY METHOD: AN APPLICATION TO THE BÉNARD PROBLEM FOR A VISCOELASTIC ROTATING FLUID

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ABSTRACT. In a previous paper of the second author the nonlinear Lyapunov stability of the thermodiffusive equilibrium of a viscoelastic rotating Walters fluid, in a horizontal rotating layer heated and salted from below was studied. If instability occurs as stationary convection a non linear stability bound coinciding with the linear one was derived. In this paper we reconsider the same problem recovering the same result without any restriction on the viscoelasticity parameter.

1. Introduction

The problem of the coincidence of linear and nonlinear stability boundaries is largely investigated, the point of loss of linear stability theory being usually a bifurcation point too (Prodi 1962; Yudovich 1965; Sattinger 1970; Yudovich 1970a,b; Yudovich 1989; Georgescu and Oprea 1994), at which subcritical instabilities may occur. In Palese (2019) a model of a second order viscoelastic Walters fluid involving one non Newtonian parameter was considered. The impossibility of predicting the behaviour of some fluids, e.g. those with high molecular weight, leads to the development of non Newtonian mechanics, of a great interest due to engineering applications, such as ground pollution by chemicals which are non Newtonian fluids, in biomedical applications, in agricolture. There are many models of viscoelastic fluids in literature, see (Beard and Walters 1964; Baris 2002; Gupta and Kumar 2010; Kumar and Singh 2010; Thirumurugan and Vasanthakumari 2013) and their extensive bibliography.

A model of a second order viscoelastic Walters fluid, that involves only a non-Newtonian parameter is considered.

In such a fluid the constitutive equation for the Cauchy stress tensor, in the case of a short memory, is given by Baris (2002):

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\eta_0\boldsymbol{D} - k_0\left[\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{D} - \boldsymbol{D}\cdot\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^T\cdot\boldsymbol{D}\right],$$

where p is the pressure, $\boldsymbol{I}$ is the identity tensor, $\boldsymbol{D}$ is the rate of strain tensor, $\boldsymbol{v}$ is the velocity, ∇ is the gradient operator,

$$\eta_0 = \int_0^\infty N(\tau) d\tau, \qquad n \geq 2,$$

is a limiting viscosity at a small rate of shear (Baris 2002), and

$$k_0 = \int_0^\infty \tau N(\tau) d\tau$$

is a short memory coefficient, where $N(\tau)$ is the distribution function of the relaxation time τ.

In this idealized model of a Walters fluid all terms involving

$$\int_0^\infty \tau^n N(\tau) d\tau, \qquad n \geq 2,$$

are neglected.

In the context of non-Newtonian fluids there are in literature many theoretical and experimental results on thermal instability of the thermodiffusive equilibrium in a fluid layer, including effects such as porosity (Baris 2002), Hall current (Gupta and Kumar 2010), dusty particles (Thirumurugan and Vasanthakumari 2013), and superposed fluids (Kumar and Singh 2010). In Palese (2019) the nonlinear Lyapunov stability problem of the thermodiffusive equilibrium for a second order viscoelastic Walters fluid in a rotating plane layer heated and salted from below was reformulated. Projecting the initial perturbation evolution equations on some suitable subspaces of the space of kinematically admissible functions, the contribution of skewsymmetric terms, such as Coriolis term was preserved, and, jointly, all the nonlinear terms vanish. The perturbation evolution equations in terms of poloidal and toroidal fields was derived and subsequently an energy relation where all nonlinear terms vanish preserving the contribution of the Coriolis term. A nonlinear stability bound, involving the viscoelasticity parameter, coinciding with the critical Rayleigh number of the linear instability, without any restriction on initial data was derived in the region of the parameter space where the principle of exchange of stabilities holds. In this paper, with a similar approach, by using some different Lyapunov functions, we recover the same result without any restriction on the viscoelasticity parameter.

2. The initial/boundary value problem for perturbation

Let us consider, in the framework of mechanics of continua and in the Oberbeck-Boussinesq approximation, a Walters fluid mixture in a horizontal layer S, bounded by the surfaces $z = 0$ and $z = 1$ in a frame of reference $\{O, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, with $\boldsymbol{k}$ unit vector in the vertical upwards direction, in rotation around the fixed vertical axis z with a constant angular velocity $\boldsymbol{\Omega} = \Omega \boldsymbol{k}$.

