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AN APPLICATION OF QUEUEING THEORY IN HYDROLOGY

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ABSTRACT. A model is considered for the flow of a river when it is high. The model is based on queueing theory. An application to the Delaware River, located in the United States, is presented. It is shown that an $M/M/1/c$ queueing model is realistic when the flow exceeds a certain threshold. Using this model, one can forecast what would happen if the rate at which events occur increases. The results can be extended by considering more general birth-and-death stochastic processes.

1. Introduction

In hydrology, a basic problem is to forecast the flow of rivers. Dam managers, in particular, need reliable forecasts to make important decisions. They must decide whether to release some water in order to avoid flooding, or to keep the water to produce more electricity. To forecast river flows, some companies in Canada use a deterministic model called PRÉVIS. This model makes use of 18 variables, such as maximum and minimum temperatures, amount of precipitation, humidity, etc., to produce its forecasts. Because PRÉVIS did not provide very good short-term forecasts (that is, up to three days in advance), various stochastic models were proposed as an alternative to this deterministic model. These stochastic models were sometimes based on diffusion processes, such as geometric Brownian motion (see (Lefebvre 2002)), or on filtered renewal processes Lefebvre (2005) and Lefebvre and Guilbault (2008) , in particular. It was found that the simpler stochastic models considered did better than PRÉVIS for short-term forecasts. However, they were not able to outperform PRÉVIS for 7-day ahead forecasts, for instance. In practice, it seems that forecasts for very short-term periods are the most important ones for people like dam managers. Here, we will try to fit a simple queueing model, namely the $M/M/1/c$ model, to real data from an important river located in the USA. The aim is not to obtain precise daily flow forecasts. Rather, we want to determine the proportion of time that the flow will spend in a given interval over a long period of time. Based on the model, we can also estimate the probability that the flow value will be located in a certain interval the day after the current observation.

2. The $M/M/1/c$ model

Let $\{X(t), t \geq 0\}$ be a continuous-time Markov chain with discrete state space $\{0, 1, \ldots, c\}$, where $X(t)$ denotes the number of *customers* in the system at time t. We assume that

- the customers arrive according to a Poisson process with rate λ;
- the service times are independent and exponentially distributed random variables with mean $1/\mu$;
- there is a single server;
- the system capacity is equal to c;
- the service discipline is FIFO (First-In First-Out).

Then $\{X(t), t \geq 0\}$ is an $M/M/1/c$ queue.

It can be shown Ross (2014), for instance that

$$\pi_j := \lim_{t\to\infty} P[X(t) = j] = \frac{\rho^j(1-\rho)}{1-\rho^{c+1}}$$

for $j = 0, 1, \ldots, c$, where

$$rho := \frac{\lambda}{\mu}.$$

In this paper, $X(t)$ will denote the discretized flow of a river at time t, when it is *high*. The states will be defined in such a way that the process $\{X(t), t \geq 0\}$ could be an $M/M/1/c$ queue. The parameter λ is the rate at which hydrological events that cause a significant increase of the river flow occur, while $1/\mu$ gives the average time that a given event caused a flow increase.

For any continuous-time Markov chain, the time τ_j that the process spends in a given state j must be a random variable having an exponential distribution with parameter ν_j. In the case of the $M/M/1/c$ queue, the process can only move from 0 to 1, from a state j to $j+1$ or $j-1$ for $j = 1, 2, \ldots, c-1$, and only from c to $c-1$. It follows that

$$\nu_0 = \lambda, \quad \nu_c = \mu \quad \text{and} \quad \nu_j = \lambda + \mu \quad \text{for} \quad j = 1, 2, ..., c-1$$

If the $M/M/1/c$ model is acceptable for a given river, then the proportion of time that the process spends in state j, over a long period of time, should be given by π_j, for all values of j. The above assumptions will be checked for the Delaware River. Then, using the model, we could forecast, in particular, what the effect on the river flow an increase in the parameter λ would be.

3. An application to the Delaware River

We considered the flow of the Delaware River, located in the United States, at the Montague station for the period from 23 January 2011 to 22 January 2014, that is, a three-year period. The data are freely available on the Web site of the U.S. Geological Survey.

During these three years, the flow was greater than or equal to 10.000 ft^3/s on 203 days. We define the states

- 0: $10.000 \leq X(t) < 15.000$;
- 1: $15.000 \leq X(t) < 20.000$;
- 2: $20.000 \leq X(t) < 30.000$;

- 3: $30.000 \leq X(t)$.

The histograms obtained for the random variables τ_j are shown in Figures 1 to 4.

