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TWO APPROACHES TO AGING AND FATIGUE MODELS IN VISCOELASTIC SOLIDS

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ABSTRACT. The paper deals with viscoelastic solids subject to aging and fatigue effects. First a scheme is set up on the basis of physical insights, provided by the analysis of a rheological model, by letting the elastic moduli and the viscosity coefficients be functions of time. Hence the functional providing the stress in terms of the strain is established. The procedure has the advantage of showing how the dependence on the present value and that on the history of the strain are influenced by the rheological parameters. The three-dimensional version is then considered and the thermodynamic restrictions are determined. Next a second approach is set up by letting the stress be given by a constitutive equation involving a fractional derivative. The coefficient characterizing the kernel is assumed to obey an evolution equation that incorporates the fatigue effects. Compatibility with thermodynamics is shown to hold with an appropriate internal structural power.

1. Introduction

Aging is a gradual process in which the properties of a material change, over time or with use, due to chemical or physical agents. Corrosion, obsolescence, and weathering are examples of aging. In metallurgical processes, aging may be induced by an heat treatment (age hardening). Consequences of aging are of various type. For instance, the damage caused by chemical or physical agents or time-deteriorating processes at constant temperature are examples for decreasing stiffness in elastic springs. Instead, solidification of concrete is an irreversible transition process where the system increases its stiffness and releases a large amount of energy per volume. The former type of aging is compatible with thermodynamics under isothermal conditions, the latter involves a latent heat and then requires a non-isothermal framework.

Two approaches to the modeling of aging effects are developed in this paper. First a scheme is set up on the basis of physical insights, provided by the analysis of a rheological model, by letting the elastic moduli and the viscosity coefficients be functions of time. The solid is viscoelastic and the aging model holds while the material is subject to chemical or physical agents at constant temperature. It is then understood that we look for the modeling of *aging isothermal viscoelasticity*. Hence the functional providing the stress in terms of the strain is established. The procedure has the advantage of showing how

the dependence on the present value and that on the history of the strain are influenced by the rheological parameters. The three-dimensional version is then considered and the thermodynamic restrictions are determined. In the second approach the stress is given by a constitutive equation involving a fractional derivative. The coefficient characterizing the kernel is assumed to obey an evolution equation that incorporates the fatigue effects. Compatibility with thermodynamics is shown to hold with an appropriate internal structural power.

Let ε and σ denote the uniaxial strain and the corresponding tensile stress at any point $\mathbf{x}$ of the reference configuration Ω of the sample. According to Boltzmann's formulation of hereditary elasticity (Boltzmann 1878), under isothermal conditions a linear viscoelastic material may be described by a stress-strain relation in the Riemann-Stieltjes integral form

$$\sigma(x,t) = \int_{-\infty}^{t} G(t-s) d_s \varepsilon(x,s) \tag{1}$$

where G is named *Boltzmann function* (or *memory kernel*) and $\varepsilon(\cdot)$ is a fading strain history, such that

$$\lim_{s\to-\infty} \varepsilon(x,s) = 0, \qquad \varepsilon(x,s) = \int_{-\infty}^{s} d_s \varepsilon(x,\zeta). \tag{2}$$

In particular, when the strain history vanishes on $(-\infty,0]$ and is differentiable almost everywhere on $(0,t)$, then (1) reduces to

$$\sigma(x,t) = \int_{0}^{t} G(t-s)\partial_s \varepsilon(x,s) ds, \tag{3}$$

where x is the space coordinate of the one-dimensional setting and ∂_s denotes the partial derivative.

A peculiar behavior of viscoelastic solid materials is named *relaxation property*: if the solid is held at a constant strain starting from a given time $t_0 \geq 0$, the stress tends (as $t \to \infty$) to a constant value which is "proportional" to the applied constant strain. Indeed, if $\varepsilon(x,\cdot)$ is continuous on $(-\infty,t_0]$ and

$$\varepsilon(x,t) = \varepsilon(x,t_0) = \varepsilon_0(x), \quad \forall t \geq t_0,$$

it follows that

$$\lim_{t\to\infty} \sigma(x,t) = \lim_{t\to\infty} G_\infty \varepsilon(x,t) + \lim_{t\to\infty} \int_{-\infty}^{t_0} [G(t-s) - G_\infty] d_s \varepsilon(x,s) = G_\infty \varepsilon_0(x),$$

where the *relaxation modulus*

$$G_\infty = \lim_{\tau\to\infty} G(\tau)$$

is assumed to be positive for a viscoelastic solid, but it vanishes for visco-plastic materials. Then, using (2) and letting

$$\hat{G}(\tau) = G(\tau) - G_\infty$$

the stress-strain relation (1) may be rewritten as

$$\sigma(x,t) = G_\infty \varepsilon(x,t) + \int_{-\infty}^{t} \hat{G}(t-s) d_s \varepsilon(x,s).$$

Of course, the choice of G is required to satisfy some basic principles, like the *fading memory principle* and the *dissipation principle*, a thermostatic version of the Second Law of

Thermodynamics (see Fabrizio and Morro (1992), for instance). In general, these conditions allow the memory kernel to be unbounded at the origin.

In modeling aging effects, we might allow for an additional variable, say ξ, affecting the values of G thus describing the increase or decrease of G in time. This approach requires a supplementary equation ruling the evolution of ξ, as devised in the second part of this paper. Otherwise we might think that for aging materials the dependence of G on t,s is not merely through the difference $t-s$ but involves t and s separately. It is a central problem to understand how to model G and this is the aim of the first approach. In the second approach, instead, we take the view that a parameter β, characterizing the the stress-strain relation, changes because of aging and fatigue effects in a manner similar to that of phase-field theories.

2. The first approach

To account for aging effects we first argue on rheological models, which in fact is not new in the literature (see, it e.g., Drozdov *et al.* (2013)). The idea here is to regard the elasticity and viscosity coefficients as time-dependent parameters and hence derive properties of the relaxation modulus.