In this paper we study the nonlinear Lyapunov stability of the motionless state, i.e. we perturb only the initial conditions, leaving unchanged all the other data of the problem. The variation of the initial conditions obviously gives rise to a perturbation at any time $t \geq 0$.

Let us now denote with $\{\boldsymbol{0}, \overline{T}, \overline{C}, \overline{P}\}$, the solution corresponding to a motionless state, where $\boldsymbol{0}$, $\overline{T}$, $\overline{C}$, $\overline{P}$, represent, respectively, the velocity, the temperature, the concentration and the pressure fields at $t = 0$. The perturbation $(\boldsymbol{u}, \theta, \gamma, p')$ at any $t \geq 0$, of velocity, temperature, concentration and pressure fields satisfies the following nonlinear nondimensional equations (Chandrasekhar 1968; Joseph 1970; Baris 2002; Thirumurugan and Vasanthakumari 2013; Palese 2019)

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla p' + \mathscr{R}\theta\boldsymbol{k} - \mathscr{R}_c\gamma\boldsymbol{k} + 2\boldsymbol{u}\times\boldsymbol{\Omega} + \left(1-\mathscr{F}\dfrac{\partial}{\partial t}\right)\Delta\boldsymbol{u}, \\ P_r\left(\dfrac{\partial\theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta\right) = \Delta\theta + \mathscr{R}w, \qquad (t,\boldsymbol{x})\in[0,\infty[\times V \\ S_c\left(\dfrac{\partial\gamma}{\partial t} + \boldsymbol{u}\cdot\nabla\gamma\right) = \Delta\gamma - \mathscr{R}_c w, \\ \nabla\cdot\boldsymbol{u} = 0, \end{cases} \tag{1}$$

in the space $\mathscr{N}$ of the functions $\boldsymbol{u}(\cdot,t),\Delta\boldsymbol{u}(\cdot,t),p'(\cdot,t),\theta(\cdot,t)$ and $\gamma(\cdot,t)$ belonging to $W^{2,2}(V)$, $\forall t\in[0,\infty[$, with $\boldsymbol{u}(\boldsymbol{x},\cdot)\in C^1[0,\infty[$, $\forall\boldsymbol{x}\in V$, verifying the following boundary conditions

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = \theta = \gamma = 0 \quad \text{ on } \partial V, \tag{2}$$

suitable for stress free and temperature and concentration perfectly conductor planes.

In $(1)_1$-$(1)_3$ $\boldsymbol{u}=(u,v,w)$, $V=\mathscr{V}\times[0,1]$ denotes the three dimensional box over the rectangle $\mathscr{V}\equiv[0,2\pi/k_1]\times[0,2\pi/k_2]$, whose boundary is denoted by ∂V, after assuming the perturbation fields, depending on the time t and space $\boldsymbol{x}=(x,y,z)$, doubly periodic functions in x and y, of period $2\pi/k_1$ and $2\pi/k_2$.

$\mathscr{R}^2$ and $\mathscr{R}_c^2$ are the Rayleigh and solute Rayleigh numbers, P_r and S_c are the Prandtl and Schmidt numbers, respectively, $\mathscr{F}$ is a dimensionless viscoelasticity parameter. The representation theorem of solenoidal vectors (Joseph 1976a) in a plane layer, if the mean values of u,v,w vanish over $\mathscr{V}$ (Schmitt and Wahl Wolf 1992), i.e. if the conditions

$$\int_{\mathscr{V}} u(x,y,z)dxdy = \int_{\mathscr{V}} v(x,y,z)dxdy = \int_{\mathscr{V}} w(x,y,z)dxdy = 0, \quad \forall z\in[0,1],$$

hold, allow us to obtain, for the velocity perturbation $\boldsymbol{u}$ the unique decomposition (Joseph 1976a; Schmitt and Wahl Wolf 1992),

$$\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2, \tag{3}$$

with

$$\nabla\cdot\boldsymbol{u}_1 = \nabla\cdot\boldsymbol{u}_2 = \boldsymbol{k}\cdot\nabla\times\boldsymbol{u}_1 = \boldsymbol{k}\cdot\boldsymbol{u}_2 = 0, \tag{4}$$

