DOI: 10.1478/AAPP.972A5

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 97, No. 2, A5 (2019)

STABILITY CONDITIONS FOR THE MIXING FLOW DYNAMICAL SYSTEM IN A PERTURBED VERSION

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(communicated by Liliana Restuccia)

ABSTRACT. This paper continues some recent work on dynamical systems models from mixing flow area. The 2d mixing flow model is taken into account, in a slightly perturbed form. A Lyapunov function is found for the system, and it is found that the same Lyapunov function could exist for the feedback linearized system too, in similar conditions.

1. Introduction

The mixing theory is a quite new branch of the fluid mechanics, with far from complete solving problems. If the most of problems concentrated on the dynamics features of various flows and mixing flows, new interesting features could be exhibited from the study of the *kinematics of mixing* models. The methods and techniques for approaching these models developed the *significant relation between turbulence and chaos*, being a typical feature of *far from equilibrium systems* (the systems with few degrees of freedom, from a mathematical standpoint).

The statistical idea of a flow is generally represented by a map:

$$\boldsymbol{x} = \Phi_t(\boldsymbol{X}),\ \boldsymbol{X} = \Phi_{t=0}(\boldsymbol{X})\,. \tag{1}$$

We say that $\boldsymbol{X}$ is *mapped* in $\boldsymbol{x}$ after a time t. In continuum mechanics, the relation (1) is named *flow*, and it is a diffeomorphism of class C^k. Moreover, (1) must satisfy the relation

$$J = \det(D(\Phi_t(\boldsymbol{X}))) = \det\left(\frac{\partial x_i}{\partial X_j}\right), \tag{2}$$

where D denotes the derivation with respect to the reference configuration, in this case $\boldsymbol{X}$. The above relation implies two particles, X_1 and X_2, which occupy the same position x at a moment.

The mixing flow phenomenon implies a basic fluid (water) in which a biological material is moving (mixing) in different conditions and with different velocities. Therefore, special phenomena like *stretching* and *folding* appear for the biological material mixing in the host fluid. With respect to $\boldsymbol{X}$ there is defined the basic measure of deformation, the *deformation* *gradient* $\boldsymbol{F}$, namely (Ottino 1989):

$$\boldsymbol{F} = (\nabla_{\boldsymbol{X}} \Phi_t(\boldsymbol{X}))^T,\ F_{ij} = \left(\frac{\partial x_i}{\partial X_j}\right). \tag{3}$$

For a material filament and correspondingly for a material surface, there are defined the basic deformation measures, *i.e.*, the *length deformation* λ and *surface deformation* η, with the relations:

$$\lambda = (\boldsymbol{C} : \boldsymbol{MM})^{1/2},\ \eta = (\det \boldsymbol{F}) \cdot (\boldsymbol{C}^{-1} : \boldsymbol{NN})^{1/2}, \tag{4}$$

where $\boldsymbol{C}$ $(= \boldsymbol{F}^T \cdot \boldsymbol{F})$ is the *Cauchy-Green deformation tensor*, and the vectors $\boldsymbol{M}$ and $\boldsymbol{N}$ are the orientation versors in length and surface respectively, defined by

$$\boldsymbol{M} = \frac{d\boldsymbol{X}}{d|\boldsymbol{X}|},\ \boldsymbol{N} = \frac{d\boldsymbol{A}}{d|\boldsymbol{A}|}. \tag{5}$$

In this context a basic feature to approach is the so-called "good mixing concept": a flow has a *good mixing* (Ottino 1989) if $\frac{D(ln\lambda)}{Dt}$ and $\frac{D(ln\eta)}{Dt}$ are not decreasing to zero, for any initial position $\boldsymbol{P}$ and any initial orientations $\boldsymbol{M}$ and $\boldsymbol{N}$. The deformation tensor $\boldsymbol{F}$ and the associated tensors $\boldsymbol{C}$ and $\boldsymbol{C}^{-1}$ form the fundamental quantities for the analysis of deformation of infinitesimal elements. The class of flows with a special form of $\boldsymbol{F}$ is of very large interest, as it contains the so-called "constant stretch history motion" (CSHM flows).

