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QUANTUM FINITE-TIME AVAILABILITY

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ABSTRACT. The availability of a thermodynamic system with respect to an environment is the maximum work, which can be gained from bringing it into equilibrium with its environment by a reversible process. If the process has to proceed in finite time, there will be unavoidable losses diminishing the availability; this consequence is captured by the Finite-Time Availability. Here we consider the consequences of an availability extracting process for a paradigmatic quantum system, the parametric harmonic oscillator. Differences and similarities between its Quantum Finite-Time Availability and the classical Finite-Time Availability of an ideal gas in a cylinder with a piston are discussed.

1. Introduction

The notion of available work from disequilibrium between a system and its surroundings dates back to classical treatise by Gibbs (1873). Gibbs' availability uses only reversible processes to extract that available work and thus is freed from any considerations of time. The subsequently developed notion of finite-time availability (Andresen *et al.* 1984; Andresen, Rubin, and Berry 1983; Tolman and Fine 1948) takes into account the fact that macroscopic processes require infinite time to proceed reversibly and thus in finite time must forego some of the available work. Recent extensions of this concept to the quantum realm (K. H. Hoffmann and Salamon 2015; K. H. Hoffmann, Schmidt, and Salamon 2015) share some of these features while displaying some important (and surprising) differences.

In the present paper, we highlight these similarities and differences between Finite-Time Availability and Quantum Finite-Time Availability using two simple examples: the classical gas in a cylinder equipped with a piston and the quantum parametric harmonic oscillator.

2. Finite-Time Availabilty

2.1. Availability. The maximum work, which can be gained from a system by bringing it into equilibrium with its environment by a reversible process, is a useful thermodynamic quantity. This work is known as availability or exergy, and is of considerable importance especially in the engineering community (R. U. Ayres, L. W. Ayres, and Martinás 1998; Badescu 2013; Izumida and Okuda 2014; Kotas 2013; Madadi, Tavakoli, and Rahimi 2015; Souidi *et al.* 2015; Szargut, Morris, and Steward 1987; Tsatsaronis 1993; Valero 1998). In its definition the environment is considered to be so large that its temperature and pressure do not change during the extraction process. In that case the reversible availability A^{rev} is given by

$$A^{\mathrm{rev}} = U - U_0 - T_0(S - S_0) + p_0(V - V_0) \tag{1}$$

where U is the system's internal energy, S its entropy, T its temperature, and zero subscripts refer to its reference state of equilibrium with the environment. We note that (1) may need to be augmented by further terms for work parameters if these are present in the system of interest. Such additional terms are needed if they lead to energy gains or losses of the environment. Finally we mention that the choice of the environment in a technological application requires careful specification (Gaggioli 2012).

2.2. Finite-Time Availability. The condition that the process considered in the definition of availability must be reversible is usually not fulfilled for processes which have to proceed in finite time. The changes needed in the availability concept in order to get a quantification of the work available from a system under such conditions led to the development of finite-time availability (Andresen *et al.* 1984; Andresen, Rubin, and Berry 1983; Tolman and Fine 1948)

$$A^{\mathrm{FT}} = \mathrm{Max}\left(-\Delta A^{\mathrm{rev}} - T_0 \int_0^{\tau} \sigma_{\mathrm{u}}(t)\mathrm{d}t\right), \tag{2}$$

where the maximum is taken over all processes of duration τ, $\sigma_{\mathrm{u}}(t)$ is the total entropy production rate over all systems participating in the process, and $\Delta A^{\mathrm{rev}} = A^{\mathrm{rev}}_{\mathrm{f}} - A^{\mathrm{rev}}_{\mathrm{i}}$, where i and f subscripts refer to the initial $t = 0$ and final $t = \tau$ states, respectively.

Technically, this optimization can be performed using optimal control theory and incorporating any constraints on the available controls (Andresen *et al.* 2011; Franz and K. H. Hoffmann 2002; Franz, K. H. Hoffmann, and Salamon 2001; K. H. Hoffmann *et al.* 2003; Leitmann 1981; Lin, Chang, and Xu 2014; Stefanatos, Ruths, and Li 2010; A. M. Tsirlin, Salamon, and K. H. Hoffmann 2011). Such constraints may not allow the final state of equilibrium with the environment to be reached in which case $A^{\mathrm{rev}}_{\mathrm{f}} \geq 0$. Usually one finds that the ratio of the available time to the intrinsic time scales of the irreversible processes is an important factor for the dissipative losses (Salamon and Berry 1983).

