DOI: 10.1478/AAPP.972A3

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 97, No. 2, A3 (2019)

CONSERVATION LAWS FOR A VISCOELASTIC MEDIUM

NATALE MANGANARO *

ABSTRACT. In this paper we consider a first order quasilinear non homogenoeus hyperbolic system describing viscoelastic processes. A full classification of first order local conservation laws admitted by such a rate-type model has been given. Classes of material response functions involved in the viscoelastic system under interest allowing the existence of the corresponding conservation laws have been obtained.

1. Introduction

Hyperbolic systems of partial differential equations are of great importance in describing different situations of interest in wave propagation phenomena as, for instance, nonlinear wave interactions (Whitham 1974; Jeffrey 1976; Boillat and Ruggeri 1979; Currò *et al.* 2013, 2015a, 2017; Currò and Manganaro 2019b), and Riemann problems (Lax 1957; Smoeller 1983; Dafermos 2010; Currò *et al.* 2011, 2012a; Currò and Manganaro 2013, 2016). Moreover exact solutions of hyperbolic equations usually can help to study qualitatively nonlinear wave propagation problems (Rozhdestvenskii and Janenko 1983; Currò *et al.* 2012b; Manganaro and Pavlov 2014; Currò *et al.* 2015b; Currò and Manganaro 2019a).

Conservation laws of PDEs are useful for investigating integrability and linearization mappings, for establishing existence and uniqueness of solutions of PDEs, for studying stability analysis and the global behavior of solutions, for developing numerical methods. In order to calculate the conservation laws of a given system of PDEs, probably the most famous approach is based on the Noether's theorem (Noether 1918) which, unfortunately, can be applied only to variational systems which admit variational symmetries. To overcome such limitations, starting from Noether'results, many different methods have been proposed for determining conservation laws (Anco and Bluman 1997; Olver 2000; Ibragimov 2007; Bluman *et al.* 2010). In particular the Direct Method proposed by Anco and Bluman (1997) (see also Bluman *et al.* 2010; Cheviakov 2010)) can be applied to any system of PDEs and, in fact, it has been used in many different contexts. For further convenience we sketch briefly the algorithm proposed in the Direct Method and we refer the interested reader to Anco and Bluman (1997), Bluman *et al.* (2010), and Cheviakov (2010).

Let us consider a system of PDEs

$$Q_i\left(x_\alpha, u^k, u^k_\alpha, u^k_{\alpha\beta}, ...,\right) = 0, \tag{1}$$

where $i, k = 1, .., n$; $\alpha, \beta = 1, .., m$; x_α denote the independent variables; $u^k(x_\alpha)$ the dependent variables while $u^k_\alpha = \frac{\partial u^k}{\partial x_\alpha}$, $u^k_{\alpha,\beta} = \frac{\partial^2 u_k}{\partial x_\alpha \partial x_\beta}$, etc. A local conservation law of (1) is a supplementary equation

$$D_\gamma \Omega_\gamma\left(x_\alpha, u_k, u^k_\alpha, u^k_{\alpha\beta}, ...\right) = 0 \tag{2}$$

which is identically satisfied for all solutions of (1). In (2) D_γ denote the total derivative with respect to x_γ ($\gamma = 1, ..., m$), the functions Ω_γ are called fluxes of the conservation law and here and in the following for the repeated indices the Einstein convention is adopted. Moreover we denote with $A_i\left(x_\alpha, u_k, u^k_\alpha, u^k_{\alpha\beta}, ...\right)$ a non singular set of multipliers whose funtional dependence must be assumed and we set $\Gamma = A_i Q_i$. Next, since the action of the Euler operator

$$E_{u^k}(\cdot) = \frac{\partial}{\partial u^k}(\cdot) - D_\alpha\left(\frac{\partial}{\partial u^k_\alpha}(\cdot)\right) + ... + (-1)^j D_{\alpha_1} ... D_{\alpha_j}\left(\frac{\partial}{\partial x^k_{\alpha_1 ... \alpha_j}}(\cdot)\right) + ...$$

on a scalar function is identically zero if such a function can be written in a divergence form, it follows that the equations

