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ON A RECURSIVE METHOD FOR FEEDBACK LINEARIZATION OF NONLINEAR SYSTEMS: THE CASE OF MIXING FLOW DYNAMICAL SYSTEM

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ABSTRACT. The paper aims to approach a design method for transforming a non-linear system in order to find a state-feedback controller form for it. The method is based on a recursive algorithm and takes into account the *controllability matrix* of the initial system. The efficiency of the recursive algorithm compared to classical feedback linearization method is tested on the 2d mixing flow dynamical system, in a perturbed form with a logistic-type term. It is outlined both the shorter time for the calculus, and the final form less complex for the global non-linear controller associated to the model.

1. Introduction

The feedback linearization method is applicable to a broad class of nonlinear control problems. It is an approach to nonlinear control design that has attracted lot of research in recent years. The central idea is to algebraically transform nonlinear systems dynamics into (fully or partly) linear ones, so that linear control techniques can be applied. It is important to notice that this differs entirely from conventional (Jacobian) linearization, because feedback linearization is achieved by *exact state transformation and feedback*, rather than by linear approximations of the dynamics. The basic idea of simplifying the form of a system by choosing a different state representation is not completely unfamiliar; rather it is similar to the choice of reference frames or coordinate systems in mechanics. Thus, we can state that "*feedback linearization = way of transforming original system models into equivalent models of a simpler form*".

The feedback linearization technique has multiple approaches. Basically the approach is based on concepts of nonlinear systems theory and contains two fundamental non-linear controller design techniques: input-output linearization and state-space linearization (Isidori 1989; Henson and Seborg 2005). A general approach of the feedback method works for differential systems of the form:

$$\dot{x} = f(x) + g(x) \cdot u \tag{1}$$

with $f, g : D \subset \mathbb{R}^n \mapsto \mathbb{R}^n$ and $u \in \mathbb{R}$.

A diffeomoprhism

$$\mathbf{T} : D \subset \mathbb{R}^n \mapsto \mathbb{R}^n \tag{2}$$

which defines a coordinate transformation $z = \mathbf{T}(x)$ is needed, in order to find for the system (1) a state space realization of the form

$$z = A \cdot z + B \cdot v \tag{3}$$

The method was presented in detail by Isidori (1989). The transformation $\mathbf{T}$ has to be obtained in special conditions for the partial derivatives of f, g. The components T_i of $\mathbf{T}$ have to fulfill the following systems of partial derivatives equations:

$$\begin{cases} \frac{\partial T_i}{\partial x} \cdot g(x) & = & 0, i = 1, 2, ...n = 1 \\ \\ \frac{\partial T_n}{\partial x} \cdot g(x) & \neq & 0 \end{cases} \tag{4}$$

and

$$\frac{\partial T_i}{\partial x} \cdot f(x) = T_{i+1}, i = 1, 2, ...n - 1 \tag{5}$$

Also, A and B are $n \times n$, respectively $n \times 1$ matrix in the controllable form, namely:

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \; B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{6}$$

If it is possible to find $\mathbf{T}$, then it is possible to find a static feedback control law of the form

$$u = \alpha(x) + \beta(x) \cdot v \tag{7}$$

where v is the new control and $\beta(x)$ is assumed to be non-zero for all x, such that the composed dynamics of the new system expressed in z coordinates, is the controllable system:

$$\begin{cases} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & z_3 \\ & \vdots & \\ \dot{z}_{n-1} & = & z_n \\ \dot{z}_n & = & v \end{cases} \tag{8}$$

Recently Ionescu (2017), it was tested a slightly perturbed version of the basic $2d$ mixing flow model (Ottino 1989):

$$\begin{cases} \dot{x}_1 & = & G \cdot x_2 \\ \dot{x}_2 & = & K \cdot G \cdot x_1 \end{cases} \tag{9}$$

namely the following system was taken into account:

$$\begin{cases} \dot{x}_1 & = & G \cdot x_2 + x_1 \\ \dot{x}_2 & = & K \cdot G \cdot x_1 - x_2 \end{cases} \tag{10}$$

where $-1 < K < 1, G > 0$.