$$\boldsymbol{u}_1 = \nabla\frac{\partial\chi}{\partial z} - \boldsymbol{k}\Delta\chi \equiv \nabla\times\nabla\times(\chi\boldsymbol{k}), \qquad \boldsymbol{u}_2 = \boldsymbol{k}\times\nabla\psi = -\nabla\times(\psi\boldsymbol{k}), \tag{5}$$

where $\nabla\times$ is the curl operator, the poloidal and toroidal potentials χ and ψ are doubly periodic functions satisfying the equations (Joseph 1976a)

$$\Delta_1\chi \equiv \frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} = -\boldsymbol{k}\cdot\boldsymbol{u}, \qquad \Delta_1\psi = \boldsymbol{k}\cdot\nabla\times\boldsymbol{u}. \tag{6}$$

From now on, we denote $\dfrac{\partial f}{\partial x} \equiv f_x$, where f is an arbitrary function. The boundary conditions for χ and ψ, for free planar surfaces, are (Joseph 1976a):

$$\chi = \chi_{zz} = \psi_z = 0, \qquad z = 0,1. \tag{7}$$

From (3)-(4) it follows that

$$\boldsymbol{u}\cdot\boldsymbol{k}=\boldsymbol{u}_1\cdot\boldsymbol{k}=-\Delta_1\chi. \tag{8}$$

In order to project the perturbation equation $(1)_1$ on some suitable subspaces of the space of kinematically admissible functions, we observe that, due to (5),

$$\boldsymbol{u}=\nabla\times\nabla\times(\chi\boldsymbol{k})-\nabla\times(\psi\boldsymbol{k}), \tag{9}$$

because of $\nabla\cdot\boldsymbol{u}=0$,

$$\Delta\boldsymbol{u}=-\nabla\times\nabla\times\boldsymbol{u}. \tag{10}$$

We apply the operator $\boldsymbol{I}-\boldsymbol{k}\otimes\boldsymbol{k}$ to the perturbation equation $(1)_1$, and we obtain

$$\frac{\partial\boldsymbol{u}^{\perp}}{\partial t}=-(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}^{\perp}-\nabla_1 p'+2\boldsymbol{u}\times\boldsymbol{\Omega}+\left(1-\mathscr{F}\frac{\partial}{\partial t}\right)\Delta\boldsymbol{u}^{\perp}, \tag{11}$$

where $\boldsymbol{u}^{\perp}=(\boldsymbol{I}-\boldsymbol{k}\otimes\boldsymbol{k})\boldsymbol{u}$ is a two dimensional vector.

Taking into account (9) and (10) it follows that

$$\boldsymbol{\Omega}\times\boldsymbol{u}^{\perp}=\boldsymbol{\Omega}\times(\nabla_1\chi_z-\nabla\times(\psi\boldsymbol{k}))=-\Omega\nabla\times(\chi_z\boldsymbol{k})-\Omega\nabla_1\psi. \tag{12}$$

By using the Weyl decomposition theorem (Mikhlin 1970; Georgescu 1985), the advective term in (11) can be uniquely written as

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\perp}=\nabla U+\nabla\times\boldsymbol{A}, \tag{13}$$

where U is a scalar function and $\boldsymbol{A}$ a vector field.
The imbedding Sobolev theorems of $W^{2,2}(V)$ in the space of continuous functions $C(\overline{V})$ (Sobolev 1963) allows us to prove the following identity

$$\nabla\times(\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\perp})\equiv\nabla\times(\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\perp}-\nabla U).$$

Let us define

$$\boldsymbol{B}=\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\perp}-\nabla U,$$

by choosing $\nabla\cdot\boldsymbol{B}=0$, the scalar function U is (up to a constant) the solution of the interior Neumann problem (Mikhlin 1970) in the periodicity cell V

$$\begin{cases}\Delta U=\Phi,\\ \dfrac{\partial U}{\partial\boldsymbol{n}}=\Gamma,\end{cases} \tag{14}$$

where $\Phi=\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u}^{\perp})$ and $\dfrac{\partial U}{\partial\boldsymbol{n}}$ is the normal derivative of U on the boundary ∂V of V and $\Gamma=-\boldsymbol{B}\cdot\boldsymbol{n}$. As the solution has continuous first derivatives in $\overline{V}$, the interior Neumann problem holds for ∂V piecewise smooth. The relation $\int_V\Phi dV-\int_{\partial V}\Gamma dV=0$, which is a necessary and sufficient (Taylor 2011) condition for the existence of a solution of (14), in our case becomes

$$\int_{\partial V}\boldsymbol{u}\cdot\nabla(\boldsymbol{u}_1^{\perp}+\boldsymbol{u}_2)\cdot\boldsymbol{n}d\sigma+\int_V\nabla\cdot\boldsymbol{B}dV=0.$$