To this purpose, we first address attention to the Maxwell element and the standard solid (Maxwell 2011). Hence we establish the functional providing the stress in terms of the strain. This procedure has the advantage of showing how the dependence on the present value and that on the history of ε are influenced by the rheological parameters. Next we generalize the model and look for the corresponding three-dimensional version. For a generic time-dependent relaxation function, a free energy is found to hold for the stress functional as a suitable Graffi-Volterra functional (Volterra 1928; Graffi 1974; Fabrizio and Morro 1992). As a consequence, the isothermal stress functional is found to be compatible with thermodynamics (in the sense that it obeys the Dissipation Principle) subject to weak restrictions on the relaxation function.

2.1. Insights from a rheological model. To get some insights about the modeling of aging viscoelastic solids we start from the classical standard linear solid where a Maxwell unit, consisting of a spring and a dashpot connected in series, is set in parallel with a lone spring. While we have in mind the behavior of the model in terms of elongation and forces, we extend the formulae to the continuum framework by the standard analogies stress-force and strain-elongation. It is understood that the model is framed within a one-dimensional picture, so that both strains and stresses are scalar fields depending on (x,t). Since the elastic and Maxwell elements are in parallel, the strain is the same for every element and the applied stress is the sum of the stress in each element (see Fig. 1).

Hereafter, the dependence on x of all the fields involved is understood and not written. For the Maxwell element, let ε_s and ε_d be the strain of the spring and that of the dashpot. Hence, denoting by ε the common strain we have

$$\varepsilon = \varepsilon_s + \varepsilon_d. \tag{4}$$

Let σ_e be the stress on the isolated spring while σ_m the stress on the Maxwell element. Then, the total applied stress is given by

$$\sigma = \sigma_e + \sigma_m.$$

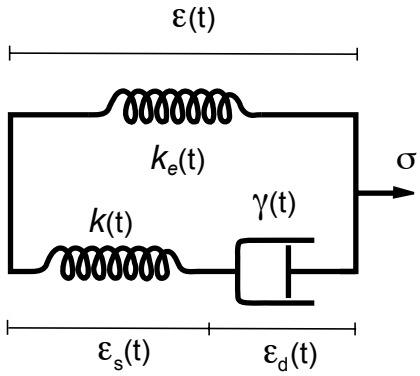

FIGURE 1. Mechanical scheme of an aging standard viscoelastic solid

Moreover let k and k_e be the elastic modulus (or rigidity) of the spring of the Maxwell element and of the spring in parallel, respectively, and γ the viscosity of the dashpot (Fig. 1). It is the essential feature of the aging effect that k, k_e and γ are positive functions of the time t. In the Maxwell unit, the spring and the dashpot are in series and hence they are subject to the same stress so that, according to the Hook's law,

$$\sigma_e = k_e \varepsilon\,, \qquad \sigma_m = k\varepsilon_s = \gamma \partial_t \varepsilon_d\,, \tag{5}$$

where ∂_t denotes partial differentiation with respect to time t. Using the last equality, from (4) we have

$$\frac{1}{\alpha}\partial_t \varepsilon_d + \varepsilon_d = \varepsilon, \qquad \alpha = \frac{k}{\gamma}. \tag{6}$$

Incidentally, if k_e and k are time independent then time differentiation of (4) and use of (5) give

$$\partial_t \varepsilon = \frac{1}{k}(\partial_t \sigma - k_e \partial_t \varepsilon) + \frac{1}{\gamma}(\sigma - k_e \varepsilon),$$

which holds for any viscoelastic standard element. Letting $g_0 = k_e + k$ and $g_\infty = k_e$, this differential equation is equivalent to

$$\partial_t \sigma = g_0 \partial_t \varepsilon - \alpha(\sigma - g_\infty \varepsilon),$$

which is commonly used in the literature. So far it is only assumed that

(A1) $k_e, k, \gamma \in \mathbb{C}^1(\mathbb{R})$ and $k_e(t), k(t) \geq 0$, $\gamma(t) \geq \gamma_0 > 0$ for every $t \in \mathbb{R}$.

(A2) $\displaystyle\int_{-\infty}^{t} \alpha(\xi)\, d\xi = \infty.$

The last condition is fulfilled when $\alpha = k/\gamma$ is a constant function, for instance.

We may regard (6) as a differential equation in the unknown $\varepsilon_d(t)$. Then, integration over $[t_0, t]$ yields

$$\varepsilon_d(t) = \varepsilon_d(t_0) \exp\Big(-\int_{t_0}^{t} \alpha(y)dy\Big) + \int_{t_0}^{t} \exp\Big(-\int_{s}^{t} \alpha(y)dy\Big)\, \alpha(s)\varepsilon(s)ds.$$

It is convenient to let $t_0 \to -\infty$. By assuming that ε_d is uniformly bounded on $(-\infty, t]$, assumption (A2) allows us to take

$$\lim_{t_0 \to -\infty} \varepsilon_d(t_0) \exp\Big(-\int_{t_0}^{t} \alpha(s) ds \Big) = 0.$$