The paper is organized as follows. Section 1 is devoted to the general mathematical framework of kinematic of mixing flow dynamical systems. In Section 2 we present recent results on mixing flow dynamical systems with an accent on feedback recursive linearization. Section 3 is devoted to stability. After a brief review of basic/classical stability methods, we present the results on searching for the global asymptotic stability (GAS) of the origin, both for the initial and for the inverse mixing flow model. Section 4 is finally devoted to concluding remarks.

2. Recent results on the mixing flow dynamical system

When studying the mixing flow phenomena, one starts from the widespread kinematic 2d mixing flow (Ottino 1989):

$$\left\{ \begin{array}{ccc} \dot{x}_1 & = & G \cdot x_2 \\ \dot{x}_2 & = & K \cdot G \cdot x_1 \end{array} \right\}, \tag{6}$$

where $-1 < K < 1$, $G \in \mathbb{R}$. To this broad isochoric flow, one can associate easily the corresponding 3d dynamical system (Ionescu 2010):

$$\left\{ \begin{array}{ccc} \dot{x}_1 & = & G \cdot x_2 \\ \dot{x}_2 & = & K \cdot G \cdot x_1 \\ \dot{x}_3 & = & c \end{array} \right\}, \tag{7}$$

with $-1 < K < 1$, $G \in \mathbb{R}$, where the third component represents the moving velocity of the system; c is a constant.

In the 3d case, the non-periodic models exhibit a complicate behavior. A lot of comparative computational analysis proved that the parameters have a great influence on the model behavior, leading to far from equilibrium models (Ionescu 2010). In the case of the perturbed model, the sensitivity of the model with respect to the parameters were outlined with better accuracy, in both the 2d and 3d cases (Ionescu 2010, 2017). The special behavior of this dynamical system under perturbations allowed one to recent take into account some approaches for linearizing the model. The feedback linearization method was approached, in order to enrich the qualitative information about the mixing flow model and its perturbation cases.

Feedback is a powerful idea that is extensively used in natural and technological systems. The principle of feedback is simple: *base correcting actions on the difference between desired and actual performance*. An interesting use of feedback is to change the dynamics of a system. Through feedback one can alter the behavior of a system to meet the needs of an application: systems that are unstable can be stabilized, systems that are sluggish can be made responsive, and systems that have drifting operating points can be held constant (Khalil 1992; Henson and Seborg 1996). The method is applicable to a broad class of non-linear control problems and in this context, control theory provides a rich collection of techniques to analyze the stability and dynamic response of complex systems and to place bounds on the behavior of such systems, by analyzing the gains of linear and nonlinear operators that describe their components. The feedback method is applied to general dynamical systems of the form:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x}) \cdot u, \tag{8}$$

where $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ and $u \in \mathbb{R}$. One is searching for a diffeomorphism $\boldsymbol{T} : \boldsymbol{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ which defines a coordinate transformation

$$\boldsymbol{z} = \boldsymbol{T}(\boldsymbol{x}) \tag{9}$$

for finding for the system (8) a *state-space realization*:

$$\dot{\boldsymbol{z}} = \boldsymbol{A} \cdot \boldsymbol{z} + \boldsymbol{B} \cdot v. \tag{10}$$

$\boldsymbol{A}$ and $\boldsymbol{B}$ are controllable form matrices and v is a new control related to u. The method was presented in detail by Isidori (1989) and by Ionescu (2017). If $\boldsymbol{f}$, $\boldsymbol{g}$ fulfill some special conditions, the transformation $\boldsymbol{T}$ turns the system into *the global non-linear controller form* (Su 1982; Liqun and Yanzhu 1998) given in the new coordinates $\boldsymbol{z}$, by

$$\left\{ \begin{array}{ccc} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & z_3 \\ & \vdots & \\ \dot{z}_{n-1} & = & z_n \\ \dot{z}_n & = h(z) + \mu(z) \cdot u & \end{array} \right\}. \tag{11}$$

Notice the special form of the equivalent model (11), where only the last component has more terms and in practice is a nonlinear expression (Isidori 1989). Here h is another function (resulting by calculus) and u is the control.