$$E_{u^k}\left(H\left(x_\alpha, u_k, u^k_\alpha, u^k_{\alpha\beta}, ...\right)\right) = 0$$

hold for arbitrary u_k if and only if $H = D_i \Psi_i$ for some functions $\Psi_i\left(x_\alpha, u_k, u^k_\alpha, u^k_{\alpha\beta}, ...\right)$. Therefore by requiring that

$$E_{u^k}(\Gamma) = 0, \qquad k = 1, ..., n \tag{3}$$

hold for arbitrary $u_k = U_k(x_\alpha)$, it follows

$$\Gamma = A_i Q_i = D_\alpha \Omega_\alpha. \tag{4}$$

for some fluxes $\Omega_\gamma(x_\alpha, u^k, ...)$. The equations (3) give an overdetermined set of equations for the multipliers A_i. Once such a multipliers are determined, then the fluxes Ω_γ of the conservation law (2) can be calculated from (4). Finally we notice that if the system (1) is of a Cauchy–Kovalevskaya type (*i.e.*, it can be solved with respect to highest derivatives of all dependent variables with respect some independent variables) then the Direct Method permits, in principle, to calculate all the local conservation laws of the system under interest.

It should be of a certain interest to notice that conservation laws are strictly related to the subject of the so called null Lagrangian. In fact it is well known that $L\left(x_\alpha, u_k, u^k_\alpha, u^k_{\alpha\beta}, ...\right)$ is a null Lagrangian if the corresponding Euler Lagrange equations vanish identically (Olver and Sivaloganathan 1988; Giaquinta and Hildebrandt 1996; Crampin and Saunders 2005). In such a case it can be proved (see Olver 1983, and references quoted therein) that L is a null Lagrangian if and only if there exist some fluxes $\Omega_\gamma\left(x_\alpha, u_k, u^k_\alpha, u^k_{\alpha\beta}, ...\right)$ such that

$$L = D_\gamma \Omega_\gamma\left(x_\alpha, u_k, u^k_\alpha, u^k_{\alpha\beta}, ...\right).$$

For instance in Continuum Mechanics if $L(F)$ is a null Lagrangian, where F is the deformation gradient, taking into account that $\partial_t F = \nabla \mathbf{v}$ with $\mathbf{v}$ denoting the eulerian velocity, it can be proved (Dafermos 2010) that the conservation law

$$\frac{\partial L}{\partial t} = \frac{\partial}{\partial x_i}\left(v_i \frac{\partial L}{\partial F}\right)$$

holds for any smooth or Lipschitz continuous solutions of the field equation. Within such a theoretical framework, here we consider the following one–dimensional quasilinear first order hyperbolic system describing a viscoelastic and/or viscoplastic continuum medium where memory effects are present (for an exhaustive review on this subject see Cristescu 2007)

$$\frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} = 0 \tag{5}$$

$$\frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial x} = 0 \tag{6}$$

$$\frac{\partial \sigma}{\partial t} - \Phi(\varepsilon, \sigma)\frac{\partial v}{\partial x} = \Psi(\varepsilon, \sigma) \tag{7}$$

In (5)–(7), v indicates the lagrangian velocity, σ the stress, ε the strain while t and x denote, respectively, time and lagrangian space coordinates. Moreover the material response functions $\Phi(\varepsilon, \sigma)$ and $\Psi(\varepsilon, \sigma)$ measure, respectively, the instantaneous and the non instantaneous response of the medium. The model (5), (6) and (7) has been widely studied and many results have been obtained concerning energy estimates and phase transformation phenomena (Suliciu 1989; Faciu 1991a,b; Suliciu 1992; Faciu and Suliciu 1994; Faciu 1996a,b; Tang *et al.* 2006), moving boundary problems (Frydrychowicz and Singh 1985; Fazio 1992), traveling waves and similarity solutions (Suliciu *et al.* 1973), reduction procedures (Fusco and Manganaro 2008; Currò and Manganaro 2017), nonlinear wave propagation problems (Manganaro 2017; Currò and Manganaro 2018), and numerical experiments (Cristescu 1972; Schuler and Nunziato 1974).

