DOI: 10.1478/AAPP.97S1A13

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 97, No. S1, A13 (2019)

ON THE NONLINEAR STABILITY OF THE THERMODIFFUSIVE EQUILIBRIUM FOR THE MAGNETIC BÉNARD PROBLEM

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ABSTRACT. In this paper we study the nonlinear Lyapunov stability of the thermodiffusive equilibrium of a viscous electroconducting horizontal fluid layer heated from below. We reformulate the nonlinear stability problem, in terms of poloidal and toroidal fields, by projecting the initial perturbation evolution equations on some suitable orthogonal subspaces of the kinematically admissible functions. In such a way, if the principle of exchange of stabilities holds, we obtain, in the classical L^2-norm, the coincidence of linear and nonlinear stability bounds.

1. Introduction

The Bénard problem, *i.e.*, the problem of the stability of the conduction state in a horizontal thermoconducting fluid layer subject to a temperature gradient, has been widely investigated, also in presence of a vertical magnetic field, due to its importance in astrophysics, geophysics, oceanography and meteorology (Chandrasekhar 1968; Joseph 1970, 1976b; Georgescu 1985; Straughan 2004), in the Oberbeck-Boussinesq approximation (Joseph 1976b; Galdi and Straughan 1985; Georgescu 1985; Rionero and Mulone 1987, 1988; Galdi and Padula 1990; Mulone and Rionero 1997; Straughan 2004) too. Linear and nonlinear stability bounds are, usually, different. The point of loss of linear stability can be a bifurcation point at which subcritical instabilities can occur (Prodi 1962; Joseph 1965; Yudovich 1965; Joseph 1966; Chandrasekhar 1968; Joseph 1970; Sattinger 1970; Yudovich 1970a,b; Joseph 1976b; Galdi and Straughan 1985; Georgescu 1985; Rionero and Mulone 1987, 1988; Yudovich 1989; Galdi and Padula 1990; Georgescu and Oprea 1994; Mulone and Rionero 1997; Mulone 2004; Straughan 2004; Palese 2005). This problem was studied by Mulone and Rionero (1997) in the hydrodynamic case, for a horizontal rotating layer, by Palese (2005) in the magnetohydrodynamic case, for a fully ionized fluid and the coincidence of linear and nonlinear stability parameters was deduced, under some restrictions on the initial data. The given problem was changed to obtain an equivalent one with better symmetry properties and the equality of linear and nonlinear stability bounds was obtained in the region of stationary convection of the linear instability theory, without any restriction on initial data, by Georgescu and Palese (1996), by Georgescu *et al.* (2000, 2001) and by Georgescu and Palese (2009) for a thermoanisotropic fluid mixture in a horizontal layer heated from below, by Georgescu and Palese (2010, 2011) and Palese (2014b) in presence of chemical surface reactions and by Palese (2014a,c) for a horizontal rotating layer.

This paper deals with the problem of the coincidence of linear and nonlinear stability bounds for the thermodiffusive equilibrium of a Newtonian fluid in a plane layer heated from below, with a vertical imposed magnetic field, *i.e.*, the classical magnetic Bénard problem. About this, to our knowledge, there is no result in the scientific literature. After formulating the initial boundary value problem (Section **2**) we derive the evolution perturbation equations (Section **3**) in terms of poloidal and toroidal fields, suitable to represent a solenoidal field in a plane layer. Later we obtain an additional evolution equation for the third component of the magnetic field, by projecting the given perturbation equation on some suitable subspaces of the space of the kinematically admissible functions. We study (Section **4**) the Lyapunov stability of the thermodiffusive equilibrium, obtaining (Section **5**) the coincidence of linear and non linear stability bounds, in the range of validity of the principle of exchange of stabilities.

2. Mathematical model

Let us consider a Newtonian thermo-electroconducting viscous fluid in a horizontal layer S, bounded by the planes $z = 0$ and $z = 1$. in a Cartesian frame of reference $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$, with $\mathbf{k}$ vertical upwards unit vector. The fluid, heated from below, is subject to a vertical temperature gradient, in an external constant magnetic field $\mathbf{H}_0 = H_0\mathbf{k}$. In the Oberbeck-Boussinesq approximation the (dimensionless) mathematical model is the following:

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla P + M^2\mathbf{H}\cdot\nabla\mathbf{H} - [1-\mathscr{R}(T-T_0)]\,\mathbf{k} + \Delta\mathbf{v}, \\ \dfrac{\partial \mathbf{H}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{H}) + \dfrac{\mathscr{P}_m}{\mathscr{P}_r}\Delta\mathbf{H}, \\ \mathscr{P}_r\left(\dfrac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T\right) = \Delta T, \\ \nabla\cdot\mathbf{v} = 0, \qquad \nabla\cdot\mathbf{H} = 0, \end{cases} \tag{1}$$

(Chandrasekhar 1968; Joseph 1970, 1976b) with the boundary conditions for stress free, thermal conducting and electrically non conducting planes (Chandrasekhar 1968):

$$\begin{aligned} &\mathbf{v}\cdot\mathbf{n} = 0, \qquad \mathbf{n}\times\mathbf{D}\cdot\mathbf{n} = 0, \qquad \mathbf{H} = \mathbf{H}_0, \qquad \mathbf{n}\cdot\nabla\times\mathbf{H} = 0, \qquad z = 0, 1, \\ &T = T^0, \qquad z = 0, \\ &T = T^1, \qquad z = 1, \end{aligned} \tag{2}$$

where $\mathbf{v}$, $\mathbf{H}$, T, P are velocity, magnetic, temperature and pressure fields, respectively. T^0 represents a reference temperature, $\mathbf{D}$ is the strain rate tensor, $\mathbf{n}$ is the outer (unit) normal to the boundary ∂S of S. M^2, $\mathscr{R}^2$, $\mathscr{P}_r$, $\mathscr{P}_m$ denote dimensionless Hartmann, Rayleigh, Prandtl and magnetic Prandtl numbers, respectively.