The feedback method worked good for the mixing flow dynamical system in a perturbed form. Namely, the following perturbed form of the model, analyzed also by Ionescu (2017), was taken into account:

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & G \cdot x_2 + x_1 \\ \dot{x}_2 & = & K \cdot G \cdot x_1 - x_2 \end{array} \right\}, \tag{12}$$

$-1 < K < 1$, $G \in \mathbb{R}$. With a suitable arrangement for the vector fields $\boldsymbol{f}$ and $\boldsymbol{g}$, it was found the diffeomorphism $\boldsymbol{T}$ and the feedback linearized model associated to (12) was found as:

$$\left\{ \begin{array}{lcl} \dot{z}_1 & = & z_2 \\ \dot{Z}_2 & = & -4K \cdot G^2 \cdot z_1 \cdot z_2 + 2G \cdot (Kz_1^2 - z_2^2) \cdot u \end{array} \right\}, \tag{13}$$

with $-1 < K < 1$. So in the slightly perturbed form (12), the mixing flow dynamical system admits a feedback linearized form. It must be noticed the special repartition of the parameters in the model (13), which actually changes the dynamics of the model and gives rise to further approaches.

3. Stability conditions for the mixing flow in a perturbed form

Since dynamical systems are a main tool for modeling in the applied sciences, Lyapunov functions appear in as various branches of science as meteorology, biology, computer science, and economics. The results have been taken as far as to prove the existence of a so-called complete Lyapunov function that characterizes the decomposition of the flow of a dynamical system into a chain-recurrent and a gradient-like part. Lyapunov functions characterize various forms of stability and, moreover, provide information about basins of attractions of local attractors, *i.e.*, give a characterization of the long-time behavior of solutions according to their initial conditions. Hence, a natural question is: How to compute a Lyapunov function for a particular system? Unfortunately, the converse theorems are non-constructive in nature, since they typically use the solution trajectories of the system to construct the Lyapunov function, and the solution trajectories are usually not known. Or, as Krasovskii wrote in 1959 (see Krasovskii 1963): *"One could hope that a method for proving the existence of a Lyapunov function might carry with it a constructive method for obtaining this function. This hope has not been realized"*. The general problem of constructing a Lyapunov function is a very hard problem. There have been numerous attempts and methods in the literature of how to compute Lyapunov functions for various kinds of systems. Some of them use a physical insight into the system to have a good intuition about a candidate for a Lyapunov function, and others use more systematic methods, including numerical algorithms. These methods have come from different communities in Engineering, Mathematics, and Informatics. The terminology is often not consistent between the various communities.

3.1. Classical methods.

Theorem 1. *Consider the dynamical system in* $\boldsymbol{R}^n$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) \tag{14}$$

and let $\boldsymbol{x} = 0$ *be its unique equilibrium point. If there exists a continuously differentiable function* $V : \mathbb{R}^n \mapsto \mathbb{R}$ *such that*

$$V(0) = 0 \tag{15}$$

$$V(\boldsymbol{x}) > 0 \quad \forall \boldsymbol{x} \neq 0 \tag{16}$$

$$\| \boldsymbol{x} \| \to \infty \Rightarrow V(\boldsymbol{x}) \to \infty \tag{17}$$

$$\dot{V}(\boldsymbol{x}) < 0 \quad \forall \boldsymbol{x} \neq 0 \tag{18}$$

Then $\boldsymbol{x} = 0$ *is global asymptotically stable.*

The condition (18) refers to the *monotonicity requirement* for the Lyapunov function. $\dot{V}(\boldsymbol{x})$ denotes the orbital derivative, given by

$$\dot{V}(\boldsymbol{x}) = \langle \frac{\partial V}{\partial \boldsymbol{x}}, f(\boldsymbol{x}) \rangle \tag{19}$$

where $\langle , \rangle$ is the inner product in $\boldsymbol{R}^n$ and $\frac{\partial V}{\partial \boldsymbol{x}}$ is the gradient of V. Also, the condition (17) refers to the requirement for V to be *radially unbounded*.