3. Classical Finite-Time Availability: an ideal gas in a cylinder

In order to highlight the special features of Quantum Finite-Time Availability (QFT Availabiliy) we will contrast the quantum example system with a classical counterpart. For that purpose we will use an ideal gas in a cylinder with a piston and an environment characterized by its intensities p_0 and temperature T_0. We assume that the gas is initially at pressure p_{i}, volume V_{i} and temperature T_{i}. The volume V of the gas can be changed by moving a piston, and thus work can be obtained from the gas or delivered to the gas. Varying the volume allows us to control the state of the gas, i.e. the pressure and the temperature of the gas can be altered by changing the volume. For the purpose of the planned comparison with the quantum system we consider only initial gas states $(T_{\mathrm{i}}, p_{\mathrm{i}})$, which under a reversible adiabatic expansion lead to final gas states $(T_{\mathrm{f}}, p_{\mathrm{f}})$ which are in equilibrium with the environment $(T_{\mathrm{f}}, p_{\mathrm{f}}) = (T_0, p_0)$, i.e. the equilibrium state is adiabatically accessible from the initial state. And finally we assume that the range of our control $V(t)$ is restricted to the interval

$$V_{\text{min}} = V_{\text{i}} \leq V(t) \leq V_{\text{f}} = V_{\text{max}}. \tag{3}$$

The reversible availability A^{rev} of the gas is the work which can be extracted in a reversible process bringing the system from its initial state to an equilibrium state with the environment. From (1) one finds

$$A^{\text{rev}} = U(p_{\text{i}}, T_{\text{i}}) - U(p_0, T_0) + p_0(V_{\text{i}} - V_0) \tag{4}$$

where $U(p,T)$ is the internal energy of the gas. This availablity is independent of the control $V(t)$ as long as it leads from its initial value V_{min} to its final value V_{max}. Note that the term $T_0(S_{\text{i}} - S_0)$ is missing due to our assumption of adiabatic accessibility, i.e $S_{\text{i}} = S_0$.

The concept of finite-time availability becomes important if the process must be performed in a finite available time τ and if it cannot be performed without dissipation. To exemplify that we here assume that any movement of the piston leads to a frictional force $F = \alpha \dot{V}$ acting on it, where α is the frictional constant. Thus some of the work gained from the expansion is consumed. The power lost due to this dissipation is $P_{\text{lost}} = \alpha \dot{V}^2$ and is considered to be completely dissipated into heat put into the environment. Thus the entropy production is $\sigma_{\text{u}} = \alpha \dot{V}^2 / T_0$ and the finite-time availability becomes

$$A^{\text{FT}} = \underset{V(t)}{\text{Max}} \left(-\Delta A^{\text{rev}} - T_0 \int_0^\tau \sigma_{\text{u}}(t) \mathrm{d}t \right) \tag{5}$$

$$= \underset{V(t)}{\text{Max}} \left(A^{\text{rev}} - \int_0^\tau \alpha \dot{V}^2 \mathrm{d}t \right) \tag{6}$$

Note that as the final state is in equilibrium with the environment the remaining availability $A^{\text{rev}}(\tau) = A_{\text{f}}^{\text{rev}} = 0$ and thus $-\Delta A^{\text{rev}} = -(A_{\text{f}}^{\text{rev}} - A_{\text{i}}^{\text{rev}}) = A_{\text{i}}^{\text{rev}} = A^{\text{rev}}$. In this case the application of control theory shows that the optimal process is an expansion with $\dot{V} = \text{const.}$, for details see for instance K. H. Hoffmann, Watowich, and Berry (1985). The speed of the volume increase is then determined by the requirement that $V(0) = V_{\text{min}}$ and $V(\tau) = V_{\text{max}}$:

$$\dot{V} = (V_{\text{max}} - V_{\text{min}})/\tau. \tag{7}$$

Inserting this into (6) we find

$$A^{\text{FT}} = A^{\text{rev}} - \tau\alpha \left(\frac{V_{\text{max}} - V_{\text{min}}}{\tau} \right)^2 = A^{\text{rev}} - (V_{\text{max}} - V_{\text{min}})^2 \frac{\alpha}{\tau}, \tag{8}$$

which shows that the loss of availability is proportional to $1/\tau$. This is a very typical feature which stresses that losses can only be avoided by going to infinite process times τ. The situation is depicted in Figure 1, where the ratio of finite-time availability and the reversible availability is shown as a function of the available process time τ. The difference between the two availabilities is the work loss due to the entropy produced and deposited in the environment.