Hence we have

$$\varepsilon_d(t) = \int_{-\infty}^{t} \exp\Big(-\int_{s}^{t} \alpha(y) dy \Big) \alpha(s) \varepsilon(s) ds,$$

and from the representation

$$\sigma = k_e \varepsilon + k \varepsilon_s = [k_e + k]\varepsilon - k\varepsilon_d,$$

we obtain the stress-strain relation

$$\sigma(t) = [k_e(t) + k(t)]\varepsilon(t) - \int_{-\infty}^{t} k(t) \exp\Big(-\int_{s}^{t} \alpha(y) dy \Big) \alpha(s) \varepsilon(s) ds \tag{7}$$

which involves both the present value $\varepsilon(t)$ and the past history $\varepsilon(s), s \in [-\infty, t)$. Since

$$\exp\Big(-\int_{s}^{t} \alpha(y) dy \Big) \alpha(s) = \partial_s \exp\Big(-\int_{s}^{t} \alpha(y) dy \Big),$$

an integration by parts allows (7) to be rewritten as

$$\sigma(t) = k_e(t)\varepsilon(t) + \int_{-\infty}^{t} k(t) \exp\Big(-\int_{s}^{t} \alpha(y) dy \Big) \partial_s \varepsilon(s) ds \tag{8}$$

provided that ε is uniformly bounded on $(-\infty, t]$. If we discard the spring in parallel by letting $k_e \equiv 0$, then we obtain the stress-strain relation for the aging Maxwell element

$$\sigma(t) = \int_{-\infty}^{t} k(t) \exp\Big(-\int_{s}^{t} \alpha(y) dy \Big) \partial_s \varepsilon(s) ds. \tag{9}$$

A change of variables $\tau = t - s$ within (7) leads to the alternate form

$$\sigma(t) = [k_e(t) + k(t)]\varepsilon(t) - \int_{0}^{\infty} k(t) \exp\Big(-\int_{0}^{\tau} \alpha(t - \xi) d\xi \Big) \alpha(t - \tau) \varepsilon(t - \tau) d\tau. \tag{10}$$

Finally, after introducing the so called *relative history*,

$$\eta^t(\tau) = \varepsilon(t) - \varepsilon(t - \tau),$$

the stress-strain relation may be rewritten as

$$\sigma(t) = k_e(t)\varepsilon(t) + \int_{0}^{\infty} k(t) \exp\Big(-\int_{0}^{\tau} \alpha(t - \xi) d\xi \Big) \alpha(t - \tau) \eta^t(\tau) d\tau. \tag{11}$$

2.2. Rescaled time and relation to other approaches. Underwood (2016) modeled continuum damage and fatigue effects for asphalt mixtures by using the convolution form (3) modified by the occurrence of a reduced time t_r in place of time t, that is

$$\sigma(t_r) = \int_{0}^{t_r} G(t_r - s) \partial_s \varepsilon(s) ds, \tag{12}$$

where

$$t_r = \int_{0}^{t} \frac{1}{a_T(\tau)} d\tau,$$

a_T being named the time-temperature shift factor. Appropriate shift factors may equally well model other fatigue or aging effects. Denote by $f(t)$ the function associated with the process applied to the body so that t_r is determined by

$$t_r = \int_0^t f(\tau) d\tau.$$

Upon an integration by parts of (12) we have

$$\sigma(t_r) = G(0)\varepsilon(t_r) + \int_0^{t_r} G'(t_r - s)\varepsilon(s) ds.$$

Hence, because of aging, the kernel G' is no longer a function of $t-s$.

To our mind the use of a rescaled time t_r is an operative way of accounting for aging effects. Hereafter, we show that the stress-strain relation we derived for the aging standard solid may be rewritten using a proper rescaled time. First, we let

$$A(s) = \int_0^s \alpha(y) dy$$

and say that A is positive and non-decreasing because of (A1). Moreover, by (A2),

$$\lim_{s \to -\infty} A(s) = -\infty.$$

Then we let

$$\varepsilon(s) = \hat{\varepsilon}(A(s))$$

and hence

$$\partial_s \varepsilon(s) = \hat{\varepsilon}'(A(s))\alpha(s).$$

As a consequence the stress-strain relation (7) becomes

$$\sigma(t) = k_e(t) + k(t) \int_{-\infty}^{A(t)} \exp\{-[A(t) - A(s)]\} \hat{\varepsilon}'(A(s))\alpha(s) ds.$$

Accordingly, aging effects can be represented by the rescaled time $\tau = A(t)$ so that

$$\sigma(t) = k_e(t) + k(t) \int_{-\infty}^{\tau} \exp[-(\tau - \xi)] \hat{\varepsilon}'(\xi) d\xi.$$

For non-aging materials we have $A(s) = \alpha s$ so that

$$\sigma(t) = k_e \varepsilon(t) + k \int_{-\infty}^{t} \exp[-\alpha(t-s)] \partial_s \varepsilon(s) ds.$$

2.3. Generalized aging constitutive equation. In the classical, homogeneous, one dimensional model of linear viscoelasticity, σ is given by

$$\sigma(t) = G_0 \varepsilon(t) + \int_0^{\infty} G'(\tau) \varepsilon(t - \tau) d\tau, \tag{13}$$

or equivalently by

$$\sigma(t) = G_0 \varepsilon(t) - \int_{-\infty}^{t} \partial_s G(t-s)\, \varepsilon(s)\, ds = G_0 \varepsilon(t) + \int_{-\infty}^{t} G'(t-s)\, \varepsilon(s)\, ds, \tag{14}$$

where the prime means differentiation with respect to the argument. The representation (14) reduces to (1) upon an integration by parts and assuming $G_0 = G(0)$ and

$$\lim_{s \to -\infty} G(t-s)\, \varepsilon(s) = 0.$$

Borrowing from the standard solid developed above, we now state the stress-strain constitutive equation so that the viscoelastic model is obtained that allows for time-dependent properties. The constitutive law (7) can be given the general form

$$\sigma(x,t) = G_0(t)\varepsilon(x,t) - \int_{-\infty}^{t} \partial_s G(t,s)\,\varepsilon(x,s)ds, \tag{15}$$

where ∂_s denotes partial differentiation with respect to the variable s, provided we let

$$G(t,s) = k_e(t) + k(t)\exp\Big(-\int_s^t \alpha(y)dy\Big).$$

The function $G(t,s)$ is defined on the half plane

$$\mathscr{D} = \{(t,s) \in \mathbb{R}^2 : s \le t\}.$$

In addition, it follows from (A1)-(A2) that

$$G_0(t) := G(t,t) = k_e(t) + k(t), \qquad G_\infty(t) := \lim_{s\to-\infty} G(t,s) = k_e(t)$$

and then

$$G_0(t) - G_\infty(t) = k(t) \ge 0 \quad \forall t \in \mathbb{R}.$$

The classical form (13) is recovered by assuming that k_e, k, and α are constants, so that $G(t,s) = G(t-s)$. A change of the integration variable in (15) yields

$$\sigma(x,t) = G_0(t)\varepsilon(x,t) + \int_0^\infty \partial_\tau G(t,t-\tau)\,\varepsilon(x,t-\tau)d\tau. \tag{16}$$