Moreover the first, third and fourth equations of the model (1) are, in the Oberbeck-Boussinesq approximation, the balance equations for momentum, energy and mass, respectively. In addition, for the electromagnetic part, the second equation of the model (1) follows by taking into account the Maxwell equations which are Galileo, not Lorentz, invariant.

A layer of fluid heated from below, for a not too large temperature gradient β, is called, in mechanical equilibrium, *conduction state* (Koschmieder 1993). When β increases the fluid has a stationary motion, periodic in the x and y directions, *i.e.*, the thermal horizontal convection, that, for increasing gradient, becomes non stationary, till the turbulence (Georgescu and Palese 2011). We consider the conduction state

$$\left(\bar{\mathbf{v}} = \mathbf{0},\ \overline{\mathbf{H}} = \mathbf{H}_0,\ \overline{T} = T^0 - \beta z,\ \overline{P} = P(z)\right), \tag{3}$$

in the periodicity cell $V = \mathscr{V} \times [0,1]$, where $\mathscr{V} = \left[0, \dfrac{2\pi}{a_x}\right] \times \left[0, \dfrac{2\pi}{a_y}\right]$ and $a^2 = a_x^2 + a_y^2$ is the wave number.

3. Perturbation model

Let us denote with $\mathbf{v} = \bar{\mathbf{v}} + \mathbf{u}$, $\mathbf{H} = \overline{\mathbf{H}} + \mathbf{h}$, $P = \overline{P} + p$, $T = \overline{T} + \vartheta$ the perturbed fields around the conduction state (3). Then the initial boundary value problem governing the evolution of the perturbation $(\mathbf{u}, \mathbf{h}, p, \vartheta)$ of (3) is the following:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + M^2 \left(\dfrac{\partial \mathbf{h}}{\partial z} + \mathbf{h} \cdot \nabla \mathbf{h} \right) + \mathscr{R} \vartheta \mathbf{k} + \Delta \mathbf{u}, \\ \dfrac{\partial \mathbf{h}}{\partial t} = \dfrac{\partial \mathbf{u}}{\partial z} + \nabla \times (\mathbf{u} \times \mathbf{h}) + \dfrac{\mathscr{P}_m}{\mathscr{P}_r} \Delta \mathbf{h}, \\ \mathscr{P}_r \left(\dfrac{\partial \vartheta}{\partial t} + \mathbf{u} \cdot \nabla \vartheta \right) = \Delta \vartheta + \mathscr{R} w, \\ \nabla \cdot \mathbf{u} = 0, \qquad \nabla \cdot \mathbf{h} = 0, \end{cases} \tag{4}$$

in the space $\mathscr{N}$ given by:

$$\mathscr{N} = \left\{ (\mathbf{u}(\cdot,t), \mathbf{h}(\cdot,t)) \in \left[L^2(V)\right]^2, (w(\cdot,t), h_3(\cdot,t), \vartheta(\cdot,t), \Delta h_3(\cdot,t)) \in \left[W^{2,2}(V)\right]^4, \forall t \geq 0 \;\middle|\; \dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = w = 0, \mathbf{h} = \mathbf{0}, \vartheta = 0 \text{ on } (\partial V)_0 \cup (\partial V)_1 \right\}, \tag{5}$$

with

$$(w(\mathbf{x},\cdot), h_3(\mathbf{x},\cdot), \vartheta(\mathbf{x},\cdot)) \in \left(C^1[0,+\infty[\right)^3, \forall \mathbf{x} \in V. \tag{6}$$

In Eqs. (4)-(6), we have $\mathbf{x}(x,y,z)$, $\mathbf{u}(u,v,w)$, $\mathbf{h}(h_1,h_2,h_3)$ and

$$(\partial V)_0 = \left\{ \mathbf{x} \in \mathbb{R}^3 \;\middle|\; 0 \leq x \leq \frac{2\pi}{a_x}, 0 \leq y \leq \frac{2\pi}{a_y}, z = 0 \right\},$$

$$(\partial V)_1 = \left\{ \mathbf{x} \in \mathbb{R}^3 \;\middle|\; 0 \leq x \leq \frac{2\pi}{a_x}, 0 \leq y \leq \frac{2\pi}{a_y}, z = 1 \right\}.$$

