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AN OPTIMAL CONTROL PROBLEM FOR A WIENER PROCESS WITH RANDOM INFINITESIMAL MEAN

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ABSTRACT. We consider a stochastic optimal control problem for one-dimensional diffusion processes with random infinitesimal mean and variance that depend on a continuous-time Markov chain. The process is controlled until it reaches either end of an interval. The aim is to minimize the expected value of a cost criterion with quadratic control costs on the way and a final cost. A particular case with a Wiener process will be treated in detail. Approximate and numerical solutions will be presented.

1. Introduction

We consider a one-dimensional controlled diffusion process $\{X(t), t \geq 0\}$ defined by the stochastic differential equation

$$dX(t) = f[X(t)]dt + b_0 u[X(t)]dt + \sqrt{v[X(t)]}dW(t), \tag{1}$$

where $b_0 \neq 0$ is a constant, $v(\cdot) > 0$ and $\{W(t), t \geq 0\}$ is a standard Brownian motion. Our aim is to minimize the expected value of the cost function

$$J(x) = \int_0^{T(x)} \left\{ \frac{1}{2} q_0 u^2[X(t)] + \lambda \right\} dt + K[X(T(x))], \tag{2}$$

where $q_0\,(> 0)$ and λ are constants, K is a general termination cost function and $T(x)$ is a random variable defined by

$$T(x) = \inf\{t \geq 0 : X(t) = a \text{ or } b \mid X(0) = x \in [a, b]\}. \tag{3}$$

Notice that if the parameter λ is positive (respectively negative), then the optimizer wants to leave the interval $[a, b]$ as soon (respectively late) as possible, while taking the quadratic control costs into account.

Whittle (1982) considered this type of problem, which he called "LQG homing", in n dimensions. He showed that, in some cases, it is possible to express the optimal control in terms of a mathematical expectation for the uncontrolled process that corresponds to $\{X(t), t \geq 0\}$. In practice, the problems that can be solved explicitly are generally for the case when $n = 1$, or the one for which we can use symmetry to reduce the problem from n to only one dimension.

Lefebvre has published many papers on LGQ homing; (Ionescu *et al.* 2016; Lefebvre and Ionescu 2016; Lefebvre and Zitouni 2016, see, for instances) for recent papers. (Kuhn 1985) has treated the case when the cost function takes the risk-sensitivity of the optimizer into account. Makasu has also published various papers on homing problems. He was able, in particular, to solve explicitly a two-dimensional problem; see (Makasu 2013).

In this paper, we will consider the case when the infinitesimal parameters of the controlled process $\{X(t), t \geq 0\}$ are random and vary according to the values taken by a continuous-time Markov chain. In the above formulation, to solve our problem we can first derive the dynamic programming equation that the value function

$$F(x) := \inf_{u[X(t)],\, 0 \leq t \leq T(x)} E[J(x)] \tag{4}$$

satisfies. To obtain an explicit expression for $F(x)$, we must solve a second-order non-linear differential equation. However, when the infinitesimal parameters are random, we will have to solve a system of equations that are second-order non-linear differential-difference equations.

Next, we will try to find approximate analytical solutions in the particular case when the uncontrolled process is a Wiener process with non-zero drift. These approximate solutions will be compared to a numerical solution to assess their quality.

2. Random infinitesimal parameters

Let $\{Y(t), t \geq 0\}$ be a continuous-time Markov chain with state space $E = \{1, \ldots, k\}$. The process spends a random exponential time with parameter ν_i in state i before making a transition to state $j \neq i$ with probability p_{ij}.

We now consider the controlled process $\{(X(t), Y(t)), t \geq 0\}$ defined by

$$dX(t) = f[X(t), Y(t)]dt + b_0 u[X(t), Y(t)]dt + \sqrt{v[X(t), Y(t)]}dW(t). \tag{5}$$

The final time becomes the first-passage time

$$T(x, i) = \inf\{t \geq 0 : X(t) = a \text{ or } b \mid X(0) = x \in [a, b], Y(0) = i\}, \tag{6}$$

for $i = 1, \ldots, k$, and the aim is to find the control $u^*[X(t), i]$ that minimizes the expected value of

$$J(x, i) := \int_0^{T(x,i)} \left\{ \frac{1}{2} q_0 u^2[X(t), i] + \lambda_i \right\} dt + K_i[X(T(x, i))]. \tag{7}$$