The rate-type equation (7) generalizes different models proposed in literature. For instance, if

$$\Phi = E, \quad \Psi = -\frac{1}{\tau}\sigma \tag{8}$$

it reduces to the Maxwell's equation where E denotes the Young modulus and τ is a relaxation time, while if

$$\Phi = E, \quad \Psi = -\frac{1}{\tau}\left(\sigma - \sigma_e(\varepsilon)\right) \tag{9}$$

it specializes to the Malvern's model (Malvern 1951a,b), where $\sigma = \sigma_e(\varepsilon)$ denotes the equilibrium stress-strain curve characterized by

$$\Psi(\varepsilon, \sigma_e(\varepsilon)) = 0.$$

Furthermore we consider the celebrated integro-differential equation proposed by Colemann and Noll (1961) for describing finite linear viscoelastic materials

$$\Sigma = \Sigma_e(E) + \int_0^\infty \Omega(E, s)\left\{E^t(s) - E\right\} ds \tag{10}$$

where Σ is the second Piola-Kirchhoff stress tensor, Σ_e an equilibrium response function tensor, Ω a fourth order tensor, while

$$E^t(s) = E(X, t-s), \qquad 0 \leq s < \infty$$

denotes the history of the Green's strain E. Herrmann and Nunziato (1972) showed that for fading memory materials (*i.e.*, materials which depend strongly on recent history and for which the influence of past events diminishes in time) the equation (10) is equivalent to (7) with

$$\Phi = \sigma_i'(\varepsilon) + \alpha(\varepsilon)(\sigma - \sigma_i(\varepsilon)) \tag{11}$$

$$\Psi = -\frac{1}{\tau}(\sigma - \sigma_e(\varepsilon)) \tag{12}$$

where $\alpha(\varepsilon)$ is a constitutive function while $\sigma = \sigma_i(\varepsilon)$ characterizes the instantaneous stress-strain curve defined by

$$\frac{d\sigma_i}{d\varepsilon} = \Phi(\sigma_i(\varepsilon), \varepsilon).$$

Therefore, following the approach proposed in the Direct Method (Anco and Bluman 1997), the main aim of this paper is to calculate the local conservation laws admitted by the hyperbolic system (5)–(7). In particular in section 2 the overdetermined set of equations determining the multipliers involved in the procedure at hand is deduced. In section 3 a general analysis of such a system has been carried on and the full classification of the local conservation laws admitted by the model (5)-(7) has been obtained. Finally in section 4 a particular case concerning a special local conservation law of interest in continuum mechanics has been characterized while in section 5 some conclusions and final remarks are given.

2. Local conservation laws

In this section we look for first order local conservation laws admitted by the the rate-type system (5)-(7) under the form

$$D_t F(x,t,v,\varepsilon,\sigma) + D_x G(x,t,v,\varepsilon,\sigma) = 0. \tag{13}$$

To this end, following the direct method approach sketched in the introduction of the present paper, we introduce the multipliers $A(x,t,v,\varepsilon,\sigma)$, $B(x,t,v,\varepsilon,\sigma)$, $C(x,t,v,\varepsilon,\sigma)$ and we set

$$\Gamma = A\left(\frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x}\right) + B\left(\frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial x}\right) + C\left(\frac{\partial \sigma}{\partial t} - \Phi(\varepsilon,\sigma)\frac{\partial v}{\partial x} - \Psi(\varepsilon,\sigma)\right). \tag{14}$$

The next step is to apply the Euler operator on Γ and to require that

$$E_v(\Gamma) = 0, \quad E_\varepsilon(\Gamma) = 0 \quad E_\sigma(\Gamma) = 0 \tag{15}$$

where

$$E_v(\cdot) = \frac{\partial}{\partial v}(\cdot) - D_t\left(\frac{\partial}{\partial v_t}(\cdot)\right) - D_x\left(\frac{\partial}{\partial v_x}(\cdot)\right)$$
$$E_\varepsilon(\cdot) = \frac{\partial}{\partial \varepsilon}(\cdot) - D_t\left(\frac{\partial}{\partial \varepsilon_t}(\cdot)\right) - D_x\left(\frac{\partial}{\partial \varepsilon_x}(\cdot)\right)$$
$$E_\sigma(\cdot) = \frac{\partial}{\partial \sigma}(\cdot) - D_t\left(\frac{\partial}{\partial \sigma_t}(\cdot)\right) - D_x\left(\frac{\partial}{\partial \sigma_x}(\cdot)\right).$$