If the mean values of the components of velocity and magnetic fields vanish over $\mathscr{V}$, that is if the conditions

$$\int_{\mathscr{V}} u(x,y,z)\,dx\,dy = \int_{\mathscr{V}} v(x,y,z)\,dx\,dy = \int_{\mathscr{V}} w(x,y,z)\,dx\,dy = 0, \qquad \forall z \in [0,1], \tag{7}$$

$$\int_{\mathscr{V}} h_1(x,y,z)\,dx\,dy = \int_{\mathscr{V}} h_2(x,y,z)\,dx\,dy = \int_{\mathscr{V}} h_3(x,y,z)\,dx\,dy = 0, \qquad \forall z \in [0,1], \tag{8}$$

are satisfied, then the velocity $\mathbf{u}$ and the magnetic field $\mathbf{h}$ have the unique decomposition (Joseph 1976a; Schmitt and von Wahl 1992):

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2, \qquad \mathbf{h} = \mathbf{h}_1 + \mathbf{h}_2, \tag{9}$$

with

$$\nabla \cdot \mathbf{u}_1 = \nabla \cdot \mathbf{u}_2 = \mathbf{k} \cdot \nabla \times \mathbf{u}_1 = \mathbf{k} \cdot \mathbf{u}_2 = 0, \tag{10}$$

$$\nabla \cdot \mathbf{h}_1 = \nabla \cdot \mathbf{h}_2 = \mathbf{k} \cdot \nabla \times \mathbf{h}_1 = \mathbf{k} \cdot \mathbf{h}_2 = 0, \tag{11}$$

$$\mathbf{u}_1 = \nabla \frac{\partial \chi}{\partial z} - \mathbf{k}\Delta\chi \equiv \nabla \times \nabla \times (\chi \mathbf{k}), \qquad \mathbf{u}_2 = \mathbf{k} \times \nabla \psi = -\nabla \times (\mathbf{k}\psi), \tag{12}$$

$$\mathbf{h}_1 = \nabla \frac{\partial \chi'}{\partial z} - \mathbf{k}\Delta\chi' \equiv \nabla \times \nabla \times (\chi' \mathbf{k}), \qquad \mathbf{h}_2 = \mathbf{k} \times \nabla \psi' = -\nabla \times (\mathbf{k}\psi'). \tag{13}$$

In (12), (13) χ, χ' and ψ, ψ', called *poloidal and toroidal potentials*, are doubly periodic functions satisfying

$$\Delta_1 \chi = -\mathbf{k} \cdot \mathbf{u} = -w, \qquad \Delta_1 \psi = \mathbf{k} \cdot \nabla \times \mathbf{u}, \tag{14}$$

$$\Delta_1 \chi' = -\mathbf{k} \cdot \mathbf{h} = -h_3, \qquad \Delta_1 \psi' = \mathbf{k} \cdot \nabla \times \mathbf{h}, \tag{15}$$

where $\Delta_1 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ (Joseph 1976a; Schmitt and von Wahl 1992).

The boundary conditions (5) written in terms of χ, ψ, χ', ψ' become (Joseph 1976a):

$$\chi = \frac{\partial^2 \chi}{\partial z^2} = \frac{\partial \psi}{\partial z} = 0, \qquad \chi' = \frac{\partial \chi'}{\partial z} = \Delta_1 \psi' = 0, \qquad z = 0, 1. \tag{16}$$

From the third component of the second equation of the system (4), we obtain:

$$\frac{\partial h_3}{\partial t} = \frac{\partial w}{\partial z} + \nabla \cdot [(\mathbf{u} \times \mathbf{h}) \times \mathbf{k}] + \frac{\mathscr{P}_m}{\mathscr{P}_r} \Delta h_3. \tag{17}$$

Taking into account (14), (15) and the imbedding of $W^{2,2}(V)$ in $C(\overline{\Omega})$, (17) can be written as follows (Sobolev 1963):

$$\nabla \cdot \left[\frac{\partial}{\partial t} \nabla_1 \chi' - \nabla_1 \frac{\partial \chi}{\partial z} + (\mathbf{u} \times \mathbf{h}) \times \mathbf{k} - \frac{\mathscr{P}_m}{\mathscr{P}_r} \nabla_1 \Delta \chi' \right] = 0, \tag{18}$$

where $\nabla_1 \equiv \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y} \right)$. It follows that there exists a vector field $\mathbf{B}$ such that

$$\frac{\partial}{\partial t} \nabla_1 \chi' - \nabla_1 \frac{\partial \chi}{\partial z} + (\mathbf{u} \times \mathbf{h}) \times \mathbf{k} - \frac{\mathscr{P}_m}{\mathscr{P}_r} \nabla_1 \Delta \chi' = \nabla \times \mathbf{B}. \tag{19}$$

From the Weyl decomposition theorem of $L^2(V)$ (Sobolev 1963; Georgescu 1985), the vector field $-(\mathbf{u} \times \mathbf{h}) \times \mathbf{k}$ can be written as

$$-(\mathbf{u} \times \mathbf{h}) \times \mathbf{k} = \nabla U + \nabla \times \mathbf{A}, \tag{20}$$

with U and $\mathbf{A}$, respectively, scalar and vector field. From (19), (20) taking into account that the only vector belonging to both the subspaces of potential and solenoidal vectors is zero (Sobolev 1963; Georgescu 1985), it follows that:

$$\frac{\partial}{\partial t}\nabla_1 \chi' = \nabla_1 \frac{\partial \chi}{\partial z} + \nabla U + \frac{\mathscr{P}_m}{\mathscr{P}_r}\nabla_1 \Delta \chi'. \tag{21}$$