4. Quantum Finite-Time Availabilty: the Parametric Harmonic Oscillator

The parametric harmonic oscillator is one of the paradigm system in physics. It can for instance be used to describe particles in potential wells, where the curvature of the well can

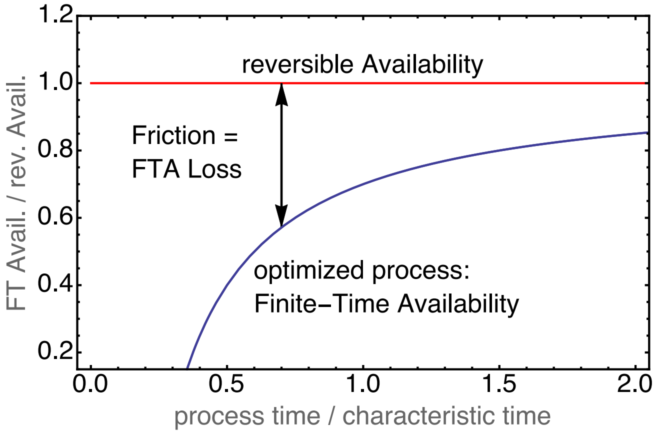

FIGURE 1. The ratio of finite-time availability and reversible availability is shown as a function of the available process time τ for a gas-in-cylinder system with friction. Note that the loss in availability due to the friction forces on the piston are non-vanishing for all finite expansion speeds. Such loss can only be avoided if the process time is extended to infinity. Note further that in the limit of zero process time the finite-time availability goes to $-\infty$.

be adjusted on time scales much faster than the dynamics of the particles themselves. The curvature determines the frequency ω of the particle oscillation, which can thus be used as a control for its dynamics.

Here we are interested in the quantum description of such an oscillator. Its Hamiltonian is given by

$$\mathscr{H} = \tfrac{1}{2m}\mathscr{P}^2 + \tfrac{1}{2}m\omega(t)^2\mathscr{Q}^2, \tag{9}$$

where m is the mass of the particle, and $\mathscr{Q}$ and $\mathscr{P}$ are the position and momentum operators, respectively.

If the system is thermally isolated its reversible dynamics follows from the Heisenberg equation

$$\frac{d\mathscr{O}(t)}{dt} = \frac{i}{\hbar}[\mathscr{H}(t),\mathscr{O}(t)] + \frac{\partial\mathscr{O}(t)}{\partial t}. \tag{10}$$

While usually taking the time derivative of an operator leads to a new operator, for the quantum harmonic oscillators there exists a set of three operators, for which the time derivative of each operator can be expressed as a linear combination of the same three operators (Boldt *et al.* 2013). In addition to the Hamiltonian these can be taken to be (Salamon *et al.* 2009) the Lagrangian $\mathscr{L} = \tfrac{1}{2m}\mathscr{P}^2 - \tfrac{1}{2}m\omega(t)^2\mathscr{Q}^2$ and the position-momentum correlation $\mathscr{C} = \frac{\omega(t)}{2}(\mathscr{Q}\mathscr{P} + \mathscr{P}\mathscr{Q})$, but other sets exist as well in Salamon, K. H. Hoffmann, and A. Tsirlin (2012). Note the explicit time dependence of these operators via $\omega(t)$.

From the equation of motion (10) and in combination with a transition to expectation values ($L = \langle \mathscr{L} \rangle$, $C = \langle \mathscr{C} \rangle$, and $E = \langle \mathscr{H} \rangle$) taken with the initial equilibrium density matrix one finds

$$\dot{E} = \frac{\dot{\omega}}{\omega}(E - L) \tag{11}$$

$$\dot{L} = -\frac{\dot{\omega}}{\omega}(E - L) - 2\omega C \tag{12}$$

$$\dot{C} = 2\omega L + \frac{\dot{\omega}}{\omega} C \tag{13}$$

If the density matrix describes an equilibrium state, the relation between the energy E, the temperature T and the frequency ω of a quantum harmonic oscillator can be determined. This relation can be obtained by calculating the partition function as well as the mean energy which can be expressed in closed form

$$E_{\mathrm{eq}}(T, \omega) = \hbar\omega \left(\frac{1}{2} + \frac{1}{e^{\hbar\omega/k_{\mathrm{B}}T} - 1} \right). \tag{14}$$