This theorem constitutes the *second Lyapunov criterion*. It must be emphasized the significance of Lyapunov's theorem: it allows stability of the system to be verified without explicitly solving the differential equation. The theorem turns the question of determining stability into a search for the so-called *Lyapunov function*, a positive definite function of the state that decreases monotonically along the trajectories. There are two immediate questions that arise. First, do we even know that Lyapunov functions always exist? Second, if they do exist, how would one go about finding one? The answer to the first question is in many situations positive, but different types on converse Krasovskii theorems assume the knowledge of the solutions of the system (14) (Krasovskii 1963) and therefore are useless in practice for finding the Lyapunov function V.

The *first Lyapunov criterion* is based on the eigenvalues analysis. Let us consider the following continuous-time nonlinear system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), u(t)) . \tag{20}$$

In the vicinity of the equilibrium point (x_0, u_0), let us consider the corresponding linearized system

$$\dot{\widetilde{\boldsymbol{x}}}(t) = A \cdot \widetilde{\boldsymbol{x}}(t) + B \cdot \widetilde{u}(t) . \tag{21}$$

The following statements hold (Giesl and Hafstein 2015):

(1) if all the eigenvalues of the matrix A have negative real part, then the equilibrium point (x_0, u_0) is asymptotically stable for the nonlinear system;
(2) if at least one of the eigenvalues of A has positive real part, then the equilibrium point (x_0, u_0) is unstable for the nonlinear system;
(3) if at least one eigenvalue of A is located on the imaginary axis and all other eigenvalues have negative real part, then it is not possible to conclude anything about the stability of the equilibrium point (x_0, u_0) for the nonlinear system (in this case the criterion is not effective).

3.2. Other methods. Besides the above basic stability statements, various methods to compute Lyapunov functions are based on computational techniques: optimal quadratic Lyapunov functions, neural network and genetic algorithms, simulation methods, and so on (Giesl and Hafstein 2015). Among them, the *polynomial methods* are particularly useful. Let us consider the continuous time dynamical system:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \tag{22}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a polynomial; let the system have an equilibrium in the origin, $\boldsymbol{f}(0) = 0$. When a polynomial function $V : \mathbb{R}^n \mapsto \mathbb{R}$ is used as a candidate Lyapunov function for stability analysis of system (22), the conditions of Lyapunov's theorem reduce to a set of polynomial inequalities. For instance, as presented above, if it is desired establishing global asymptotic stability of the origin, one would require a radially unbounded Lyapunov function candidate to satisfy the conditions (15) - (18) above, where $\dot{V}$ denotes the derivative of V along the trajectories of the system (22). The problem arising from this analysis approach is that even though polynomials of a given degree are finitely parameterized, the computational problem of searching for a polynomial V satisfying inequalities of the type (16) and (18) is intractable. In fact, even deciding if a given polynomial V of degree four or larger satisfies (18) is hard (Ahmadi and Parrilo 2017). An approach pioneered by Parrilo (2000) and quite popular by now is to replace the positivity or nonnegativity conditions by the requirement of existence of a *sum of squares (sos) decomposition*:

$$\left\{ \begin{array}{r} V\,\mathrm{sos} \\ -\dot{V} = -\langle \nabla V, f \rangle \mathrm{sos} \end{array} \right\}. \tag{23}$$