4. Lyapunov stability

If we multiply the first equation of (4) by $\mathbf{u}$, the second equation of (4) by $M^2\mathbf{h}$ and the third equation of (4) by $b\vartheta/\mathscr{P}_r$, with b a scalar parameter, adding the resulting equations and integrating over V we obtain the classical energy relation

$$\frac{dE}{dt} = \mathscr{I} - \mathscr{D}, \tag{22}$$

where

$$E(t) = \frac{1}{2}\left(\|\mathbf{u}\|^2 + M^2\|\mathbf{h}\|^2 + b\,\|\vartheta\|^2\right), \tag{23}$$

$$\mathscr{I} = \mathscr{R}\left(1 + \frac{b}{\mathscr{P}_r}\right)(\vartheta, w), \tag{24}$$

$$\mathscr{D} = \|\nabla \mathbf{u}\|^2 + M^2\frac{\mathscr{P}_m}{\mathscr{P}_r}\|\nabla \mathbf{h}\|^2 + \frac{b}{\mathscr{P}_r}\|\nabla\vartheta\|^2, \tag{25}$$

and $\|\cdot\|$ and $(\cdot,\cdot)$ are the norm and the scalar product in $L^2(V)$.

Indeed, from the boundary conditions (5) it follows that the non linear terms vanish. From (21) we have:

$$\frac{\partial U}{\partial z} = 0, \tag{26}$$

and

$$\nabla_1\left(\frac{\partial \chi'}{\partial t} - \frac{\partial \chi}{\partial z} - U - \frac{\mathscr{P}_m}{\mathscr{P}_r}\Delta\chi'\right) = \mathbf{0}. \tag{27}$$

Therefore the function in parentheses does not depend on the variables x and y. We can introduce a function $F(z)$ such that

$$\frac{\partial \chi'}{\partial t} = \frac{\partial \chi}{\partial z} + \frac{\mathscr{P}_m}{\mathscr{P}_r}\Delta\chi' + U(x,y) + F(z). \tag{28}$$

If we consider the scalar product of $\Delta_1\chi'$ with the second derivative of (28) respect to z, owing the boundary conditions (16) we obtain:

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla_1\frac{\partial \chi'}{\partial z}\right\|^2 = \left(\Delta_1\chi', \frac{\partial^3\chi}{\partial z^3}\right) + \frac{\mathscr{P}_m}{\mathscr{P}_r}\left(\Delta\frac{\partial^2\chi'}{\partial z^2}, \Delta_1\chi'\right), \tag{29}$$

because, by using (15) and (8),

$$\left(\frac{d^2F}{dz^2}, \Delta_1\chi'\right) = \int_0^1 \frac{d^2F}{dz^2}\,dz \int_{\mathscr{V}} \Delta_1\chi'(x,y,z)\,d\mathscr{V} = 0. \tag{30}$$

From now on, for the sake of simplicity, we will use the subscript notation for the partial differentiation, that is $f_x \equiv \dfrac{\partial f}{\partial x}$ for a function f depending on x. Now we consider the function

$$E^*(t) = E(t) + d\,\frac{M^2}{2}\left\|\nabla_1 \chi'_z\right\|^2, \tag{31}$$

which, in terms of poloidal and toroidal fields, can be written as follows

$$\begin{aligned} E^*(t) = \frac{1}{2}\Big[&\|\chi_{xz}\|^2 + \left\|\chi_{yz}\right\|^2 + \|\Delta_1\chi\|^2 + \|\psi_x\|^2 + \left\|\psi_y\right\|^2 + \\ &+(1+d)M^2\left(\left\|\chi'_{xz}\right\|^2 + \left\|\chi'_{yz}\right\|^2\right) + M^2\left(\left\|\Delta_1\chi'\right\|^2 + \|\psi'_x\|^2 + \left\|\psi'_y\right\|^2\right) + b\,\|\vartheta\|^2\Big] \end{aligned} \tag{32}$$

where d is a constant to be determined later. From (22), (23), (24), (25), (29) and (31) we have:

$$\frac{dE^*}{dt} = \mathscr{I}^* - \mathscr{D}^* = -\mathscr{D}^*\left(1 - \frac{\mathscr{I}^*}{\mathscr{D}^*}\right), \tag{33}$$

where, in terms of poloidal and toroidal fields,

$$\mathscr{I}^* = -\mathscr{R}\left(1 + \frac{b}{\mathscr{P}_r}\right)(\vartheta, \Delta_1\chi) + d\,M^2(\chi'_{zz}, \Delta_1\chi_z) + M^2\frac{\mathscr{P}_m}{\mathscr{P}_r}d\,\alpha(\Delta\Delta_1\chi', \chi'_{zz}), \tag{34}$$

$$\begin{aligned} \mathscr{D}^* = &\|\nabla\chi_{xz}\|^2 + \left\|\nabla\chi_{yz}\right\|^2 + \|\nabla\Delta_1\chi\|^2 + \|\nabla\psi_x\|^2 + \left\|\nabla\psi_y\right\|^2 + \frac{b}{\mathscr{P}_r}\|\nabla\vartheta\|^2 + \\ &+M^2\frac{\mathscr{P}_m}{\mathscr{P}_r}\Big\{[1+d(1-\alpha)]\left[\left\|\nabla\chi'_{xz}\right\|^2 + \left\|\nabla\chi'_{yz}\right\|^2\right] + \left\|\nabla\Delta_1\chi'\right\|^2 + \|\nabla\psi'_x\|^2 + \left\|\nabla\psi'_y\right\|^2\Big\}, \end{aligned} \tag{35}$$