For later convenience, we let $\check{G} : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ be defined by

$$\check{G}(t,\tau) = G(t,t-\tau)$$

so that

$$G_0(t) = \check{G}(t,0), \qquad G_\infty(t) = \lim_{\tau\to\infty}\check{G}(t,\tau), \qquad \partial_\tau G(t,t-\tau) = \partial_\tau\check{G}(t,\tau)$$

and then

$$\int_0^\infty \partial_\tau\check{G}(t,\tau)d\tau = G_\infty(t) - G_0(t).$$

Accordingly, Eq. (16) becomes

$$\sigma(x,t) = G_0(t)\varepsilon(x,t) + \int_0^\infty \partial_\tau\check{G}(t,\tau)\varepsilon(x,t-\tau)d\tau, \tag{17}$$

or, equivalently,

$$\sigma(x,t) = G_\infty(t)\varepsilon(x,t) - \int_0^\infty \partial_\tau\check{G}(t,\tau)[\varepsilon(x,t) - \varepsilon(x,t-\tau)]d\tau. \tag{18}$$

Starting from eq. (15), an integration by parts yields

$$\sigma(x,t) = \int_{-\infty}^{t} G(t,s)\partial_s\varepsilon(x,s)ds \tag{19}$$

provided that

$$\lim_{s\to-\infty} G(t,s)\varepsilon(s) = 0.$$

This condition is equivalent to

$$\lim_{\tau\to\infty}\check{G}(t,\tau)\varepsilon(x,t-\tau) = 0,$$

so that, upon an integration of (17) by parts, we obtain

$$\sigma(x,t) = \int_0^\infty \check{G}(t,\tau)\partial_t \varepsilon(x,t-\tau)d\tau.$$

We now look for a general, though linear, three-dimensional version, where all fields depend on the space-time pair $(\mathbf{x},t) \in \mathbb{R}^3 \times \mathbb{R}$. Following the standard notation, we let $\mathbf{u}$ be the displacement vector and $\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ be the infinitesimal strain tensor. Borrowing from (15) we then let the Cauchy stress tensor $\mathbf{T}$ be given by

$$\mathbf{T}(\mathbf{x},t) = \mathbf{G}_0(t)\mathbf{E}(\mathbf{x},t) - \int_{-\infty}^t \partial_s \mathbf{G}(t,s)\mathbf{E}(\mathbf{x},s)ds, \tag{20}$$

where $\mathbf{G}$ stands for the t-dependent relaxation function and

$$\mathbf{G} : \mathscr{D} \to \mathrm{Lin(Sym)}, \qquad \mathbf{G}_0(t) := \mathbf{G}(t,t).$$

Let $\check{\mathbf{G}} : \mathbb{R} \times \mathbb{R}^+ \to \mathrm{Lin(Sym)}$ be defined by

$$\check{\mathbf{G}}(t,\tau) = \mathbf{G}(t,t-\tau). \tag{21}$$

A change of the integration variable in (20) yields the alternate stress-strain relation

$$\mathbf{T}(\mathbf{x},t) = \mathbf{G}_0(t)\mathbf{E}(\mathbf{x},t) + \int_0^\infty \partial_\tau \check{\mathbf{G}}(t,\tau)\mathbf{E}(\mathbf{x},t-\tau)d\tau$$

or, equivalently,

$$\mathbf{T}(\mathbf{x},t) = \mathbf{G}_\infty(t)\mathbf{E}(\mathbf{x},t) - \int_0^\infty \partial_\tau \check{\mathbf{G}}(t,\tau)[\mathbf{E}(\mathbf{x},t) - \mathbf{E}(\mathbf{x},t-\tau)]d\tau.$$

Moreover,

$$\mathbf{G}_0(t) = \check{\mathbf{G}}(t,0), \quad \mathbf{G}_\infty(t) = \lim_{\tau\to\infty} \check{\mathbf{G}}(t,\tau), \quad \partial_\tau \check{\mathbf{G}}(t,\tau) = \partial_\tau \mathbf{G}(t,t-\tau).$$

Remark. For non-aging materials, $\mathbf{G}(t,s)$ and $\check{\mathbf{G}}(t,\tau)$ reduce to $\mathbf{G}(t-s)$ and $\mathbf{G}(\tau)$, respectively. Therefore, when considering $\check{\mathbf{G}}(t,\tau)$ the dependence on t actually translates the effect of the aging.

To simplify the notation, hereafter we use the symbol $\mathscr{G}$ to denote the function

$$\mathscr{G}(t,\tau) = -\partial_\tau \check{\mathbf{G}}(t,\tau), \qquad (t,\tau) \in \mathbb{R} \times \mathbb{R}^+ \tag{22}$$

which is assumed to satisfy the following properties.