It was found that the transformation $\mathbf{T}$ exists in the conditions (4)–(5), and the inverse system of (10), in its controllable form was found as:

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = 2 \cdot G\left(K \cdot z_1^2 - z_2^2\right) \cdot u - 4 \cdot K \cdot G^2 \cdot z_1 \cdot z_2 \end{cases} \tag{11}$$

2. Recursive method for feedback linearization

A common method of controlling a non-linear system involves linearizing the system about an operating point, and then using linear feedback control methods to design a controller. It is a successful approach in the case when the operation of the system is restricted to a small region about the chosen operating point. For the case of a wider range of operation this technique could fail. Therefore, some alternative methods are required, for linearizing the system about a series of operating points. There are a number of quite recent papers that suggest new approaches to control of non-linear systems (Sommer 1980; Su 1982). We shall focus in what follows on transformation of non-linear systems into quasi-controller canonical forms. It is generally known (Kailath 1980) that if a linear system represented by the triplet $\{A, b, c\}$ is completely controllable, then it can be reduced via a non-singular transformation to an equivalent controllable form. Often for practice, in order to determine a control law, it is advantageous to work with such an equivalent form rather than with the original one. There are few different approaches to simplification of a non linear system. We shall focus in what follows on that proposed by Su (1982). The class of systems considered can be described by the equation:

$$\dot{x(t)} = a(x(t)) + b(x(t))u(t) \tag{12}$$

where $a, b \in \mathbb{R}^n$ are analytic in a neighbourhood of the origin, and $a(0) = 0$. We are looking to find sufficient conditions on a and b so that there exists a C^∞ transformation $z^* = \mathbf{T}(z)$ such that the system (12) can be transformed into the *global non-linear controller form*:

$$\frac{d}{dt}\begin{bmatrix} z_1^* \\ z_2^* \\ \vdots \\ z_{n-1}^* \\ z_n^* \end{bmatrix} = \begin{bmatrix} z_2^* \\ \vdots \\ z_n^* \\ f(z^*) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{13}$$

The recursive algorithm that we shall use is based on the controllability matrix of the system (12). In this sense is useful to remind the following definitions.

For two vector fields f and g on R^n, the *Lie bracket* $[f, g]$ is a vector field defined by

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g \tag{14}$$

where $\partial f / \partial x, \partial g / \partial x$ are the jacobians of f and g respectively.

The Lie bracket is also denoted as $[f, g] = \left(ad^1 f, g\right)$. Thus we define iteratively

$$\left(ad^k f, g\right) = \left[f, \left(ad^{k-1} f, g\right)\right] \tag{15}$$

where $\left(ad^0 f, g\right) = g$.

For a scalar field h and a vector field $f = (f_1, f_2, \dots, f_n)^T$ the *Lie derivative of h with respect to f* is

$$\langle dh, f\rangle = \frac{\partial h}{\partial x_1} f_1 + \cdots + \frac{\partial h}{\partial x_n} f_n \tag{16}$$

Note that $\langle dh, f\rangle = \nabla h f$.

The recursive algorithm is now as follows (Zak 2003):

i) Construct the controllability matrix of the system:

$$C = \left[b\ (ad^1 a, b)\ (ad^2 a, b) \dots (ad^{(n-1)} a, b) \right] \tag{17}$$

ii) If there exist C^{-1} denote by q its last row;

iii) Solve the equation

$$\frac{\partial T_1}{\partial x} = \mu(x) \cdot q(x) \tag{18}$$

and obtain the first component of T;

iv) Construct the other components of $\mathbf{T}$, applying the recursive relation

$$T_{i+1} = \langle dT_i, a\rangle\,,\ i = 1, \dots, n-1\,. \tag{19}$$

It must be noticed that in this recursive method we need to find only the first component T_1 of the transformation $\mathbf{T}$. This makes all the calculus easier, comparing to the form presented in the section **2**, where the conditions (4)–(5) for finding T_i become really complex for $n \geq 3$.