taking into account that $(\Delta\Delta_1\chi', \chi'_{zz}) = -\left[\left\|\nabla\chi'_{xz}\right\|^2 + \left\|\nabla\chi'_{yz}\right\|^2\right]$ and α is an arbitrary parameter that we shall determine later in order to have $\mathscr{D}^*$ positive definite. Let us define

$$\frac{1}{\sqrt{\mathscr{R}^*_a}} = \max_{\mathscr{K}}\frac{\mathscr{I}^*}{\mathscr{D}^*} \tag{36}$$

in the class $\mathscr{K}$ of the kinematically admissible functions.

From (33), if $E^*(t)$ is positive definite, it follows that the inequality

$$\sqrt{\mathscr{R}^*_a} \geq 1 \tag{37}$$

is a sufficient condition of the linear and nonlinear Lyapunov stability of the conduction state. We shall determine now, explicitely, the region of the parameters space where the inequality (37) is satisfied.

5. The nonlinear stability bound

To deduce, in the parameter space, the nonlinear stability bound, we study now the variational problem (36) in terms of the independent fields $(\chi, \chi', \psi, \psi', \vartheta)$ verifying the boundary conditions (5) and (16).

The Euler equations associated to the maximum problem (36) are:

$$\begin{cases} -\mathscr{R}\left(1+\dfrac{b}{\mathscr{P}_r}\right)\Delta_1\vartheta - M^2 d\Delta_1\chi'_{zzz} + \dfrac{2}{\sqrt{\mathscr{R}_a^*}}\Delta\Delta\Delta_1\chi = 0, \\ -\mathscr{R}\left(1+\dfrac{b}{\mathscr{P}_r}\right)\Delta_1\chi + \dfrac{2}{\sqrt{\mathscr{R}_a^*}}\dfrac{b}{\mathscr{P}_r}\Delta\vartheta = 0, \\ d\,\Delta_1\chi_{zzz} + 2\dfrac{\mathscr{P}_m}{\mathscr{P}_r}d\,\alpha\Delta\Delta_1\chi'_{zz} + \dfrac{2}{\sqrt{\mathscr{R}_a^*}}\dfrac{\mathscr{P}_m}{\mathscr{P}_r}\left[\Delta\Delta\Delta_1\chi' + d(1-\alpha)\Delta\Delta_1\chi'_{zz}\right] = 0, \\ \Delta\Delta_1\psi = 0, \\ \Delta\Delta_1\psi' = 0. \end{cases} \tag{38}$$

In the class of normal mode perturbations

$$(\chi,\chi',\vartheta,\psi,\psi') = (X(z),K(z),\Theta(z),\Psi(z),\Psi'(z))\exp[i(a_x x + a_y y)+\sigma t], \tag{39}$$

with $\sigma \in \mathbb{C}$, the Euler equations (38) become:

$$\begin{cases} \mathscr{R}\left(1+\dfrac{b}{\mathscr{P}_r}\right)a^2\Theta + M^2 d\,a^2 D^3 K - \dfrac{2}{\sqrt{\mathscr{R}_a^*}}(D^2-a^2)^2 a^2 X = 0, \\ \mathscr{R}\left(1+\dfrac{b}{\mathscr{P}_r}\right)a^2 X + \dfrac{2}{\sqrt{\mathscr{R}_a^*}}\dfrac{b}{\mathscr{P}_r}(D^2-a^2)\Theta = 0, \\ d\,a^2 D^3 X + 2\dfrac{\mathscr{P}_m}{\mathscr{P}_r}\left[d\left(\alpha + \dfrac{1}{\sqrt{\mathscr{R}_a^*}}(1-\alpha)\right)D^2 + \dfrac{1}{\sqrt{\mathscr{R}_a^*}}(D^2-a^2)\right](D^2-a^2)a^2 K = 0. \\ (D^2-a^2)\Psi = (D^2-a^2)\Psi' = 0, \end{cases} \tag{40}$$

where $D = \dfrac{d}{dz}$.

From (40) we have

$$\begin{aligned} &\left\{2\left[\mathscr{R}^2\left(1+\frac{b}{\mathscr{P}_r}\right)^2 a^4\frac{\mathscr{P}_r}{b}\frac{\sqrt{\mathscr{R}_a^*}}{2} + \frac{2}{\sqrt{\mathscr{R}_a^*}}(D^2-a^2)^3 a^2\right]\cdot\right. \\ &\cdot\left[d\left(\alpha+\frac{1}{\sqrt{\mathscr{R}_a^*}}(1-\alpha)\right)D^2 + \frac{1}{\sqrt{\mathscr{R}_a^*}}(D^2-a^2)\right](D^2-a^2)a^2 + \\ &\left. + d^2M^2\frac{\mathscr{P}_r}{\mathscr{P}_m}a^4(D^2-a^2)D^6\right\}X = 0. \end{aligned} \tag{41}$$

Owing to the boundary conditions (16) we can assume

$$X(z) = \sum_{n=0}^{\infty} X_n \sin(n\pi z), \tag{42}$$

where $X_n \equiv (X, \sin(n\pi z))$ are the Fourier coefficients.