Let

$$F(x, i) := \inf_{u[X(t), i],\, 0 \leq t \leq T(x,i)} E[J(x, i)]. \tag{8}$$

By making use of Bellman's principle of optimality, we can write that

$$\begin{aligned} F(x,i) &= \inf_{u[X(t),i],\, 0\le t\le \Delta t} E\left[\int_0^{\Delta t}\left\{\frac{1}{2}q_0u^2[X(t),i]+\lambda_i\right\}dt\right. \\ &\quad +F\left(x+[f(x,i)+b_0u(x,i)]\Delta t+\sqrt{v(x,i)}W(\Delta t),Y(\Delta t)\right) \\ &\quad \left.+o(\Delta t)\right] \\ &= \inf_{u[X(t),i],\, 0\le t\le \Delta t} E\left[\frac{1}{2}q_0u^2(x,i)\Delta t+\lambda_i\Delta t\right. \\ &\quad +F\left(x+[f(x,i)+b_0u(x,i)]\Delta t+\sqrt{v(x,i)}W(\Delta t),i\right)(1-\nu_i\Delta t) \\ &\quad +\sum_{j\neq i}F\left(x+[f(x,j)+b_0u(x,j)]\Delta t+\sqrt{v(x,j)}W(\Delta t),j\right)\nu_i\Delta t\,p_{ij} \\ &\quad \left.+o(\Delta t)\right]. \end{aligned} \tag{9}$$

Next, with $W(0)=0$, we can write that

$$E[W(\Delta t)]=0 \tag{10}$$

and that

$$E[W^2(\Delta t)]=V[W(\Delta t)]=\Delta t. \tag{11}$$

Hence, assuming that F is twice differentiable with respect to x, we deduce from Taylor's formula that, for any $i\in E$,

$$\begin{aligned} &E\left[F\left(x+[f(x,i)+b_0u(x,i)]\Delta t+\sqrt{v(x,i)}W(\Delta t),i\right)\right] \\ &=F(x,i)+[f(x,i)+b_0u(x,i)]\Delta t\frac{dF(x,i)}{dx}+\frac{1}{2}v(x,i)\Delta t\frac{d^2F(x,i)}{dx^2} \\ &\quad +o(\Delta t). \end{aligned} \tag{12}$$

It follows that (because $\sum_{j\neq i}p_{ij}=1$)

$$\begin{aligned} 0 &= \inf_{u[X(t),i],\, 0\le t\le \Delta t}\left\{\frac{1}{2}q_0u^2(x,i)\Delta t+\lambda_i\Delta t\right. \\ &\quad +[f(x,i)+b_0u(x,i)]\Delta t F'(x,i)+\frac{1}{2}v(x,i)\Delta t F''(x,i) \\ &\quad \left.+\sum_{j\neq i}[F(x,j)-F(x,i)]\,\nu_i\Delta t\,p_{ij}+o(\Delta t)\right\}. \end{aligned} \tag{13}$$

Finally, dividing both sides of the previous equation by Δt, and letting Δt decrease to 0, we obtain the following dynamic programming equation (DPE):

$$\begin{aligned} 0 &= \inf_{u(x,i)}\left\{\frac{1}{2}q_0u^2(x,i)+\lambda_i+[f(x,i)+b_0u(x,i)]F'(x,i)\right. \\ &\quad \left.+\frac{1}{2}v(x,i)F''(x,i)+\sum_{j\neq i}\nu_i p_{ij}\,[F(x,j)-F(x,i)]\right\}. \end{aligned} \tag{14}$$

Now, differentiating with respect to $u(x,i)$, we deduce that the optimal control $u^*(x,i)$ is given by

$$u^*(x,i) = -\frac{b_0}{q_0} F'(x,i). \tag{15}$$

Notice that we only need to find the derivative $F'(x,i)$ to obtain the optimal control. Substituting this expression into the DPE, we find that we must solve the system of non-linear second-order differential-difference equations

$$\begin{aligned} 0 &= \lambda_i + f(x,i)F'(x,i) - \frac{1}{2}\frac{b_0^2}{q_0}\left[F'(x,i)\right]^2 + \frac{1}{2}v(x,i)F''(x,i) \\ &\quad + \sum_{j \neq i} \nu_i p_{ij}\left[F(x,j) - F(x,i)\right] \end{aligned} \tag{16}$$

for $i = 1, \ldots, k$.