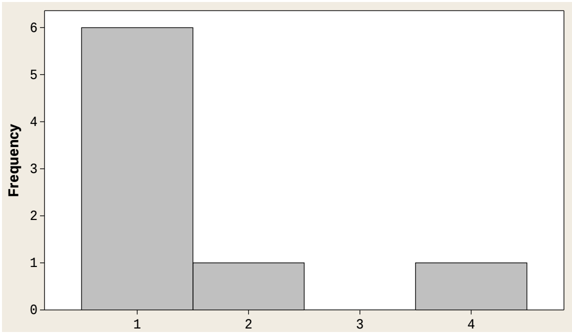

FIGURE 1. Histogram of τ_0

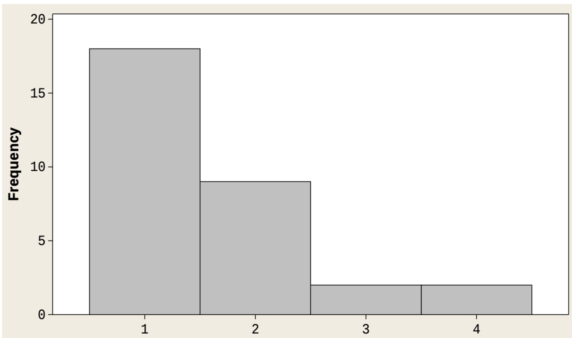

FIGURE 2. Histogram of τ_1.

From the means of the observations of the random variables τ_j, for $j = 1,2,3$, we deduce that

- $\hat{\lambda} + \hat{\mu} \simeq 0,62$
- $\hat{\lambda} + \hat{\mu} \simeq 0,60$
- $\hat{\mu} \simeq 0,39$

Hence, $\hat{\lambda} \simeq 0,22$ and

$$\hat{\rho} = \frac{\hat{\lambda}}{\hat{\mu}} \simeq 0,56.$$

Next, we computed the number $n_{i,j}$ of transitions from state i to state j:

$i \setminus j$	0	1	2	3
0	–	4	4	0
1	22	–	5	4
2	5	14	–	3
3	0	0	9	–

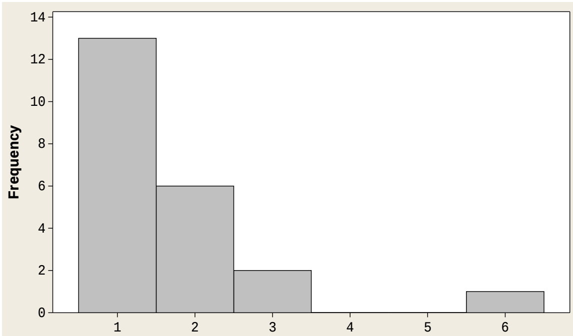

FIGURE 3. Histogram of τ_2.

Finally, we computed the number m_j of days that the process spent in state j during the 203 days considered:

- 0: 92;
- 1: 51;
- 2: 37;
- 3: 23.

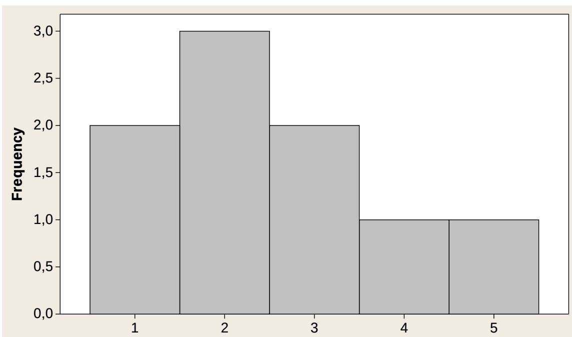

FIGURE 4. Histogram of τ_3.

From the ratio $92/203$, we deduce that

$$\hat{\rho} \simeq 0,61.$$

Making use of this value in the formula for π_j, we find that m_j should have been equal to

- 0: 92;
- 1: 56 (versus 51);
- 2: 34 (versus 37);
- 3: 21 (versus 23).

Hence, we can conclude that the model is actually quite good to estimate the limiting probabilities π_j for $j = 0,1,2,3$.

4. Concluding remarks

We saw that an $M/M/1/3$ queueing model provided a good fit to the data representing the flow of the Delaware River when it is greater than or equal to 10.000 ft^3/s. Based on this model, it is easy to see what would happen if the rate λ at which important hydrological events occur increases significantly. In general, we could try to fit a continuous-time Markov chain to the data for which $p_{i,j} \geq 0$ for any pair of states i and j.

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