Substituting (42) into (41) we obtain

$$
\begin{aligned}
&\frac{1}{\sqrt{\mathscr{R}_a^*}}\left[\mathscr{R}^2\mathscr{R}_a^*\left(1+\frac{b}{\mathscr{P}_r}\right)^2\frac{\mathscr{P}_r}{b}a^4-4(n^2\pi^2+a^2)^3a^2\right]= \\
&=-M^2\frac{\mathscr{P}_r}{\mathscr{P}_m}\frac{d^2n^4\pi^4a^2}{d\left(\alpha+\frac{1}{\sqrt{\mathscr{R}_a^*}}(1-\alpha)\right)+\frac{1}{\sqrt{\mathscr{R}_a^*}}\frac{n^2\pi^2+a^2}{n^2\pi^2}}.
\end{aligned}
\tag{43}
$$

Differentiating the r.h.s. of (43) with respect to d we obtain:

$$
d=-\frac{2}{\sqrt{\mathscr{R}_a^*}}\frac{\frac{n^2\pi^2+a^2}{n^2\pi^2}}{\alpha+\frac{1}{\sqrt{\mathscr{R}_a^*}}(1-\alpha)}, \qquad \forall n\in\mathbb{N}. \tag{44}
$$

Taking into account (44), equation (43) becomes:

$$
\mathscr{R}^2\left(1+\frac{b}{\mathscr{P}_r}\right)^2\frac{\mathscr{P}_r}{b}a^2=\frac{4(n^2\pi^2+a^2)}{\mathscr{R}_a^*}\left\{(n^2\pi^2+a^2)^2+\frac{\mathscr{P}_r}{\mathscr{P}_m}\frac{M^2n^2\pi^2}{\left[\alpha\left(1-\frac{1}{\sqrt{\mathscr{R}_a^*}}\right)+\frac{1}{\sqrt{\mathscr{R}_a^*}}\right]^2}\right\}. \tag{45}
$$

By choosing

$$
\alpha=-\frac{\sqrt{\mathscr{R}_a^*}+1}{\sqrt{\mathscr{R}_a^*}-1}, \tag{46}
$$

equivalent to $\left[\alpha\left(1-\frac{1}{\sqrt{\mathscr{R}_a^*}}\right)+\frac{1}{\sqrt{\mathscr{R}_a^*}}\right]=-1$, and differentiating the l.h.s. of (45) with respect to b which gives $b=\mathscr{P}_r$, from (45) we have:

$$
\mathscr{R}_a^*(M^2,\mathscr{P}_r,\mathscr{P}_m,n^2,a^2)=\frac{1}{\mathscr{R}^2}\frac{n^2\pi^2+a^2}{a^2}\left[(n^2\pi^2+a^2)^2+M^2\frac{\mathscr{P}_r}{\mathscr{P}_m}n^2\pi^2\right]. \tag{47}
$$

We observe that, if $\sqrt{\mathscr{R}_a^*}\geq 1$, d and α, given by (44) and (46), satisfy

$$
1+d>0 \quad\wedge\quad 1+d(1-\alpha)>0,
$$

whence $E^*(t)$ and $\mathscr{D}^*$ are both definite positive. The minimum of (47) with respect to $n\in\mathbb{N}$ is attained for $n=1$, therefore, as a function of $x=\frac{a^2}{\pi^2}$,

$$
\mathscr{R}_a^*(M^2,\mathscr{P}_r,\mathscr{P}_m,x)=\frac{\pi^4}{\mathscr{R}^2}\frac{1+x}{x}\left[(1+x)^2+\frac{M^2}{\pi^2}\frac{\mathscr{P}_r}{\mathscr{P}_m}\right]. \tag{48}
$$

We have so proved the following

Theorem 1. *If the principle of exchange of stabilties holds, the inequality*

$$
1\leq\mathscr{R}_a^*,
$$

with $\mathscr{R}_a^*$ given by (48), is a sufficient condition of linear and non linear Lyapunov stability, that is, the linear and non linear stability bounds coincide if instability occurs as stationary convection.

Indeed

$$\mathscr{R}_a^* \geq 1 \iff \mathscr{R}^2 \leq \pi^4 \frac{1+x}{x} \left[(1+x)^2 + \frac{M^2}{\pi^2} \frac{\mathscr{P}_r}{\mathscr{P}_m} \right],$$

where

$$\pi^4 \frac{1+x}{x} \left[(1+x)^2 + \frac{M^2}{\pi^2} \frac{\mathscr{P}_r}{\mathscr{P}_m} \right]$$

is exactly the Rayleigh function of the linear instability theory, if the principle of exchange of stabilities holds.

We observe that, owing (44), d is a function of n, whence (31) really defines a sequence of Lyapunov functions, but we can, *a posteriori*, consider only the Lyapunov function of that sequence corresponding to $n = 1$, obtaining the same result.