3. Particular case

We will consider the particular case when $E = \{1,2\}$, so that the system (16) is reduced to

$$\begin{aligned} 0 &= \lambda_1 + f(x,1)F'(x,1) - \frac{1}{2}\frac{b_0^2}{q_0}\left[F'(x,1)\right]^2 + \frac{1}{2}v(x,1)F''(x,1) \\ &\quad + \nu_1\left[F(x,2) - F(x,1)\right], \end{aligned} \tag{17}$$

$$\begin{aligned} 0 &= \lambda_2 + f(x,2)F'(x,2) - \frac{1}{2}\frac{b_0^2}{q_0}\left[F'(x,2)\right]^2 + \frac{1}{2}v(x,2)F''(x,2) \\ &\quad + \nu_2\left[F(x,1) - F(x,2)\right]. \end{aligned} \tag{18}$$

We will treat the case when $\{X(t), t \geq 0\}$ is a controlled Wiener process.

We assume that $f(x,1) = 1$, $f(x,2) = -1$ and $v(x,1) = v(x,2) = 1$, so that only the infinitesimal mean of the controlled process is random. Moreover, we take $\lambda_i = \nu_i = 1$ and $K_i[X(T(x,i))] \equiv 0$, for $i = 1,2$. That is, there is no termination cost. Finally, we take $a = -1$ and $b = 1$ in the definition of $T(x,i)$.

Letting

$$c^2 := \frac{b_0^2}{2q_0}, \tag{19}$$

the system that we must solve is then

$$\begin{aligned} 0 &= 1 + F'(x,1) - c^2\left[F'(x,1)\right]^2 + \frac{1}{2}F''(x,1) \\ &\quad + \left[F(x,2) - F(x,1)\right], \end{aligned} \tag{20}$$

$$\begin{aligned} 0 &= 1 - F'(x,2) - c^2\left[F'(x,2)\right]^2 + \frac{1}{2}F''(x,2) \\ &\quad + \left[F(x,1) - F(x,2)\right]. \end{aligned} \tag{21}$$

The boundary conditions are $F(-1,i) = F(1,i) = 0$ for $i = 1,2$.

Notice that in this particular case, using symmetry we can write that

$$F(x,2) = F(-x,1). \tag{22}$$

Hence, the above system can be rewritten as follows:

$$\begin{aligned} 0 &= 1+F'(x,1)-c^2\left[F'(x,1)\right]^2+\frac{1}{2}F''(x,1) \\ &\quad +\left[F(-x,1)-F(x,1)\right], \qquad (23) \\ 0 &= 1-F'(-x,1)-c^2\left[F'(-x,1)\right]^2+\frac{1}{2}F''(-x,1) \\ &\quad +\left[F(x,1)-F(-x,1)\right]. \qquad (24) \end{aligned}$$

Since the two equations are equivalent, we only have to solve the differential-difference equation

$$0=1+F'(x,1)-c^2\left[F'(x,1)\right]^2+\frac{1}{2}F''(x,1)+\left[F(-x,1)-F(x,1)\right]. \qquad (25)$$

Three methods will be used to solve the above equation:

(1) First, we will try to find approximate solutions of the equation by making use of Taylor's formula to simplify the term $F(-x,1)-F(x,1)$. A special case will be considered.
(2) We will also appeal to the method of series solutions.
(3) Finally, a numerical method will be used to solve a particular problem. We will use the numerical solution to check whether the approximate solutions are satisfactory.

3.1. Taylor's formula. We deduce from Taylor's formula:

$$F(-x,1)=F(x,1)+F'(x,1)(-2x)+\frac{1}{2}F''(x,1)(4x^2)+o(x^2) \qquad (26)$$

that Eq. (25) can be approximated as follows:

$$\frac{1}{2}\left(4x^2+1\right)G'(x,1)-c^2G^2(x,1)-(2x-1)G(x,1)=-1, \qquad (27)$$

where $G(x,1):=F'(x,1)$. The mathematical software *Maple* is able to solve this equation. The solution is in terms of the special function HeunC. As mentioned above, we only need $F'(x,1)$ to obtain the optimal control $u^*(x,1)$. However, the boundary conditions are in terms of $F(x,1)$. Therefore, we must either integrate the function $G(x,1)$ and use the boundary conditions $F(1,1)=F(-1,1)=0$, or find a condition on the function $G(x,1)$.

3.1.1. *A special case.* By definition, $c^2=b_0^2/(2q_0)$ is a constant. If we assume instead that c is a function of x given by

$$c^2(x)=\alpha\left(2x^2+\frac{1}{2}\right), \qquad (28)$$

where $\alpha>0$ is a constant, then we find that the transformation

$$\Phi(x,1)=e^{-\alpha F(x,1)} \qquad (29)$$

linearises Eq. (27). Indeed, the function $\Phi(x,1)$ satisfies the second-order linear differential equation

$$\frac{1}{2}\left(4x^2+1\right)\Phi''(x,1)-(2x-1)\Phi'(x,1)=\alpha\Phi(x,1). \qquad (30)$$

The boundary conditions become $\Phi(1,1)=\Phi(-1,1)=1$.