The energy change resulting from an adiabatic change in ω involves work only, as per definition no heat exchange occurs. Thus the corresponding power P_{adi} going into the oscillator can be expressed as

$$P_{\mathrm{adi}} = \dot{E}. \tag{15}$$

Similarly we can obtain dynamical equations describing the contact with a heat bath with temperature T at *constant* frequency ω based on the approach by Lindblad (1976) to open quantum system dynamics

$$\dot{E} = -g \left(E - E_{\mathrm{eq}}(T, \omega) \right) \tag{16}$$

$$\dot{L} = -g\, L - 2\omega C \tag{17}$$

$$\dot{C} = -g\, C + 2\omega L, \tag{18}$$

where g is the coupling constant to the heat bath. The energy change during a thermal contact involves heat only, as the constant ω does not allow any work exchange. Thus the heat flux q into the oscillator can be expressed as

$$q = \dot{E}. \tag{19}$$

4.1. FEATs: Fastest Effectively Adiabatic Transitions. From the dynamical equations introduced we can now determine the heat and work exchange of the system for an arbitrary frequency schedule $\omega(t)$. In analogy with the limits for the work parameter V in the classical case the range of the work parameter ω is also subject to constraints

$$\omega_{\mathrm{min}} = \omega_{\mathrm{f}} \leq \omega(t) \leq \omega_{\mathrm{i}} = \omega_{\mathrm{max}}. \tag{20}$$

Of special interest are now those processes which lead from the initial equilibrium state with $E_{\mathrm{eq}}(T_{\mathrm{i}}, \omega_{\mathrm{i}})$ with $\omega_{\mathrm{i}} = \omega_{\mathrm{max}}$ to the final equilibrium state with $E_{\mathrm{eq}}(T_0, \omega_{\mathrm{min}})$. As in the classical example we choose the environmental temperature such that after a reversible adiabatic process the system is in equilibrium with the environment. This can be achieved by setting T_0 such that $E_{\mathrm{eq}}(T_0, \omega_{\mathrm{min}}) = E_{\mathrm{eq}}(T_{\mathrm{i}}, \omega_{\mathrm{max}})\ \omega_{\mathrm{min}}/\omega_{\mathrm{max}}$.

Surprisingly, the final equilibrium energy can even be achieved by processes of finite duration. Those processes which lead from one equilibrium state to another equilibrium state *in shortest time* have been dubbed fastest effectively adiabatic transitions or FEATs (Boldt, Salamon, and K. H. Hoffmann 2016; K. H. Hoffmann and Salamon 2015; Salamon *et al.* 2009).

A natural question to ask is then by which schedule for the frequency change this total gain of the reversible availability in finite time becomes possible. Based on a control theoretic analysis (Salamon *et al.* 2009; Salamon, K. H. Hoffmann, and A. Tsirlin 2012; Stefanatos, Ruths, and Li 2010; Stefanatos, Schaettler, and Li 2011; A. M. Tsirlin, Salamon, and K. H. Hoffmann 2011) it turns out that the time-optimal control is of the bang-bang type consisting of two branches with constant ω, where only the extreme values ω_{min} and ω_{max} are used.

In particular starting from an equilibrium state with $\omega_{\text{i}} = \omega_{\text{max}}$, for a FEAT the frequency is immediately switched to ω_{min}, followed with a fixed frequency branch of duration τ_1. Then the second frequency change brings the frequency back to ω_{max} followed by another fixed frequency branch of duration τ_2. And finally the third frequency jump takes the frequency to its lowest value $\omega_{\text{f}} = \omega_{\text{min}}$.

Such a FEAT leads from the initial equilibrium state to a final equilibrium state in minimum time when the branch times are chosen appropriately

$$\begin{aligned} \tau_1 &= \frac{1}{2\omega_{\text{f}}} \arccos\phi \\ \tau_2 &= \frac{1}{2\omega_{\text{i}}} \arccos\phi, \end{aligned} \tag{21}$$

with

$$\phi = \frac{\omega_{\text{i}}^2 + \omega_{\text{f}}^2}{(\omega_{\text{i}} + \omega_{\text{f}})^2}. \tag{22}$$

The total FEAT time is

$$\tau = \tau_1 + \tau_2 = \frac{1}{2}\left(\frac{1}{\omega_{\text{i}}} + \frac{1}{\omega_{\text{f}}}\right) \arccos\phi. \tag{23}$$

Such a FEAT is shown in Figure 2, where the optimized time dependence of the frequency and the resulting energy of the oscillator are depicted as functions of time. Note that during a FEAT the energy of the system changes discontinuously with jumps in the optimal control $\omega_{\text{opt}}(t)$. This is due to the explicit ω-dependence of the Hamiltonian.