6. Conclusions

In this paper we have dealt with the point of loss of stability of the conduction state of the magnetic Bénard problem, when the thermal convection appears. To study the non linear Lyapunov stability of the conduction state we have considered the L^2 norm of the perturbation fields, by projecting the perturbation equations on some orthogonal subspaces of the space of kinematically admissible functions. The problem has been formulated in terms of poloidal and toroidal fields, suitable to represent the solenoidal velocity and magnetic fields in a plane layer. By solving the Euler equation associated to the variational problem arising from the deduction of the non linear stability bound, and, later, by differentiating with respect to some parameters involved in the Lyapunov function, we have obtained a non linear stability bound that coincides with the Rayleigh function of the linear instability theory, in the subspace of the parameter space where instability occurs as stationary convection. The obtained additional equation that has allowed us to obtain the coincidence of the linear and non linear stability bounds is a linear one. In a similar way, for the classical rotating Bénard problem, Palese (2018) has shown the absence of subcritical instabilities, in the region of the parameter space in which the principle of exchange of stabilities holds. In this way we was able to avoid the introduction of more complicated Lyapunov functions that can provide the absence of subcritical instabilities, but only under some restrictions on the initial data.

References

Chandrasekhar, S. (1968). *Hydrodynamic and Hydromagnetic Stability*. Dover, p. 704. URL: http://store.doverpublications.com/048664071x.html (visited on 05/08/2018).

Galdi, G. P. and Padula, M. (1990). "A new approach to energy theory in the stability of fluid motion". *Archive for Rational Mechanics and Analysis* **110**(3), 187–286. DOI: 10.1007/BF00375129.

Galdi, G. P. and Straughan, B. (1985). "A nonlinear analysis of the stabilizing effect of rotation in the Bénard problem". *Proceedings of the Royal Society A* **402**(1823), 257–283. DOI: 10.1098/rspa.1985.0118.

Georgescu, A. (1985). *Hydrodynamic Stability Theory*. 2nd ed. Mechanics: Analysis 9. Dordrecht: Springer, p. 307. DOI: 10.1007/978-94-017-1814-1.

Georgescu, A. and Oprea, I. (1994). *Bifurcation Theory from the Applications Point of View*. Monografii Matematice 51. Timişoara: Tipografia Universităţii din Timişoara, p. 261.

Georgescu, A. and Palese, L. (1996). "Extension of a Joseph's criterion to the non linear stability of mechanical equilibria in the presence of thermodiffusive conductivity". *Theoretical and Computational Fluid Dynamics* **8**(6), 403–413. DOI: 10.1007/BF00455992.

Georgescu, A. and Palese, L. (2009). *Stability Criteria for Fluid Flows*. Series on Advances in Mathematics for Applied Sciences 81. Singapore: World Scientific, p. 420. DOI: 10.1142/7513.

Georgescu, A. and Palese, L. (2010). "A linearization principle for the stability of the chemical equilibrium of a binary mixture". *Romai Journal* **6**(2), 131–138. URL: http://rj.romai.ro/arhiva/2010/2/RJvol6-nr2-Georgescu-Palese.pdf (visited on 05/08/2018).

Georgescu, A. and Palese, L. (2011). "On the nonlinear stability of a binary mixture with chemical surface reactions". *Annals of the Academy of Romanian Scientists Series on Mathematics and its Applications* **3**(1), 106–115. URL: http://www.mathematics-and-its-applications.com/preview/july2011/data/6_Adelina_Lidia_abstract.pdf (visited on 05/08/2018).

Georgescu, A., Palese, L., and Redaelli, A. (2000). "On a new method in hydrodynamic stability theory". *Math. Sci. Res. Hotline* **4**(7), 1.

Georgescu, A., Palese, L., and Redaelli, A. (2001). "The complete form for the Joseph extended criterion". *Annali dell'Università di Ferrara, Sezione VII - Scienze Matematiche* **47**(1), 9–22. URL: https://link.springer.com/article/10.1007/BF02838172 (visited on 05/08/2018).

Joseph, D. D. (1965). "On the stability of the Boussinesq equations". *Archive for Rational Mechanics and Analysis* **20**(1), 59–71. DOI: 10.1007/BF00250190.

Joseph, D. D. (1966). "Nonlinear stability of the Boussinesq equations by the method of energy". *Archive for Rational Mechanics and Analysis* **22**(3), 163–184. DOI: 10.1007/BF00266474.

Joseph, D. D. (1970). "Global stability of the conduction-diffusion solution". *Archive for Rational Mechanics and Analysis* **36**(4), 285–292. DOI: 10.1007/BF00249516.

Joseph, D. D. (1976a). *Stability of Fluid Motions*. Vol. I. Berlin: Springer, pp. xiii+282. DOI: 10.1007/978-3-642-80991-0.

Joseph, D. D. (1976b). *Stability of Fluid Motions*. Vol. II. Berlin: Springer, pp. xiv+276. DOI: 10.1007/978-3-642-80994-1.

Koschmieder, E. L. (1993). *Bénard Cells and Taylor Vortices*. Cambridge Monographs on Mechanics. Cambridge: Cambridge University Press, p. 350. URL: http://www.cambridge.org/it/academic/subjects/mathematics/fluid-dynamics-and-solid-mechanics/benard-cells-and-taylor-vortices (visited on 05/08/2018).

Mikhlin, S. G. (1970). *An Advanced Course of Mathematical Physics*. North-Holland Series in Applied Mathematics and Mechanics. Amsterdam, London: North Holland, p. 561.