(M1) For every fixed $t \in \mathbb{R}$, the map $\tau \mapsto \mathscr{G}(t,\tau)$ is positive semi-definite, absolutely continuous and summable on $\mathbb{R}^+$. Then, for every $t \in \mathbb{R}$,

$$\int_0^\infty \mathscr{G}(t,\tau)d\tau = \mathbf{G}_0(t) - \mathbf{G}_\infty(t) \geq 0.$$

(M2) For every compact set $\mathscr{C} \subset \mathbb{R} \times \mathbb{R}^+$, the map $\tau \mapsto \mathscr{G}(t,\tau)$ is differentiable for all $\tau \in \mathbb{R}^+$ and

$$(t,\tau) \mapsto \partial_\tau \mathscr{G}(t,\tau) \in L^\infty(\mathscr{C}),$$

the map $t \mapsto \mathscr{G}(t,\tau)$ is differentiable for all $t \in \mathbb{R}$ and

$$(t,\tau) \mapsto \partial_t \mathscr{G}(t,\tau) \in L^\infty(\mathscr{C}).$$

(M3) For every $(t,\tau) \in \mathbb{R} \times \mathbb{R}^+$

$$\partial_t \mathscr{G}(t,\tau) + \partial_\tau \mathscr{G}'(t,\tau) \leq 0$$

According to (M1), the t-dependent relaxation function $\check{\mathbf{G}}$ can be represented by

$$\check{\mathbf{G}}(t,\tau) = \mathbf{G}_0(t) - \int_0^\tau \mathscr{G}(t,\sigma)\, d\sigma.$$

As a consequence, by (22) we have

$$\mathscr{G}(t,s) = -\mathbf{G}'(t,s), \qquad \mathbf{G}_0(t) = \mathbf{G}(t,0), \qquad \mathbf{G}_0(t) - \mathbf{G}_\infty(t) = \int_0^\infty \mathscr{G}(t,s) ds \geq 0.$$

As a consequence, from (20) we have

$$\mathbf{T}(t) = \mathbf{G}_\infty(t)\mathbf{E}(t) + \int_0^\infty \mathscr{G}(t,s)[\mathbf{E}(t) - \mathbf{E}(t-s)] ds, \tag{23}$$

which represents the three-dimensional version of (18). By analogy with the scalar case, $\mathbf{G}_\infty(t)$ is assumed to be positive semidefinite for every $t \in \mathbb{R}$, namely

$$\mathbf{G}_\infty(t)\mathbf{E}\cdot\mathbf{E} \geq 0 \quad \forall \mathbf{E} \in \mathrm{Sym}.$$

2.4. Free energies and thermodynamics. We first derive the motion equation related to the time-dependent viscoelastic stress-strain relation (23) and then we examine its compatibility with thermodynamics. Let $\mathbf{u} : \Omega \times \mathbb{R} \to \mathbb{R}^3$ be the displacement field, relative to the reference configuration $\Omega \in \mathbb{R}^3$. The field $\mathbf{u}$ is subject to the *equation of motion*

$$\rho\, \partial_{tt}\mathbf{u} = \nabla\cdot\mathbf{T} + \mathbf{f},$$

where $\mathbf{f}$ is the body force, per unit volume. Hence, from (23) we obtain

$$\rho\, \partial_{tt}\mathbf{u}(\mathbf{x},t) - \nabla\cdot\mathbf{G}_\infty(t)\nabla\mathbf{u}(\mathbf{x},t) - \nabla\cdot\int_0^\infty \mathscr{G}(t,s)\nabla[\mathbf{u}(\mathbf{x},t) - \mathbf{u}(\mathbf{x},t-s)]\, ds = \boldsymbol{f}(\mathbf{x},t). \tag{24}$$

In order to introduce the initial boundary value problem for this equation we have to take in mind that it is not invariant under time shift.

Consistent with linear viscoelasticity, we restrict attention to isothermal processes namely those where the temperature is constant and uniform. Again for consistency with the linearity of the model, we let the mass density be constant and take the approximation $\mathbf{D} \simeq \partial_t \mathbf{E}$. Hence the local form of the second law inequality reduces to the *dissipation inequality*

$$-\rho \frac{d}{dt}\psi + \mathbf{T}\cdot\mathbf{D} \geq 0,$$

where ρ is the mass density, ψ is the Helmholtz free energy per unit volume, and $\mathbf{D}$ is the stretching tensor. By the constancy of ρ, we let $\Psi = \rho\psi$ and take the second law inequality in the form

$$-\partial_t \Psi + \mathbf{T}\cdot\partial_t\mathbf{E} \geq 0.$$

Let $\tau \in \mathbb{R}$ be arbitrarily fixed. We define the *relative displacement history* $\zeta^t(s)$, with $(t,s) \in [t_0, T] \times \mathbb{R}^+$, by

$$\zeta^t(s) = \begin{cases} \mathbf{u}(\mathbf{x},t) - \mathbf{u}(\mathbf{x},t-s), & s \leq t-\tau, \\ \zeta_{t_0}(\mathbf{x}, s-t+t_0) + \mathbf{u}(\mathbf{x},t) - \mathbf{u}(\mathbf{x},t_0), & s > t-t_0, \end{cases} \tag{25}$$

where ζ_{t_0} is the prescribed initial (relative) past history of $\mathbf{u}$ up to t_0,

$$\zeta_\tau(s) = \mathbf{u}(\tau) - \mathbf{u}(\tau - s) \qquad s \in [0, +\infty).$$