For completeness we mention that there exist also time-continuous processes that lead in finite time from one equilibrium to another equilibrium state. One approach leading to such solutions is also using optimal control theory with additional constraints on the rate at which we are allowed to change the frequency, i.e. $|\dot{\omega}| \leq \dot{\omega}_{\text{max}}$ (K. H. Hoffmann *et al.* 2011). This approach leads to the same branches as the bang-bang solutions connected by ramps with $|\dot{\omega}| = \dot{\omega}_{\text{max}}$. These solutions are still FEAT solutions with the additional constraint on $\dot{\omega}$.

A second source of time-continuous solutions is based on so-called Ermakov invariants and leads to STA (shortcuts to adiabaticity) processes (Chen and Muga 2010; Chen *et al.* 2010; Schaff *et al.* 2011). Of course all of these solutions are slower than the associated FEAT process.

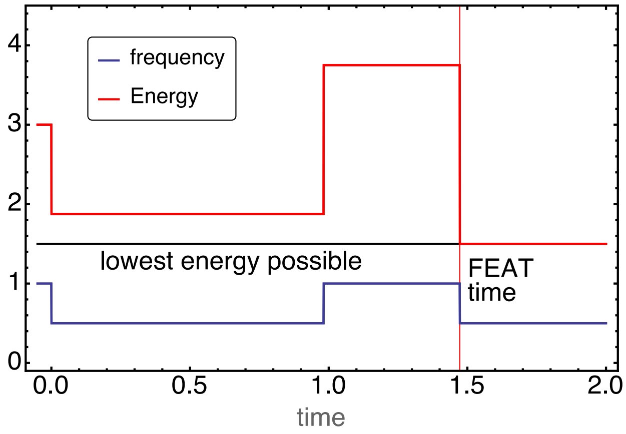

FIGURE 2. The optimized time dependence of the frequency and the resulting energy of the oscillator are shown as a function of time for a FEAT. The vertical red line indicates the FEAT time τ, which allows one to bring the energy of the oscillator to its lowest possible energy in the shortest time.

Finally we note that the process optimization of the quantum parametric harmonic oscillator is still an area of ongoing research Stefanatos 2015, 2017. Also, just which quantum processes have an associated FEAT control is not known. The existence of a FEAT time was recently shown not to be unique to the quantum parametric harmonic oscillator; it exists also in a quantum spin system (Boldt *et al.* 2012; K. H. Hoffmann, Schmidt, and Salamon 2015).

4.2. MATs: Maximum Availability Transitions. The FEATs discussed above allow one to bring the quantum parametric harmonic oscillator to its lowest energy state in the shortest possible time, and thus for all process times longer than the FEAT time the entire availability can be extracted from the system in a reversible process. If however the process time t_P is shorter than the FEAT time, this extraction is not possible and the finite-time availability has to be determined via

$$A^\mathrm{FT} = \operatorname*{Max}_{\omega(t)} \left(-\Delta A^\mathrm{rev} - T_0 \int_0^{t_\mathrm{P}} \sigma_\mathrm{u}(t) \mathrm{d}t \right) \tag{24}$$

$$= \operatorname*{Max}_{\omega(t)} \int_0^{t_\mathrm{P}} -\dot{E}\, \mathrm{d}t. \tag{25}$$

Here we made use of the fact that the entropy production is vanishing and thus the change in availability can be directly calculated from the change of the energy of the system, i.e. from the power delivered. Note that as ω is not a quantity which needs to be deposited to or taken from the environment, no term corresponding to $p_0(V - V_0)$ exists.

The controls for which this maximum is attained can again be shown to be of the bang-bang type (K. H. Hoffmann, Schmidt, and Salamon 2015). Here we restrict our

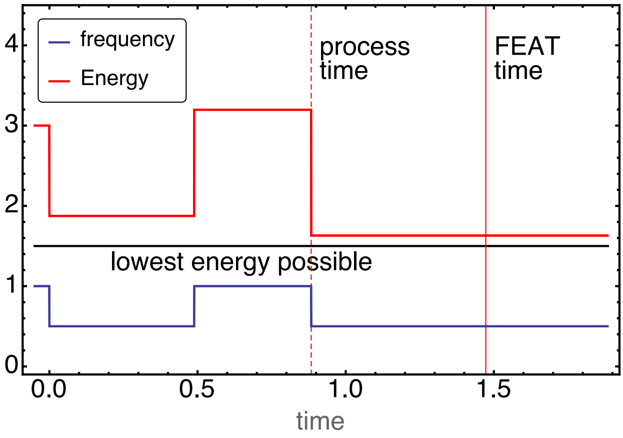

FIGURE 3. The optimized time dependence of the frequency and the resulting energy of the oscillator are shown as a function of time for a MAT. The dashed vertical line indicates the process time which is shorter than the FEAT time. The vertical red line indicates the FEAT time τ, which would allow one to bring the oscillator to its lowest possible energy. In a MAT this is no longer the case, and the final energy is above the lowest possible energy.