It follows that exists a vector field $\boldsymbol{A}$ such that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, i.e. (13). Projecting the perturbation equation (11) on the subspace of solenoidal vectors, taking into account (9), (12) and (13) we have

$$-\frac{\partial}{\partial t}\nabla \times (\psi \boldsymbol{k}) = -\nabla \times \boldsymbol{A} + 2\Omega \nabla \times (\chi_z \boldsymbol{k}) - \nabla \times \Delta(\psi \boldsymbol{k}) + \mathscr{F}\frac{\partial}{\partial t}\nabla \times \Delta(\psi \boldsymbol{k}). \tag{15}$$

From (15) it follows that exists a scalar field F such that

$$-\frac{\partial}{\partial t}(\psi \boldsymbol{k}) = -\boldsymbol{A} + 2\Omega \chi_z \boldsymbol{k} - \Delta(\psi \boldsymbol{k}) + \mathscr{F}\frac{\partial}{\partial t}\Delta(\psi \boldsymbol{k}) + \nabla F. \tag{16}$$

Since the vector field $\boldsymbol{A}$ is defined up to the gradient of a scalar function, we can write (16) in the following form

$$-\frac{\partial}{\partial t}(\psi \boldsymbol{k}) = -\boldsymbol{A} + 2\Omega \chi_z \boldsymbol{k} - \Delta(\psi \boldsymbol{k}) + \mathscr{F}\frac{\partial}{\partial t}\Delta(\psi \boldsymbol{k}). \tag{17}$$

From (15) it follows that $\nabla \times \boldsymbol{A}$ is a two-dimensional vector, indeed

$$\boldsymbol{k} \cdot \nabla \times \boldsymbol{A} = 0, \tag{18}$$

as we can deduce from (16) too. Taking into account (18) and (13) we obtain

$$\boldsymbol{k} \cdot \nabla U = 0.$$

So the Neumann problem is a two dimensional one for the rectangle $\mathscr{V}$.
We can assume the third component of $\boldsymbol{A}$ equal to zero. The third component of (17) is

$$\frac{\partial \psi}{\partial t} = -2\Omega \chi_z + \Delta \psi - \mathscr{F}\frac{\partial}{\partial t}\Delta \psi. \tag{19}$$

3. Lyapunov stability

If we multiply $(1)_1$ by $\boldsymbol{u}$, $(1)_2$ by $\frac{b}{P_r}\theta$ and $(1)_3$ by $\frac{d}{S_c}\gamma$, where b, d are some positive parameters, adding the resulted equations and integrating over V, we obtain

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int_V \left[\boldsymbol{u}^2 + b\theta^2 + d\gamma^2\right] dV = \mathscr{R}\left(1 + \frac{b}{P_r}\right)(\theta, w) - \mathscr{R}_c\left(1 + \frac{d}{S_c}\right)(\gamma, w) + \\ + \frac{b}{P_r}(\theta, \Delta\theta) + \frac{d}{S_c}(\gamma, \Delta\gamma) + (\boldsymbol{u}, \Delta\boldsymbol{u}) - \mathscr{F}\left(\boldsymbol{u}, \frac{\partial}{\partial t}\Delta\boldsymbol{u}\right). \end{aligned} \tag{20}$$

Let us introduce the function

$$E_L(t) = \frac{1}{2}\int_V \left[\boldsymbol{u}^2 - \mathscr{F}\nabla^2\boldsymbol{u} + b\theta^2 + d\gamma^2\right] dV,$$

that, in terms of poloidal and toroidal fields, becomes

$$\begin{aligned} E_L(t) = \frac{1}{2}\Big[\|\chi_{xz}\|^2 + \|\chi_{yz}\|^2 + \|\Delta_1\chi\|^2 + \|\psi_y\|^2 + \|\psi_x\|^2 + b\|\theta\|^2 + d\|\gamma\|^2 \\ -\mathscr{F}\left(\|\nabla\chi_{xz}\|^2 + \|\nabla\chi_{yz}\|^2 + \|\nabla\Delta_1\chi\|^2 + \|\nabla\psi_y\|^2 + \|\nabla\psi_x\|^2\right)\Big]. \end{aligned}$$