Clearly, if a polynomial is a sum of squares of other polynomials, then it must be nonnegative. Moreover, it is well known that an *sos* decomposition constraint on a polynomial can be cast as semidefinite programming (SDP) problem (Ahmadi and Parrilo 2017), which can be solved efficiently. Over the last decade, Lyapunov analysis with sum of squares techniques has become a relative well-established approach for a variety of problems in controls. Examples include stability analysis of switched and hybrid systems, design of nonlinear controllers, just to name a few. For the purpose of this paper it suffices to note that the existence of a polynomial Lyapunov function immediately implies existence of a Lyapunov function which is a sum of squares. The so-called converse theorems state this with details (Ahmadi and Parrilo 2017).We shall only recall the following:

Theorem 2. *Given a polynomial vector field, suppose there exists a polynomial Lyapunov function V such that V and $-\dot{V}$ are positive definite. Then, there also exists a polynomial Lyapunov function W such that W and $-\dot{W}$ are positive definite and W is sos.*

3.3. Stability analysis for the mixing flow dynamical system. It is established that for the mixing flow dynamical system, if there are added *similar terms* for perturbing the model, it becomes *far from equilibrium*. Many computational analyses confirmed this (Ionescu 2010). Therefore, after finding a linearized form of the mixing flow model, the problem of stabilizing it was approached. Does it exist a Lyapunov function for the model (12) and, if so, in what conditions? And does the respective function work also for the inverse model? We shall show in the following, that in similar conditions, it can be found a Lyapunov function working for both the initial mixing flow model and its inverse.

1) Let us re-number the system (12):

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & G \cdot x_2 + x_1 \\ \dot{x}_2 & = & K \cdot G \cdot x_1 - x_2 \end{array} \right\}, \tag{24}$$

$-1 < K < 1$. The origin is a solution for (24). We search for a Lyapunov function in a suitable form for investigating the stability of the solution. Taking into account some geometric features concerning the solutions of the basic mixing model (Ottino 1989), let us define the following Lyapunov function:

$$V : \mathbb{R}^2 \mapsto \mathbb{R}, V(\boldsymbol{x}) = x_1^2 + \frac{1}{|K|} \cdot x_2^2, \quad \boldsymbol{x} = (x_1, x_2). \tag{25}$$

It is immediate that V is positive definite and the conditions (15)-(17) are fulfilled. We have that for our differential model, the vectorial field $\boldsymbol{f}$ in (24) is

$$\boldsymbol{f} = \left(\begin{array}{c} G \cdot x_2 + x_1 \\ KG \cdot x_1 - x_2 \end{array} \right). \tag{26}$$

Calculating $\dot{V} = \langle \nabla V, \boldsymbol{f} \rangle$ we obtain:

$$\dot{V} = 2 \cdot (x_1^2 - \frac{1}{K} \cdot x_2^2) + 4G \cdot x_1 \cdot x_2 . \tag{27}$$

Arranging the terms in a suitable way we obtain further:

$$\dot{V} = -2 \cdot (\sqrt{K} \cdot G \cdot x_1 + \frac{1}{\sqrt{K}} \cdot x_2)^2 + 2 \cdot x_1^2 \cdot (1 + KG^2). \tag{28}$$

It is easy to observe that, in order to fulfill the condition (18) of a negative definite form in (28), it is necessary to impose the condition

$$1 + KG^2 < 0, \quad K < \frac{-1}{G^2}. \tag{29}$$

Hence, we have a Lyapunov function V of the form (25), in the conditions (29) which can be realized from the model standpoint.

2) Let us now consider the feedback linearized model associated to (24):

$$\left\{ \begin{array}{lcl} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & -4KG^2 \cdot z_1 \cdot z_2 + 2G \cdot (K \cdot z_1^2 - z_2^2) \cdot u \end{array} \right\} \quad -1 < K < 1, G \in \mathbb{R}. \tag{30}$$

We want to test if the same Lyapunov function (25) works for the inverse model (30). It is known that in practice it is desired generally a simple control; therefore, let us take for the feedback control u a simple form on one component, like:

$$u(\boldsymbol{x}) = m \cdot x_1, \quad m > 0. \tag{31}$$

We can re-note the system (30) for the simplicity of calculus. Thus, we have:

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -4KG^2 \cdot x_1 \cdot x_2 + 2G \cdot (K \cdot x_1^2 - x_2^2) \cdot m \cdot x_1 \quad -1 < K < 1, m > 0 \end{array} \right\}. \tag{32}$$

Now as the vector field $\boldsymbol{f}$ changed like

$$\boldsymbol{f} = \left(\begin{array}{c} x_2 \\ -4KG^2 \cdot x_1 \cdot x_2 + 2G \cdot (K \cdot x_1^2 - x_2^2) \cdot m \cdot x_1 \end{array} \right), \tag{33}$$

when calculating the deirvative $\dot{V} = \langle \nabla V, \boldsymbol{f} \rangle$ of the function (25), we obtain in this case:

$$\dot{V} = [1-4G]\cdot[1-x_2] + m\cdot[2G\cdot x_1^2 - \frac{1}{|K|}\cdot x_2^2]\,. \tag{34}$$

In order for the conditions (18) to be fulfilled, it is necessary to take into account *the same sign* for x_1 and x_2 and, consequenty, the following conditions appear:

$$1-4G<0, \quad 2G\cdot x_1^2 - \frac{1}{|K|}\cdot x_2^2 < 0\,. \tag{35}$$

This implies

$$G > \frac{1}{4}, \quad |x_1| < \frac{1}{\sqrt{2GK}}|x_2|\,. \tag{36}$$

The first condition is actually feasible as G is a real (free) parameter and the second one imposes that the streamlines have the geometrics of an ellipse interior, the axes ratio being:

$$\frac{|x_1|}{|x_2|} < \frac{1}{\sqrt{2G|K|}}\,. \tag{37}$$

This condition is feasible too because it is known (Ottino 1989) that the streamlines for the basic mixing flow model, in its initial form, form an ellipse with the axes ratio

$$\frac{|x_1|}{|x_2|} < \frac{1}{\sqrt{|K|}}\,. \tag{38}$$

Thus, we can assess that the origin is a global asymptotically stable equilibrium point for the mixing flow model, in the conditions (29) for the direct model, and respective (36) - (37) for the inverse model.

4. Conclusions

The basic conclusion of the above analysis is that, it exist a Lyapunov function for *both* the initial and feedback linearized dynamical systems, with some special conditions for the parameters. Taking into account the geometry of the streamlines of the mixing flow, it was found the Lyapunov function (25), which works in similar conditions for the initial and inverse perturbed mixing flow dynamical system. The control $u(\boldsymbol{x})$ helped in stabilizing the inverse model, and some other forms for it can be tested. Here it must be noticed that a control on each component of the system works very good in stabilizing the dynamical system (Liqun and Yanzhu 1998). It must be also noticed that the perturbed form of mixing flow model is a suitable form for constructing and analyzing a controlled stochastic model associated to it. The approach of stochastic optimal control using a control $u(t)$, together with a standard Brownian motion $B(t)$, was realized by Lefebvre and Ionescu (2016) whose model has a form similar to the form (13) of the feedback controlled model of the mixing flow.

The same form of the Lyapunov function for the initial and inverse system is an interesting fact of the analysis, which could give further useful information concerning the *global stability* of the model. Also it confirms the fact that one can study a model behavior *via its feedback linearized model* (Su 1982). This is a little feature on the way of checking the fact assessed by the theory, that the initial system and its feedback linearized are equivalent.

The parameter repartition in the dynamical system (24) makes it more difficult to choose a suitable form for the Lyapunov function. Although a Lyapunov function could be found, the specific set of conditions to be fulfilled shows that it is not immediate to check the stability of the mixing flow in the perturbed form. Hence, one should take into consideration a next step of the analysis, *i.e.*, searching for possible singularities, from the bifurcation standpoint.

Acknowledgments

This work was partially supported by the UE grant 777911-Dynamics-MSCA-RISE-2017.

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Communicated 6 December 2018; manuscript received 26 January 2019; published online 8 November 2019