The requirement that relations (15) are satisfied for all v_t, v_x, ε_t, ε_x, σ_t, σ_x leads to the following conditions determining the multipliers A, B and C

$$A_\varepsilon = 0, \quad B_v = 0, \quad A_\sigma = C_v \tag{16}$$
$$A_v = \frac{\partial}{\partial \sigma}(B + C\Phi), \quad \frac{\partial}{\partial \varepsilon}(B + C\Phi) = 0 \tag{17}$$
$$B_\sigma = C_\varepsilon, \quad A_t - B_x - \Phi C_x + \Psi C_v = 0 \tag{18}$$
$$B_t + \frac{\partial}{\partial \varepsilon}(C\Psi) = 0, \quad C_t - A_x + \frac{\partial}{\partial \sigma}(C\Psi) = 0 \tag{19}$$

where here and in the following the subscripts means for partial derivative with respect to the indicated argument. Finally, once $A(x,t,v,\varepsilon,\sigma)$, $B(x,t,v,\varepsilon,\sigma)$ and $C(x,t,v,\varepsilon,\sigma)$ are calculated according to (16)–(19), then from

$$D_t F + D_x G = \Gamma$$

the conservation laws (13) admitted by the viscoelastic model (5)–(7) are determined by solving the equations

$$\frac{\partial F}{\partial v} = A, \quad \frac{\partial F}{\partial \varepsilon} = B, \quad \frac{\partial F}{\partial \sigma} = C \tag{20}$$
$$\frac{\partial G}{\partial v} = -(B + \Phi C), \quad \frac{\partial G}{\partial \varepsilon} = 0, \quad \frac{\partial G}{\partial \sigma} = -A \tag{21}$$
$$\frac{\partial F}{\partial t} + \frac{\partial G}{\partial x} = -C\Psi. \tag{22}$$

which, of course, result to be compatible because of (16)-(19).

3. Classification

Here our aim is to classify all the possible local conservation laws admitted by (5)-(7) so that a general analysis of the overdetermind system (16)–(19) will be given. From equations (16) and (17), after some algebra, we get

$$A = \hat{m}(x,t)v + a(x,t) \tag{23}$$
$$B = \hat{m}(x,t)\sigma + \tilde{m}(x,t) - \nu(x,t,\varepsilon,\sigma) \tag{24}$$
$$C = \varphi(\varepsilon,\sigma)\nu(x,t,\varepsilon,\sigma) \tag{25}$$

where $\hat{m}(x,t)$, $a(x,t)$ and $\tilde{m}(x,t)$ are unspecified functions, while, for further convenience, we set

$$\varphi(\varepsilon,\sigma) = \frac{1}{\Phi(\varepsilon,\sigma)}. \tag{26}$$

A further insertion of (23)-(25) into (18) and (19) gives

$$\nu_\sigma = m_0 - \frac{\partial}{\partial\varepsilon}(\varphi\nu) \tag{27}$$

$$\nu_t = \tilde{m}_t + \frac{\partial}{\partial\varepsilon}(\mu\nu) \tag{28}$$

$$(\varphi\mu_\varepsilon - \mu\varphi_\varepsilon + \mu_\sigma)\,\nu = a_x - \tilde{m}_t\varphi - m_0\mu \tag{29}$$

along with

$$\hat{m} = m_0, \qquad a_t = \tilde{m}_x. \tag{30}$$

In (27)-(30) m_0 is an arbitrary constant, while we set

$$\mu(\varepsilon,\sigma) = \frac{\Psi(\varepsilon,\sigma)}{\Phi(\varepsilon,\sigma)}. \tag{31}$$

Therefore, owing to (23)-(25), by solving equations (27)-(29), the set of multipliers A, B, C obeying (16)-(19) are determined. Moreover a direct inspection shows that the equations (27) and (28) are compatible iff the relation (29) is satisfied (*i.e.*, all solutions of equation (29) assure the differential compatibility between (27) and (28)) so that, from (29), two possible cases arise depending if $\varphi\mu_\varepsilon - \mu\varphi_\varepsilon + \mu_\sigma = 0$ or $\varphi\mu_\varepsilon - \mu\varphi_\varepsilon + \mu_\sigma \neq 0$.