Accordingly, we have

$$\partial_t \zeta^t(\mathbf{x},s) = \partial_t \mathbf{u}(\mathbf{x},t) + \partial_s \zeta^t(\mathbf{x},s), \tag{26}$$

and (24) becomes

$$\begin{cases} \rho\ddot{\mathbf{u}}(\mathbf{x},t) - \nabla\cdot\mathbf{G}_\infty(t)\nabla\mathbf{u}(\mathbf{x},t) - \nabla\cdot\int_0^\infty \mathscr{G}(t,s)\nabla\zeta^t(\mathbf{x},s)\,ds = \mathbf{f}(\mathbf{x},t), \\ \partial_t \zeta^t(\mathbf{x},s) = \partial_t \mathbf{u}(\mathbf{x},t) + \partial_s \zeta^t(\mathbf{x},s), \end{cases} \tag{27}$$

with initial conditions prescribed at $t_0 \in \mathbb{R}$ in the form

$$\begin{cases} \mathbf{u}(\mathbf{x},t_0) = \mathbf{u}_{t_0}(\mathbf{x}), \\ \partial_t \mathbf{u}(\mathbf{x},t_0) = \mathbf{v}_{t_0}(\mathbf{x}), \\ \zeta^{t_0}(\mathbf{x},s) = \zeta_{t_0}(\mathbf{x},s), \quad s\in[0,+\infty). \end{cases}$$

For every $t \geq t_0$, we consider the family of memory spaces

$$\mathscr{M}_t = L^2_{\mathscr{G}}(\mathbb{R}^+;[L^2(\Omega)]^3),$$

equipped with the weighted L^2-inner products

$$\langle \zeta,\xi\rangle_{\mathscr{M}_t} = \int_0^\infty \langle \mathscr{G}(t,s)\boldsymbol{\zeta}(s),\boldsymbol{\xi}(s)\rangle\,ds,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $[L^2(\Omega)]^3$. Taking advantage of these spaces, Conti *et al.* (2018) give a proper notion of solution to (27) and provide a global well-posedness result.

Hereafter the dependence on $\mathbf{x}$ is understood and not written. Then, we consider the time dependent quadratic free energy density

$$\Psi(\nabla\mathbf{u}(t),\nabla\zeta^t,t) = \frac{1}{2}\mathbf{G}_\infty(t)\nabla\mathbf{u}(t)\cdot\nabla\mathbf{u}(t) + \frac{1}{2}\int_0^\infty \mathscr{G}(t,s)\nabla\zeta^t(s)\cdot\nabla\zeta^t(s)\,ds.$$

Upon integrating over Ω we end up with the free energy functional

$$\begin{aligned} \Psi^t(\mathbf{u}(t),\zeta^t,t) &= \int_\Omega \psi^t(\nabla\mathbf{u}(t),\nabla\zeta^t)\,dv \\ &= \frac{1}{2}\langle \mathbf{G}_\infty(t)\nabla\mathbf{u}(t),\nabla\mathbf{u}(t)\rangle + \frac{1}{2}\|\nabla\zeta^t\|^2_{\mathscr{M}_t}. \end{aligned} \tag{28}$$

First we observe that

$$\frac{d}{dt}\langle \mathbf{G}_\infty(t)\nabla\mathbf{u}(t),\nabla\mathbf{u}(t)\rangle = 2\langle \mathbf{G}_\infty(t)\nabla\mathbf{u}(t),\nabla\partial_t\mathbf{u}(t)\rangle + \langle \mathbf{G}'_\infty(t)\nabla\mathbf{u}(t),\nabla\mathbf{u}(t)\rangle.$$

Then, by virtue of (26) and some integrations by parts, we obtain

$$\frac{d}{dt}\|\nabla\zeta^t\|^2_{\mathscr{M}_t} = \int_0^\infty \langle \big[\partial_t\mathscr{G}(t,s) + \partial_s\mathscr{G}(t,s)\big]\nabla\zeta^t(s),\nabla\zeta^t(s)\rangle\,ds + 2\langle\nabla\zeta^t,\nabla\partial_t\mathbf{u}(t)\rangle_{\mathscr{M}_t},$$

and, in view of (25),

$$\begin{aligned} \langle \mathbf{T}(t),\nabla\partial_t\mathbf{u}(t)\rangle &= \langle \mathbf{G}_\infty(t)\nabla\mathbf{u}(t) + \int_0^\infty \mathscr{G}(t,s)\nabla\zeta^t(s)\,ds,\nabla\partial_t\mathbf{u}(t)\rangle \\ &= \mathbf{G}_\infty(t)\langle\nabla\mathbf{u}(t),\nabla\partial_t\mathbf{u}(t)\rangle + \langle\nabla\zeta^t,\nabla\partial_t\mathbf{u}(t)\rangle_{\mathscr{M}_t}. \end{aligned} \tag{29}$$

In summary, we end up with

$$\begin{aligned}\frac{d}{dt}\Psi_t(\mathbf{u}(t),\zeta^t) &= \langle \mathbf{T}(t), \nabla\partial_t\mathbf{u}(t)\rangle + \frac{1}{2}\langle \mathbf{G}'_\infty(t)\nabla\mathbf{u}(t), \nabla\mathbf{u}(t)\rangle \\ &\quad + \frac{1}{2}\int_0^\infty \langle \big[\partial_t\mathscr{G}(t,s) + \partial_s\mathscr{G}(t,s)\big]\nabla\zeta^t(s), \nabla\zeta^t(s)\rangle\, ds.\end{aligned} \tag{30}$$

Finally, owing to (M3),

$$\frac{d}{dt}\Psi_t(\mathbf{u}(t),\zeta^t) \le \langle \mathbf{T}(t), \nabla\partial_t\mathbf{u}(t)\rangle + \frac{1}{2}\langle \mathbf{G}'_\infty(t)\nabla\mathbf{u}(t), \nabla\mathbf{u}(t)\rangle.$$

The second law inequality then holds provided that

$$\mathbf{G}'_\infty(t) \le 0, \quad \forall t \in \mathbb{R}. \tag{31}$$

Of course, this condition is fulfilled for viscoplastic materials ($\mathbf{G}_\infty \equiv 0$).