We define now the function

$$E_L^* = E_L + \frac{c}{2}\left[\|\psi_x\|^2 + \|\psi_y\|^2 - \mathscr{F}\left(\|\psi_{xz}\|^2 + \|\psi_{yz}\|^2 + \|\Delta_1\psi\|^2\right)\right],$$

where c is a parameter that can be determined later.

From (19) and (20) we obtain the energy relation

$$\frac{dE_L^*}{dt} = \mathscr{I}^* - \mathscr{E}^*, \tag{21}$$

where

$$\mathscr{I}^* \equiv -\mathscr{R}\left(1+\frac{b}{P_r}\right)(\theta, \Delta_1\chi) + \mathscr{R}_c\left(1+\frac{d}{S_c}\right)(\gamma, \Delta_1\chi) +$$
$$+ 2c\Omega(\chi_z, \Delta_1\psi) - c\alpha(\Delta\psi, \Delta_1\psi),$$

$$\mathscr{E}^* \equiv \frac{b}{P_r}\|\nabla\theta\|^2 + \frac{d}{S_c}\|\nabla\gamma\|^2 + \|\nabla\chi_{xz}\|^2 + \left\|\nabla\chi_{yz}\right\|^2 + \|\nabla\Delta_1\chi\|^2 +$$
$$+ [1 + c(1-\alpha)]\left(\|\nabla\psi_x\|^2 + \left\|\nabla\psi_y\right\|^2\right).$$

Let us consider the definite positive functionals

$$E_1(t) = \frac{1}{2}\left[\|\chi_{xz}\|^2 + \left\|\chi_{yz}\right\|^2 + \|\Delta_1\chi\|^2 + \|\psi_x\|^2 + \left\|\psi_y\right\|^2 + b\|\theta\|^2 + d\|\gamma\|^2\right] +$$
$$+ \frac{c}{2}\left[\|\psi_x\|^2 + \left\|\psi_y\right\|^2\right],$$

$$E_2(t) = \frac{1}{2}\left[\|\nabla\chi_{xz}\|^2 + \left\|\nabla\chi_{yz}\right\|^2 + \|\nabla\Delta_1\chi\|^2 + \left\|\nabla\psi_y\right\|^2 + \|\nabla\psi_x\|^2\right]$$
$$+ \frac{c}{2}\left[\|\psi_{xz}\|^2 + \left\|\psi_{yz}\right\|^2 + \|\Delta_1\psi\|^2\right].$$

The energy relation (21) becomes

$$\mathscr{F}\frac{dE_2}{dt} - \frac{dE_1}{dt} = -\mathscr{I}^* + \mathscr{E}^*. \tag{22}$$

Applying the Chetaev instability theorem (La Salle and Lefschetz 1961) it follows that

$$\frac{1}{\sqrt{R_a^*}} = \max\frac{\mathscr{I}^*}{\mathscr{E}^*} > 1, \tag{23}$$

where the maximum is computed in the class of admissible functions, is a sufficient condition for instability.

Therefore, a necessary condition for the nonlinear global stability of the basic motion is

$$1 \leq \sqrt{R_a^*}, \tag{24}$$

and we obtain again the result of Palese (2019) without any condition on the viscoelasticity coefficient.

4. The maximum problem and the stability bound

The Euler- Lagrange equations associated with the maximum problem (23) are (Joseph 1976a):

$$\begin{cases} -\mathscr{R}\left(1+\dfrac{b}{P_r}\right)\Delta_1\theta+\mathscr{R}_c\left(1+\dfrac{d}{S_c}\right)\Delta_1\gamma-2\Omega c\Delta_1\psi_z+\dfrac{2}{\sqrt{R_a^*}}\Delta\Delta_1\chi=0,\\ -\mathscr{R}\left(1+\dfrac{b}{P_r}\right)\Delta_1\chi+\dfrac{b}{P_r}\dfrac{2}{\sqrt{R_a^*}}\Delta\theta=0,\\ \mathscr{R}_c\left(1+\dfrac{d}{S_c}\right)\Delta_1\chi+\dfrac{d}{S_c}\dfrac{2}{\sqrt{R_a^*}}\Delta\gamma=0,\\ 2\Omega c\Delta_1\chi_z-2c\alpha\Delta\Delta_1\psi-\dfrac{2}{\sqrt{R_a^*}}[1+c(1-\alpha)]\Delta\Delta_1\psi=0. \end{cases} \tag{25}$$