considerations to only such processes, which consist of a minimum frequency branch followed by a maximum frequency branch. We termed these processes maximum availability transitions (MATs) (K. H. Hoffmann, Schmidt, and Salamon 2015). In Figure 3 the optimized time dependence of the frequency $\omega_{\text{opt}}(t)$ is shown together with the resulting change in energy. Note that at the end of a MAT – contrary to a FEAT – the lowest possible energy of the oscillator is *not* attained, and thus the system has not reached equilibrium with the environment.

4.3. Quantum Finite-Time Availability. Once the optimized time dependence of the control $\omega_{\text{opt}}(t)$ is known, the resulting Quantum Finite-Time Availability can be calculated from (25). The particular availability will depend on certain features like the initial energy and the frequency ratio $\omega_{\max}/\omega_{\min}$ and of course on the available process time t_{P}. In Figure 4 we show the ratio of the QFT availability and the FEAT availability as a function of the process time. The difference between the two availabilities represents the foregone availability, which could not be extracted during the MAT. Note that such a difference occurs only for process times which are shorter than the FEAT time.

5. Comparing Classical and Quantum Finite-Time Availability

We now focus on the similarities and differences between the classical finite-time availability and the QFT Availability. Comparing Figures 1 and 4 the similarities are obvious: At shorter times both finite-time availabilities are smaller than the reversible

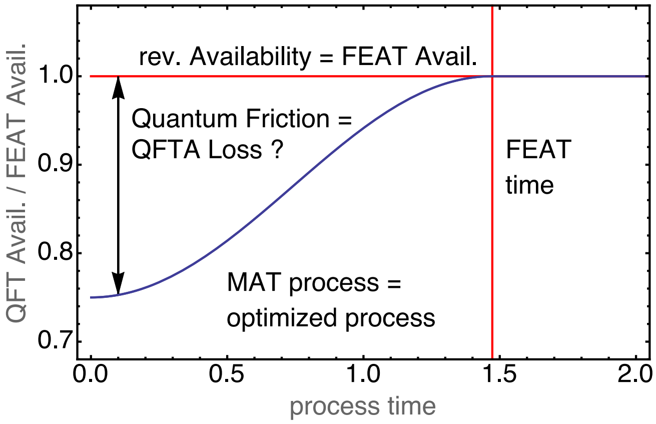

FIGURE 4. The QFT Availability in units of the FEAT availability is shown as a function of the process time. Note that contrary to the classical case, the QFT Availability is equal to the reversible availabilty (FEAT availability) for process times longer than the FEAT time. The foregone availability due to quantum friction is indicated by the double arrow. The question mark reflects its possible fate as unrecoverable loss.

availability. And in both cases the differences can be attributed to so-called friction, regular friction in the classical case and quantum friction in the quantum case.

With that, the pecularities of the quantum case start: The first difference is that even though the term quantum friction (Diósi, Feldmann, and Kosloff 2006; Feldmann and Kosloff 2003; Kosloff and Feldmann 2002; Rezek 2010) seems to indicate a dissipative process as in the classical case, the situation is different in the sense that the loss through quantum friction at this stage has not yet happened. This feature is exemplified below in more detail.

The second very clear difference is that in the quantum case there exists a characteristic time, the FEAT time, τ, such that for processes with a longer process time $t_{\mathrm{P}} > \tau$ the QFT Availability is exactly equal to the reversible availability.

Both features, the existence of a FEAT time, as well as the fact that quantum friction represents a not yet manifested loss can be traced back to the fact, that the process in the quantum case is thermodynamically fully adiabatic and reversible.

6. The Fate of the Residual Availability

In this section we turn to the question: What is the state of the system after the MAT is executed? Above we stressed already that the potential dissipation of the work missing due to quantum friction has not yet occurred. These potential losses become real losses however as soon the system is brought into contact with a heat bath. The ensuing process can be nicely followed by looking at the von Neumann and the energy entropies (Boldt *et al.* 2013; Rezek *et al.* 2009) of the system.