Again, *Maple* is able to solve Eq. (30). The solution is in terms of the generalized hypergeometric function. Moreover, we can determine the value of the two arbitrary constants in the general solution of the equation.

3.2. Series solutions. We can try to find series solutions of the original difference-differential equation (25) when c is a constant. However, to compare the various approximate solutions, we will consider the case when $c^2 = \alpha(2x^2 + \frac{1}{2})$, with $\alpha = 1$, so that the equation becomes

$$0 = 1 + F'(x,1) - \left(2x^2 + \frac{1}{2}\right)\left[F'(x,1)\right]^2 + \frac{1}{2}F''(x,1) + \left[F(-x,1) - F(x,1)\right]. \tag{31}$$

We assume that the function $F(x,1)$ can be expressed as

$$F(x,1) = \sum_{k=0}^{\infty} a_k x^k, \tag{32}$$

so that

$$F'(x,1) = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad \text{and} \quad F''(x,1) = \sum_{k=1}^{\infty} k(k+1) a_{k+1} x^{k-1}. \tag{33}$$

Substituting these expressions into Eq. (31), we find that the equation that we must solve is

$$\begin{aligned} 0 \quad = \quad & 1 + \sum_{k=1}^{\infty} k a_k x^{k-1} - \left(2x^2 + \frac{1}{2}\right)\left[\sum_{k=1}^{\infty} k a_k x^{k-1}\right]^2 \\ & + \frac{1}{2}\sum_{k=1}^{\infty} k(k+1) a_{k+1} x^{k-1} + \left[\sum_{k=0}^{\infty} (-1)^k a_k x^k - \sum_{k=0}^{\infty} a_k x^k\right]. \end{aligned} \tag{34}$$

We will find the approximate solutions expressed as a polynomial of degree 5, and of degree 6. The coefficients of the polynomial of degree 5 (respectively 6) are determined by taking into account the boundary conditions $F(-1,1) = 0$ and $F(1,1) = 0$, and are the solutions of the following system (35) (resp. (37)).

• <u>*Polynomial of degree 5.*</u>

$$\left.\begin{aligned} 0 &= \tfrac{2}{3} + \tfrac{5}{3}a_1 - a_1^2 + \tfrac{1}{3}a_1^3, \\ a_2 &= -1 - a_1 + \tfrac{1}{2}a_1^2, \\ a_3 &= \tfrac{2}{3} + \tfrac{2}{3}a_1 - a_1^2 + \tfrac{1}{3}a_1^3, \\ a_4 &= \tfrac{2}{3}a_1 + \tfrac{4}{3}a_1^2 - \tfrac{5}{6}a_1^3 + \tfrac{1}{4}a_1^4, \\ a_5 &= -a_1 - a_3, \\ a_0 &= -a_2 - a_4. \end{aligned}\right\} \tag{35}$$

The first equation of the above system is non-linear. We can determine its root a_1 by using one of the iterative methods to solve non-linear equations, for example the bisection method (see (Fortin 2016)). Once a_1 is determined, the other coefficients are computed easily using the other equations. Thus, the explicit form of the polynomial is

$$P_5(x) = \frac{579}{1039} - \frac{541}{1397}x - \frac{235}{437}x^2 + \frac{287}{1200}x^3 - \frac{120}{6151}x^4 + \frac{97}{655}x^5. \tag{36}$$

• Polynomial of degree 6.

$$\left.\begin{array}{l} 0=\frac{2}{5}\;-\;\frac{9}{5}a_1^2\;+\;\frac{34}{15}a_1^3\;-\;a_1^4\;-\;\frac{1}{5}a_1^5,\\ a_2=-1\;-\;a_1\;+\;\frac{1}{2}a_1^2,\\ a_3=\frac{2}{3}\;+\;\frac{2}{3}a_1\;-\;{a_1}^2\;+\;\frac{1}{3}a_1^3,\\ a_4=\frac{2}{3}a_1\;+\;\frac{7}{6}a_1^2\;-\;a_1^3\;+\;\frac{1}{4}a_1^4,\\ a_5=-a_1\;-\;a_3,\\ a_6=\frac{34}{45}\;+\;2a_1\;-\;\frac{46}{45}a_1^2\;-\;\frac{109}{45}a_1^3\;+\;\frac{227}{90}a_1^4\;-\;a_1^5\;+\;\frac{1}{6}a_1^6,\\ a_0=-a_2\;-\;a_4\;-\;a_6. \end{array}\right\} \tag{37}$$