In the class of normal mode perturbations

$$(\chi(\boldsymbol{x}),\psi(\boldsymbol{x}),\theta(\boldsymbol{x}),\gamma(\boldsymbol{x}))=(W(z),Z(z),\Theta(z),\Gamma(z))e^{i(k_1x+k_2y)+\sigma t},$$

the equations (25) become

$$\begin{cases} k^2\mathscr{R}\left(1+\dfrac{b}{P_r}\right)\Theta-k^2\mathscr{R}_c\left(1+\dfrac{d}{S_c}\right)\Gamma+2\Omega ck^2DZ-\dfrac{2}{\sqrt{R_a^*}}(D^2-k^2)^2k^2W=0,\\ k^2\mathscr{R}\left(1+\dfrac{b}{P_r}\right)W+\dfrac{b}{P_r}\dfrac{2}{\sqrt{R_a^*}}(D^2-k^2)\Theta=0,\\ -k^2\mathscr{R}_c\left(1+\dfrac{d}{S_c}\right)W+\dfrac{d}{S_c}\dfrac{2}{\sqrt{R_a^*}}(D^2-k^2)\Gamma=0,\\ -\Omega ck^2DW+\left\{c\left[\alpha\left(1-\dfrac{1}{\sqrt{R_a^*}}\right)+\dfrac{1}{\sqrt{R_a^*}}\right]+\dfrac{1}{\sqrt{R_a^*}}\right\}(D^2-k^2)k^2Z=0, \end{cases} \tag{26}$$

where $k^2=k_1^2+k_2^2$ is the wave number.

To (26) we add the following boundary conditions:

$$W=D^2W=\Theta=D^2\Theta=\Gamma=D^2\Gamma=DZ=0. \tag{27}$$

Owing to (27) we choose (Chandrasekhar 1968; Georgescu 1985),

$$\begin{aligned} W(z)&=\sum_{n=1}^{\infty}W_n\sin(n\pi z), & Z(z)&=\sum_{n=1}^{\infty}Z_n\cos(n\pi z),\\ \Theta(z)&=\sum_{n=1}^{\infty}\Theta_n\sin(n\pi z), & \Gamma(z)&=\sum_{n=1}^{\infty}\Gamma_n\sin(n\pi z). \end{aligned} \tag{28}$$

From (26), (27) and (28) by using the backward integration technique and the ortogonality of the set $\{\sin n\pi z\}_{n\in\mathbb{N}}$ total in $L^2([0,1])$, we have, $\forall n \in \mathbb{N}$,

$$k^4\mathscr{R}^2\frac{P_r}{b}\left(1+\frac{b}{P_r}\right)^2+k^4\mathscr{R}_c^2\frac{S_c}{d}\left(1+\frac{d}{S_c}\right)^2=4\frac{1}{R_a^*}k^2(n^2\pi^2+k^2)^3+ \\ +4\Omega^2n^2\pi^2k^2\frac{1}{\sqrt{R_a^*}}\frac{c^2}{c\left[\alpha\left(\frac{1}{\sqrt{R_a^*}}-1\right)-\frac{1}{\sqrt{R_a^*}}\right]-\frac{1}{\sqrt{R_a^*}}}. \tag{29}$$

Differentiating with respect to c we obtain:

$$c=\frac{2}{\sqrt{R_a^*}}\frac{1}{\alpha\left(\frac{1}{\sqrt{R_a^*}}-1\right)-\frac{1}{\sqrt{R_a^*}}}, \tag{30}$$

whence (29) becomes:

$$k^4\mathscr{R}^2\frac{P_r}{b}\left(1+\frac{b}{P_r}\right)^2+k^4\mathscr{R}_c^2\frac{S_c}{d}\left(1+\frac{d}{S_c}\right)^2=4\frac{1}{R_a^*}k^2(n^2\pi^2+k^2)^3+ \\ +16\Omega^2n^2\pi^2k^2\frac{1}{R_a^*}\frac{1}{\left[\alpha\left(\frac{1}{\sqrt{R_a^*}}-1\right)-\frac{1}{\sqrt{R_a^*}}\right]^2}. \tag{31}$$