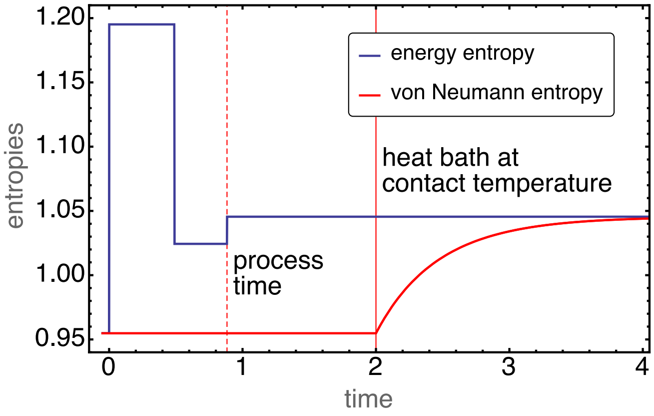

FIGURE 5. The von Neumann and the energy entropies are shown as functions of time for a MAT followed by a thermal contact with a heat bath at the contact temperature T_{con}. For times shorter than the process time (indicated by the red vertical dashed line) the difference of the two entropies reflects the non-equilibrium states the system attains. After contact with a heat bath at the contact temperature, the von Neumann entropy increases due to ongoing entropy production while the energy entropy remains constant.

These entropies are given by a function

$$S(X) = \ln\left(\sqrt{X - \frac{1}{4}}\right) + \sqrt{X}\,\operatorname{asinh}\left(\frac{\sqrt{X}}{X - \frac{1}{4}}\right), \tag{26}$$

where X takes different values for the von Neumann and the energy entropies. In particular we have for the von Neumann case $S_{\text{vN}} = S(X)$ with

$$X = \frac{E^2 - L^2 - C^2}{\hbar^2 \omega^2}, \tag{27}$$

where X is the *Casimir companion* (Boldt *et al.* 2013). The energy entropy is obtained from $S_{\text{E}} = S(X_{\text{E}})$ with

$$X_{\text{E}} = \frac{E^2}{\hbar^2 \omega^2}. \tag{28}$$

The time behavior of these entropies is shown in Figure 5. The MAT is shown in the left part of the figure, where its end is indicated by a dashed vertical red line. Since any process $\omega(t)$ leads to Hamiltonian dynamics as represented by (13), the von Neumann entropy of the system is perforce constant. Then since S is a monotonic function of X, the constancy of S_{vN} implies that X must also remain constant. We mention that this invariance can be exploited in many ways (Salamon, K. H. Hoffmann, and A. Tsirlin 2012; A. M. Tsirlin, Salamon, and K. H. Hoffmann 2011).

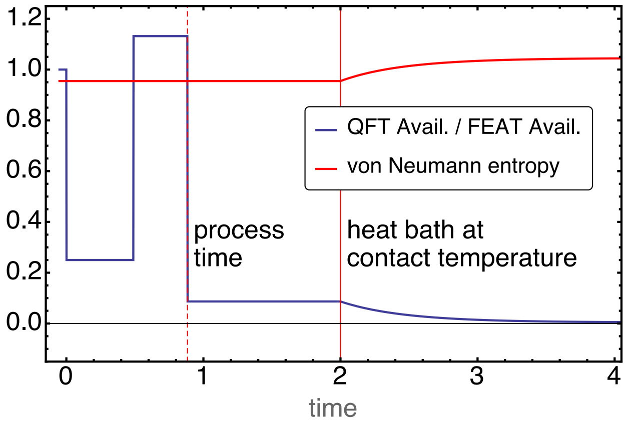

FIGURE 6. The QFT availability and the von Neumann entropy are shown as a function of time for a MAT followed by a thermal contact with a heat bath at the contact temperature T_{con}. For times shorter than the process time (indicated by the red vertical dashed line) the varying QFT availability reflects work exchange associated with the frequency changes during the MAT. After contact with a heat bath at the contact temperature, the QFT availability decreases due to the ongoing internal equilibration (decoherence) of the system. Although the asymptotic value appears to be zero its non-zero value cannot be distinguished from zero on the scale shown. The increase of the von Neumann entropy also reflects this equilibration.