2.5. A relevant example. Starting from (19) and assuming

$$G(t,s) = \frac{1}{\beta(t)}\exp[-\beta(t)(t-s)]$$

we obtain

$$\sigma(t) = \frac{1}{\beta(t)}\int_{-\infty}^t \exp[-\beta(t)(t-s)]\partial_s\varepsilon(s)ds. \tag{32}$$

In this stress-strain relation the aging of the (linear) viscoelastic material is modeled by the time-dependent decay exponent β. When $\beta = 0$ we have an elastic material, whereas $\beta > 0$ corresponds to a viscoplastic material. Actually, if β increases, the aging effect translates into a decrease of the stiffness of the material.

According to (21) and (22) we have

$$\check{G}(t,\tau) := G(t,t-\tau) = \frac{1}{\beta(t)}\exp[-\beta(t)\tau]$$

where $\tau = t-s$, $\tau < t$, and

$$\mathscr{G}(t,\tau) := -\partial_\tau\check{G}(t,\tau) = \exp[-\beta(t)\tau].$$

Properties (M1)-(M2) are fulfilled provided that $\beta(t) > 0$ for all t. Condition (M3) reads

$$\partial_t\mathscr{G}(t,\tau) + \partial_\tau\mathscr{G}(t,\tau) = -\exp[-\beta(t)\tau]\big[\beta(t) + \partial_t\beta(t)\tau\big] \le 0, \qquad \tau < t.$$

Then (M3) is equivalent to

$$\beta(t) + \partial_t\beta(t)\tau \ge 0 \qquad \tau < t. \tag{33}$$

This inequality is identically fulfilled when $\partial_t\beta(t) > 0$. On the contrary, if $\partial_t\beta(t) \le 0$, then

$$\beta(t) + \partial_t\beta(t)t \ge 0 \quad \Rightarrow \quad \beta(t) + \partial_t\beta(t)\tau \ge 0.$$

Accordingly, we may choose $\beta(t) = at^{-b}$, $a > 0, b \in [0,1]$, so that

$$\beta(t) + \partial_t\beta(t)\tau = at^{-b} - abt^{-b-1}\tau \ge at^{-b} - abt^{-b} = at^{-b}(1-b) \ge 0$$

and then (33) is satisfied.

3. Fatigue and damage by a fractional model

We now outline an alternative approach for a non-isothermal model involving a fractional operator which is likely to describe both viscoplastic materials and to represent damage effects resulting from fatigue processes. The stress-strain relation is assumed in the form (32), where the aging exponent β is regarded as a scalar phase field. The exponent β is then assumed to obey a Ginzburg-Landau equation.

The material is allowed to have memory in time and this is taken to be expressed by a fractional derivative. Following Amendola *et al.* (2016) and Caputo and Fabrizio (2015b) we assume that the stress-strain relation is given by (see also Caputo and Fabrizio (2015a))

$$\sigma(x,t) = A(x) D_t^{\beta(x,t)} \varepsilon(x,t) = \frac{A(x)}{\beta(x,t)} \int_{-\infty}^{t} \partial_\tau \varepsilon(x,\tau) \exp[-\beta(x,t)(t-\tau)] d\tau \tag{34}$$

where $\beta(x,t) \in [0,\infty)$ and A is a positive-valued function. Moreover, we take the past history of ε up to $t = 0$ as known. Accordingly, the model represents an elastic material if $\beta = 0$ and a visco-plastic material if $\beta \to \infty$. Here we restrict attention to $\beta < 1$. This ensures that the material is a visco-plastic solid (see Caputo and Fabrizio 2015a).

To fix ideas, the aging of the material is described by $\beta(x,t)$, which is assumed to be a smooth function of x and t. Indeed, we let $\beta = 0$ at $t = 0$ so that the growth of $\beta(x,t)$ describes the failure of the material stiffness, according to the final example of the previous section.

The fatigue phenomenon occurs when a material is subjected to cyclic loading with small amplitude but extended in time. Its outcome is the progressive and localized structural damage of the sample. Although it is quite different from aging, fatigue has close similarities with it. In particular, we assume here that fatigue induces a progressive increase of β, and then a decrease of the material stiffness, up to a threshold value β_c at which the tensile stress vanishes and the total damage (fracture) is reached. Indeed, in a domain $\Omega \in \mathbb{R}^3$ we use the following constitutive equation

$$\sigma(x,t) = (\beta_{\mathrm{c}}(x) - \beta(x,t))^2 A(x) D_t^{\beta(x,t)} \varepsilon(x,t)$$

where $0 \le \beta(t) \le \beta_{\mathrm{c}} < 1$, and if $\beta(t) = \beta_{\mathrm{c}}$ we have fracture. Let $\mathscr{F}$ be the fatigue function defined by

$$\mathscr{F}(x,t) = \int_{t_0}^{t} (\beta_{\mathrm{c}}(x) - \beta(x,\tau)) A(x) D_t^{\beta(x,\tau)} \varepsilon(x,\tau)\, \partial_\tau \varepsilon(x,\tau) d\tau. \tag{35}$$

Hence letting $\beta^t(x)$ and $\varepsilon^t(x)$ be the past histories of $\beta(x)$ and $\varepsilon(x)$ up to time t we can say that

$$\mathscr{F}(x,t) = \hat{\mathscr{F}}(\beta^t(x), \varepsilon^t(x)).$$

In addition to the equation of motion

$$\rho_0(x) \partial_t v(x,t) = \partial_x \sigma(x,t) + \rho_0(x) b(x,t), \tag{36}$$

the material evolution is governed by the Ginzburg-Landau equation, which describes the structural variations of the phase field $\beta(x,t)$, in the form (see Caputo and Fabrizio (2015a),