Substituting in (31) a value of α satisfying the equation

$$\left[\alpha\left(\frac{1}{\sqrt{R_a^*}}-1\right)-\frac{1}{\sqrt{R_a^*}}\right]^2=1,$$

we have

$$\frac{R_a^*}{4}=\frac{(n^2\pi^2+k^2)^3+4\Omega^2n^2\pi^2}{k^2\mathscr{R}^2\frac{P_r}{b}\left(1+\frac{b}{P_r}\right)^2+k^2\mathscr{R}_c^2\frac{S_c}{d}\left(1+\frac{d}{S_c}\right)^2}. \tag{32}$$

By choosing

$$\alpha\left(\frac{1}{\sqrt{R_a^*}}-1\right)-\frac{1}{\sqrt{R_a^*}}=1,$$

from (30) it follows that $c>0$.

Differentiating with respect to the parameters b and d (Joseph 1965, 1966) we obtain

$$\frac{b}{P_r}=1, \qquad \frac{d}{S_c}=1. \tag{33}$$

Substituting (33) in (32) we obtain

$$R_a^*(n^2,k^2,\Omega^2,\mathscr{R}^2,\mathscr{R}_c^2)=\frac{(n^2\pi^2+k^2)^3+4\Omega^2n^2\pi^2}{k^2\mathscr{R}^2+k^2\mathscr{R}_c^2},\forall n\in\mathbb{N}. \tag{34}$$

The minimum, with respect to n, is obtained for $n=1$, whence

$$R_a^*(1,k^2,\Omega^2,\mathscr{R}^2,\mathscr{R}_c^2)=\frac{(\pi^2+k^2)^3+4\Omega^2\pi^2}{k^2\mathscr{R}^2+k^2\mathscr{R}_c^2}.$$

That is we have proved the following theorem:

Theorem 1. *If the zero solution of (1) in $\mathscr{N}$ is nonlinearly globally stable then*

$$1 \leq R_a^*(1, k^2, \Omega^2, \mathscr{R}^2, \mathscr{R}_c^2),$$

that is

$$k^2\mathscr{R}^2 + k^2\mathscr{R}_c^2 \leq (\pi^2 + k^2)^3 + 4\Omega^2\pi^2,$$

where, if the principle of exchange of stabilities holds,the function on the right hand side is just the critical function of the linear instability theory.

Moreover, again from (22), if $\mathscr{F} = 0$, the perturbation equations (1) are the same of the Newtonian case and we have

Theorem 2. *Theorem 1 holds entirely for an ordinary rotating Newtonian mixture.*

Conclusions

In this paper we have studied the nonlinear Lyapunov stability of the thermodiffusive equilibrium for a rotating Walters' fluid in a horizontal layer.

Following Palese (2019) the initial perturbation evolution equations are written in terms of toroidal and poloidal fields, on some orthogonal subspaces of the kinematically admissible functions, to get an additional equation that allows us to preserve the Coriolis force, while all the nonlinear terms disappear, by using the standard L^2 norm.

We study the same problem considered in (Palese 2019) by varying the Lyapunov function in order to eliminate the condition on the viscoelasticity coefficient.

We obtain, as in Palese (2019), the same critical functions, in both linear and non linear cases, if the principle of exchange of stabilities holds.

We observe that, anyhow, in this paper we applied an idea similar to that followed in Georgescu and Palese (1996), Georgescu *et al.* (2000, 2001), Georgescu and Palese (2009), Georgescu and Palese (2010, 2011), and Palese (2014a,b,c), changing the given problem governing the perturbation evolution in order to obtain an optimum energy relation.

In a similar way in Labianca and Palese (2019) is obtained the equality of linear and nonlinear stability bounds for the classical magnetic Bénard problem.

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Paper contributed to the international workshop entitled "New approaches to study complex systems",
which was held in Messina, Italy (27–28 november 2017), under the patronage of the *Accademia Peloritana dei Pericolanti*
Manuscript received 30 July 2018; published online 20 December 2019