On the right hand side of Figure 5 the further development of the entropies is shown. Here at time $t = 2$ the system is brought into contact with a heat bath at the so-called contact temperature T_{con} of the system (Muschik and Brunk 1977; Muschik 2014). This temperature is characterized by the fact that the final energy $E(t_{\text{P}})$ of the system after the MAT corresponds to the equilibrium energy at T_{con} with $\omega_{\min}$

$$E(t_{\text{P}}) = E_{\text{eq}}(T_{\text{con}}, \omega_{\min}). \tag{29}$$

As a consequence of (16), the system does not exchange any energy with the heat bath, i.e. the heat flux is zero. Nonetheless the heat bath contact allows the system to relax towards equilibrium in an irreversible process, as is indicated in Figure 5 by the increase of the von Neumann entropy without any influx of heat. Note that during this process the energy entropy is constant, contrary to its behavior during the MAT. Finally in Figure 6 the corresponding availability decrease is shown as a function of time. This decrease corresponds to the increase in the von Neumann entropy.

We note that the equilibrium approached by the system in the limit $t \to \infty$ while maintaining contact with the heat bath at the contact temperature is *not* the equilibrium with the environment at T_0 as $T_{\text{con}} \neq T_0$. Thus the system could be brought into equilibrium with the

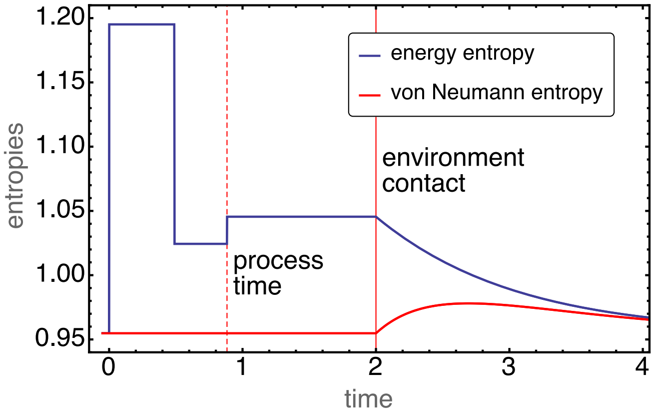

FIGURE 7. The von Neumann and the energy entropies are shown as a function of time for a MAT followed by a thermal contact with the environment at the temperature T_0. For times shorter than the process time (indicated by the red vertical dashed line) the difference of the two entropies reflects the non-equilibrium states the system attains. After contact with the environment the von Neumann entropy initially increases due to entropy production but soon declines due to the simultaneous outflux of heat. The availability remaining after the MAT (not shown) also decays and is lost to the environment as heat.

environment by a reversible process extracting heat, some of which could be converted to work.

The situation is different if, following a MAT, a thermal contact to the environment at T_0 is established. In that case there is in addition to the internal dissipative processes a heat transport from the system into the environment. These simultaneous processes of entropy production and heat flux can be seen for $t > 2$ in Figure 7. The von Neumann entropy starts to increase as soon as the environmental contact is established at $t = 2$, but soon begins to decline due to heat flowing out of the system. As both effects – entropy production and heat loss – lead to a decrease of the ability of the system to do work, the QFT Availability starts to decrease as soon as the thermal contact is established as shown in Figure 7.

7. Summary

In the above presentation we used two simple examples to compare and contrast finite-time availability in a classical and a quantum context. Our classical example used a traditional gas in a cylinder equipped with a piston. Our quantum example was the parametric quantum harmonic oscillator. Frictional losses are considered for both systems as a work parameter (the volume of the gas or the frequency of the oscillator) is controlled to move the system to a lower energy value. Proceeding at a finite rate costs availability for the gas at any finite piston speed. On the other hand for the oscillator there exists a

FEAT protocol which is frictionless and captures all of the availability for times larger than a certain FEAT time τ. For process times less than τ, quantum friction is unavoidable and some availability is necessarily foregone. The energy "lost" to quantum friction however is in-principle recoverable but rather fragile; it decays to zero and is really lost only after a thermal contact.

A promising route towards further insights is to generalize the above considerations for quantum finite-time availability to the finite-time analogue of work deficiency (K. H. Hoffmann, Andresen, and Salamon 1989, 2015). Work deficiency is a concept closely related to availability in that it counts the work loss independent from its deposition as heat to the environment.

We close our present discussion with a somewhat curious observation: QFT availability and quantum friction are in fact *not* restricted to quantum systems and should perhaps be renamed. We base our observation on a representation theorem (K. Hoffmann, Andresen, and Salamon 2013). which showed that the parametric quantum harmonic oscillator is isomorphic to an ensemble of independent classical harmonic oscillators sharing the same frequency. All of the behaviors of the quantum harmonic oscillator described above can be seen also in such an ensemble, where loss of coherence is represented by the decay of position-momentum correlations.

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