Mulone, G. (2004). "Stabilizing effects in dynamical systems: linear and nonlinear stability conditions". *Far East Journal of Applied Mathematics* **15**(2), 117–135. URL: http://www.pphmj.com/abstract/566.htm (visited on 05/08/2018).

Mulone, G. and Rionero, S. (1997). "The rotating Bénard problem: new stability results for any Prandtl and Taylor numbers". *Continuum Mechanics and Thermodynamics* **9**(6), 347–363. DOI: 10.1007/s001610050076.

Palese, L. (2005). "On the nonlinear stability of the MHD anisotropic Bénard problem". *International Journal of Engineering Science* **43**(15-16), 1265–1282. DOI: 10.1016/j.ijengsci.2005.03.007.

Palese, L. (2014a). "On the Lyapunov function for some convection problems in hydrodynamics". *Bollettino di Matematica Pura e Applicata* **VII**, 81–92. URL: http://www.aracneeditrice.it/aracneweb/index.php/estratto.html?item=10.4399/97888548780139&isbn=9788854878013 (visited on 05/08/2018).

Palese, L. (2014b). "On the nonlinear stability of a binary mixture with chemical surface reactions in the general case". *Electronic Journal of Differential Equations* **2014**(187), 1–12. URL: http://ejde.math.txstate.edu/Volumes/2014/187/abstr.html (visited on 05/08/2018).

Palese, L. (2014c). "On the stability of the rotating Bénard problem". *Annals of the Academy of Romanian Scientists Series on Mathematics and its Applications* **6**(1), 4–20. URL: http://www.mathematics-and-its-applications.com/preview/june2014/data/first-page/articolul1.pdf (visited on 05/08/2018).

Palese, L. (2018). "Thermosolutal convection in a rotating viscoelastic Walters fluid". *Atti della Accademia Peloritana dei Pericolanti. Classe di Scienze Fisiche, Matematiche e Naturali*. to be published. DOI: 10.1478/AAPP.xxxxx.

Prodi, G. (1962). "Teoremi di tipo locale per il sistema di Navier-Stokes e stabilità delle soluzioni stazionarie". *Rendiconti del Seminario Matematico dell'Università di Padova* **32**, 374–397. URL: http://www.numdam.org/item?id=RSMUP_1962__32__374_0 (visited on 05/08/2018).

Rionero, S. and Mulone, G. (1987). "On the non-linear stability of a thermo-diffusive fluid mixture in a mixed problem". *Journal of Mathematical Analysis and Applications* **124**(1), 165–188. DOI: 10.1016/0022-247X(87)90032-1.

Rionero, S. and Mulone, G. (1988). "A nonlinear stability analysis of the magnetic Bénard problem through the Lyapunov direct method". *Archive for Rational Mechanics and Analysis* **103**(4), 347–368. DOI: 10.1007/BF00251445.

Sattinger, D. H. (1970). "The mathematical problem of hydrodynamic stability". *Journal of Mathematics and Mechanics* **19**(9). (now Indiana University Mathematics Journal), 797–817. DOI: 10.1512/iumj.1970.19.19064.

Schmitt, B. J. and von Wahl, W. (1992). "Decomposition of solenoidal fields into poloidal fields, toroidal fields and the mean flow. Applications to the Boussinesq-equations". In: *The Navier-Stokes equations II - Theory and numerical methods*. Ed. by J. G. Heywood, K. Masuda, R. Rautmann, and V. A. Solonnikov. Vol. 1530. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, pp. 291–305. DOI: 10.1007/BFb0090349.

Sobolev, S. L. (1963). *Applications of Functional Analysis in Mathematical Physics*. 3rd. Translations of Mathematical Monographs 7. American Mathematical Society, p. 239.

Straughan, B. (2004). *The Energy Method, Stability and Nonlinear Convection*. 2nd. Series in Applied Mathematical Sciences 91. New York: Springer-Verlag, pp. xii+449. DOI: 10.1007/978-0-387-21740-6.

Yudovich, V. I. (1965). "On the stability of stationary flow of viscous incompressible fluids". *Doklady Akademii Nauk SSSR* **161**(5). English translation by AIP: Soviet Physics - Doklady, vol. 10, n. 4, 1965, pp. 293-295, 1037–1040.

Yudovich, V. I. (1970a). "On the stability of forced oscillations of the fluids". *Doklady Akademii Nauk SSSR* **195**(2). English translation by AMS: Soviet Mathematics, vol. 11, n. 6, 1970, pp. 1473-1477, 292–295.

Yudovich, V. I. (1970b). "On the stability of self-oscillations of the fluids". *Doklady Akademii Nauk SSSR* **195**(3). English translation by AMS: Soviet Mathematics, vol. 11, n. 6, 1970, pp. 1543-1546, 574–576.

Yudovich, V. I. (1989). *The Linearization Method in Hydrodynamical Stability Theory*. Translations of Mathematical Monographs 74. Providence, R. I.: American Mathematical Society, p. 170.

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Paper contributed to the conference entitled "Thermal theories of continua: survey and developments (Thermocon 2016)", which was held in Messina, Italy (19–22 April 2016) under the patronage of the *Accademia Peloritana dei Pericolanti*

Manuscript received 5 December 2017; published online 20 May 2019