In the same way, we can determine the coefficients of the polynomial of degree 6, and its explicit form is

$$P_6(x)=\frac{1018}{1947}-\frac{541}{1397}x-\frac{235}{437}x^2+\frac{287}{1200}x^3-\frac{120}{6151}x^4+\frac{97}{655}x^5+\frac{115}{3342}x^6. \tag{38}$$

3.3. Numerical method. Finally, to evaluate the accuracy of the above approximate solutions, we will compute a numerical solution, which is considered here to be the most precise solution of the original differential-difference equation (31).

Because we have a differential-difference equation and the problem is a Boundary Value Problem (BVP), the finite difference method (see (Fortin 2016)) is the suitable numerical technique to choose, rather than the shooting method.

The principle of the finite difference method is based on the discretization of the differential-difference equation (31) in the interval $[-1,1]$. More precisely, if we consider the numerical solution $\tilde{F}$ computed with this method, the vector $(\tilde{F}_i)_{1\le i\le n+1}$, whose elements are given by

$$\tilde{F}_i=F(x_i,1)\quad\text{for}\quad i=1,\dots,n+1, \tag{39}$$

is the solution of the following system

$$\begin{aligned}\tilde{F}_i \;&=\; \frac{h^2}{h^2+1}\left(1+\frac{\tilde{F}_{i+1}-\tilde{F}_{i-1}}{2h}-\frac{\frac{1}{2}+2x_i^2}{4h^2}\left(\tilde{F}_{i+1}-\tilde{F}_{i-1}\right)^2\right.\\ &\quad+\left.\frac{\tilde{F}_{i+1}+\tilde{F}_{i-1}}{2h^2}+\tilde{F}_{n-i+1}\right)\end{aligned} \tag{40}$$

for $i=2,\dots,n$, where h denotes the discretization time step of the interval $[-1,1]$ and satisfies $h=x_j-x_{j-1}=\frac{2}{n}$ for $j=2,\dots,n+1$. Moreover, $x_i=x_1+(i-1)h$, for $i=1,\dots,n+1$, are the equidistant points of this interval, such that

$$-1=x_1<x_2<\dots<x_n<x_{n+1}=1. \tag{41}$$

The approximate solutions are shown in Fig.1. As we can see, the accuracy of these three approximate solutions is roughly the same. Indeed, the solution based on Taylor's formula does well near the boundaries of the interval $[-1,1]$ (particularly when $x\in[-1;-0,6]$), but not so well near the origin. In the case of the polynomials, that of degree 5 (respectively 6) provides a good approximation when x is positive (resp. negative).

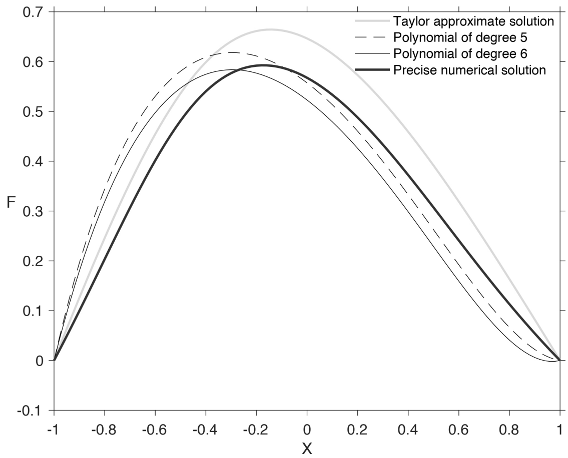

FIGURE 1. Curves of the approximate solutions of the function F.

4. Concluding remarks

In this paper, we considered an LQG homing problem for a one-dimensional diffusion process whose infinitesimal parameters depend on a continuous-time Markov chain. This type of problem is useful in financial mathematics. Indeed, the stock markets move from a "bull" to a "bear" market (or vice versa) at random times. We obtained analytical approximate solutions in the simplest case possible, and we compared these approximate solutions to a numerical solution. We saw that the approximate solutions were relatively good. We would like to obtain an exact solution to such a problem in a special case. This could be achieved by assuming that the function $F(x,i)$ is of a given form (for instance, a quadratic function), and by determining whether this solution is valid for a certain choice of the parameters in the problem.

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