3. Recursive algorithm for mixing flow dynamical system. Comparative analysis

In this section we present the feedback linearization of the dynamical system associated to the mixing flow model, for a slightly perturbed form of this model. It is outlined the comparison of the results in the two forms of the method. First let us note that the dynamical systems (1) and (12) on which the feedback method works in the sections **2** and **3** respectively, are similar. Let us take into account the mixing flow dynamical system perturbed with a logistic type term (Ionescu 2012):

$$\begin{cases} \dot{x_1} &= Gx_2 \\ \dot{x_2} &= KGx_1 + G(x_2 - x_1) \end{cases} \tag{20}$$

where $-1 < K < 1$, $G > 0$.

i) Following the method described in the first section, the transformation $\mathbf{T} = (T_1, T_2)^T$ is found as

$$\mathbf{T}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 + KG \\ 2Gx_1x_2 \end{pmatrix} \tag{21}$$

and following the calculus, the global controller form associated to (20) is

$$\begin{cases} \dot{z_1} = & z_2 \\ \dot{z_2} = & 2G^2(z_2^2 + Kz_1^2 + 2G^2 z_1(z_2 - z_1) \end{cases} \tag{22}$$

where $-1 < K < 1$, $G > 0$.

ii) Following the recursive method we find that

$$\frac{\partial T_1}{\partial x_1} = 1,\ T_2 = < dT_1, a >$$

and therefore the transformation $\mathbf{T} = (T_1, T_2)^T$ is found as

$$T_1(x_1,x_2) = x_1,\ T_2(x_1,x_2) = Gx_1x_2 \tag{23}$$

or, equivalently,

$$\mathbf{T}(\mathbf{x}) = \begin{pmatrix} x_1 \\ Gx_1x_2 \end{pmatrix}$$

The controllability matrix C has the form:

$$C = \begin{pmatrix} 0 & -G^2(x_2 - x_1) \\ G(x_2 - x_1) & G^2Kx_1 - G^2x_2 \end{pmatrix}$$

Further, the global non-linear controller form (13) of the system is in this case the following:

$$\begin{cases} \dot{z_1} &= z_2 \\ \dot{z_2} &= KG^2z_1 + G(z_2 - Gz_1)u \end{cases} \tag{24}$$

Looking at (22) and (24), it is easily to observe that the form (24) of the global controller form for the system (20), is significantly less complex, as it does not contain second order terms. Thus, by this recursive method, the number of partial derivatives equations to be solved is considerably reduced, and in fact the problem of constructing the desired T is reduced to find appropriate T_1. The form of the transformation T is simpler, too.

4. Conclusions

A basic conclusion is that the mixing flow dynamical system perturbed with a logistic-type term admits an *equivalent non-linear controller form* in the mathematical context of state-space linearization theory. The condition on the controllability matrix is basic in this sense. It is important to notice that the parameters distribution in the model is very different in this equivalent form; in fact the last equation of the model takes all the information. This is true both for the controllable equivalent forms (8) and (13) of the dynamical system. Often for practice, in order to determine a control law, it is advantageous to work with such an equivalent form rather than with the original one. Therefore a next stage of the analysis is a comparative analysis of the solution behavior for the initial and equivalent form of the model. The two-dimensional case offers the interesting feature that the feedback linearized model can reduce in fact to the *second order non-linear oscillator* with polynomial nonlinearities. The forms (11), (22) and (24) can all be reduced to a second order non-linear oscillator form. This is pointed out also in Ionescu and Munteanu (2017), for another wide spread 2d dynamical model, and brings into attention some new directions for further approach and analysis, such as a comparative analysis of the phase portrait for the equivalent systems. Finding the global non-linear controller form of a dynamical system implies in the theory of feedback linearization, finding the control u. At this point we shall state that we achieve an exact linearization. Finding the control u could be important in the study of the dynamical system, in order to better analyse the behaviour of the controlled system, and to try to reduce the chaos, when it appears. This is also an important target of the analysis of the mixing flow dynamical system, necessary to complete the events panel for this model and to generalize the results for the associated three dimensional model.

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