Eq. (2.6))

$$\begin{aligned}\rho_0(x)\partial_t\beta(x,t) &= \gamma(x)\partial_x^2\beta(x,t)+\mathscr{F}_0(x)G'(\beta(x,t))\\ &\quad -\hat{\mathscr{F}}(\beta^t(x),\varepsilon^t(x))F'(\beta(x,t)),\end{aligned} \tag{37}$$

where the function γ is positive valued and the potentials F and G are defined by

$$F(\beta)=\beta_c-\beta\ ,\quad G(\beta)=8(\beta_c^2-\frac{3}{4}\beta^4).$$

Finally, we suppose that, on the boundary $\partial\Omega$,

$$\partial_n\beta(x,t)=0. \tag{38}$$

Thermodynamic restrictions on the model are placed by the Dissipation Principle. According to Caputo and Fabrizio (2015a), we require that the free energy $\psi(\cdot)$ satisfy the inequality

$$\rho_0(x)\partial_t\psi(x,t)\le\mathscr{P}^i_m(x,t)+\mathscr{P}^i_s(x,t), \tag{39}$$

where $P^i_m(x,t)$ is the internal mechanical power and $P^i_s(x,t)$ is the internal structural power. The latter term is essentially due to the occurrence of a the structure-order parameter β. By means of Eqs. (36) and (37) we have

$$\mathscr{P}^i_m(x,t)=\sigma(x,t)\,\partial_t\varepsilon(x,t)=(\beta_{\mathrm{c}}-\beta(x,t))^2A(x)D_{t_0}^{\beta(x,t)}\,\varepsilon(x,t)\,\partial_t\varepsilon(x,t), \tag{40}$$

$$\begin{aligned}\mathscr{P}^i_s(x,t) &= \rho_0(x)\partial_t\beta^2(x,t)+\gamma(x)\partial_x\beta(x,t)\,\partial_x\partial_t\beta(x,t)\\ &\quad -\mathscr{F}(\beta^t(x),\varepsilon^t(x))\partial_tF(\beta(x,t))+\mathscr{F}_0(x)\partial_tG(\beta(x,t)).\end{aligned} \tag{41}$$

By Eqs. (39), (40) and (41) we obtain

$$\begin{aligned}\rho_0\partial_t\psi(t)\le(\beta_C-\beta(t))^2A(x)D_{t_0}^{\beta(t)}\varepsilon(t)\,\partial_t\varepsilon(t)+\rho_0\partial_t\beta^2(t)+\gamma\partial_x\beta(t)\,\partial_x\partial_t\beta(t)\\ +\mathscr{F}(\beta^t(x),\varepsilon^t(x))\partial_tF(\beta(t))+\mathscr{F}_0\partial_tG(\beta(t)).\end{aligned} \tag{42}$$

Denote by $\tilde{\sigma}(t)$ the stress defined in (34). We find that

$$\begin{aligned}\rho_0\partial_t\psi(t) &\le (\beta_c-\beta(t))\frac{\partial}{\partial t}\int_{t_0}^t(\beta_c-\beta(\tau))\tilde{\sigma}(\tau)\cdot\partial_{\mathrm{t}}\varepsilon(\tau)d\tau+\rho_0\partial_t\beta^2(t)\\ &\quad +\frac{\gamma}{2}\partial_t[\partial_x\beta(t))^2+\partial_tF(\beta(t))\mathscr{F}(\beta^t(x),\varepsilon^t(x)]+\mathscr{F}_0\partial_tG(\beta(t))\\ &= \partial_t[F(\beta(t))\mathscr{F}(\beta^t,\varepsilon^t)]+\frac{\gamma}{2}(\partial_x\beta(t))^2+\mathscr{F}_0G(\beta(t))+\rho_0\partial_t\beta^2(t).\end{aligned} \tag{43}$$

Hence inequality (39) holds provided we take the free energy in the form

$$\psi(t)=F(\beta(t))\mathscr{F}(\beta^t,\varepsilon^t)+\frac{\gamma}{2}(\partial_x\beta(t))^2+\mathscr{F}_0G(\beta(t).$$

In this framework, a model for both fatigue and aging effects might be set up on the basis of the following evolution equation for the structure-order parameter

$$\begin{aligned}\rho_0(x)\partial_t\beta(x,t) &= \gamma(x)\partial_x^2\beta(x,t)+\mathscr{F}_0(x)G'(\beta(x,t))\\ &\quad -\mathscr{F}(\beta^t(x),\varepsilon^t(x))F'(\beta(x,t))+g(x,t)\end{aligned} \tag{44}$$

where the function $g(x,t)$ is assumed to be positive and increasing for $t>0$. According to equation (44), in addition to fatigue effects, the action of the function $g(x,t)$ affects the increase of the function $\beta(x,t)$, which represents the damage of the system. If fatigue effects are absent, in which case $F_0(x)=0$ and $F[\beta^t(x),\varepsilon^t(x)]=0$, Eq. (44) together with the boundary condition (38) ensures the growth of $\beta(x,t)$, which translates the spontaneous aging.

4. Conclusions

The paper develops two approaches to the modelling of aging and fatigue in viscoelastic solids. The first one starts with the analysis of a rheological model, of the standard solid, where the elastic moduli and the viscosity coefficients be functions of time. As a consequence, the stress-strain relation is derived in the form (10). The kernel

$$k(t)\exp\Big(-\int_0^\tau \alpha(t-\xi)d\xi\Big)\alpha(t-\tau)$$

shows that, because of time dependence of the ratio $\alpha=k/\gamma$, the kernel is no longer a function of τ only. In the three-dimensional setting the constitutive relation (23) is considered and the thermodynamic restriction (31) is found to be necessary and sufficient for the validity of the second law while the free energy is given by a functional of the Graffi-Volterra type.

The second approach describes the stress-strain relation via a fractional derivative, see (34). The function $\beta(x,t)$ is taken to characterize the material in that $\beta=0$ models elastic materials whereas $\beta\to\infty$ describes viscoplastic materials. Accordingly the increasing of β accounts for the aging effects of the material. The evolution of β is taken to be governed by the Ginzburg-Landau type equation (37). Compatibility of the approach with thermodynamics is found to hold with an appropriate internal structural power.

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