3.1. First case. We require

$$\varphi\mu_\varepsilon - \mu\varphi_\varepsilon + \mu_\sigma = 0 \tag{32}$$

so that (29) is satisfied if

$$a_x - \tilde{m}_t\varphi - m_0\mu = 0. \tag{33}$$

From relation (33) the following cases arise

$$I) \quad a = a_0x + c_0t, \quad \tilde{m} = k_0t + c_0x, \tag{34}$$

$$\varphi = -\frac{m_0}{k_0}\mu + \frac{a_0}{k_0} \quad \text{with} \quad k_0 \neq 0 \tag{35}$$

$$II) \quad a = c_0t, \quad \tilde{m} = c_0x, \quad m_0 = 0 \tag{36}$$

$$III) \quad \varphi = \varphi_0, \quad a_x = \varphi_0\tilde{m}_t, \quad m_0 = 0 \tag{37}$$

$$IV) \quad a = a_0x + c_0t, \quad \tilde{m} = c_0x, \quad \mu = \frac{a_0}{m_0} \quad \text{with} \quad m_0 \neq 0. \tag{38}$$

In (34)–(38) a_0, c_0, k_0, φ_0 are constants. Moreover in order to guarantee the hyperbolicity of the system (5)-(7) we require $\varphi_0 > 0$.

Next, integration of (32) and, in turn, of (27) and (28) gives in the *I)–IV)* cases above the following results.

$$I) \quad \mu = \mu(z), \quad z = \varepsilon - \frac{a_0}{k_0}\sigma \tag{39}$$

$$v = \frac{1}{\mu(z)}\left(-k_0 z + h(x,\Omega)\right) \tag{40}$$

where

$$\Omega = t + \int \frac{dz}{\mu(z)} + \frac{m_0}{k_0}\sigma \tag{41}$$

while $h(x,\Omega)$ denotes an arbitrary function.

$$II) \quad \frac{1}{\mu} = \eta_\varepsilon, \quad \frac{\varphi}{\mu} = -\eta_\sigma \tag{42}$$

$$v = \frac{v_0(x,\xi)}{\mu}, \quad \xi = t + \eta(\varepsilon,\sigma) \tag{43}$$

where the function $\eta(\varepsilon,\sigma)$ is determined by solving the equation

$$\eta_\sigma + \varphi\eta_\varepsilon = 0. \tag{44}$$

$$III) \quad \mu = \mu(y), \quad y = \varepsilon - \varphi_0\sigma \tag{45}$$

$$v = \frac{1}{\mu(y)}\left(-\int \tilde{m}_t\left(x, t(\tau,y)\right)dy + q(x,\tau)\right) \tag{46}$$

with

$$\tau = t + \int \frac{dy}{\mu(y)} \tag{47}$$

$$IV) \quad \varphi = \varphi(\sigma), \quad v = m_0\sigma + v_1(x,\hat{z}) \tag{48}$$

where

$$\hat{z} = \varepsilon - \int \varphi(\sigma)d\sigma + \frac{a_0}{m_0}t. \tag{49}$$

Moreover in the *III)* case, from (30) and (37) we get

$$\tilde{m} = R(X) + S(T), \quad a = \sqrt{\varphi_0}\left(S(T) - R(X)\right) \tag{50}$$

where

$$X = x - \frac{1}{\sqrt{\varphi_0}}t, \quad T = x + \frac{1}{\sqrt{\varphi_0}}t. \tag{51}$$

Finally, taking (23)-(25) into account, integration of (20)-(22) provides classes of conservation laws admitted by (5)-(7). In fact in the *I)* case we have

$$\begin{aligned} F &= \frac{m_0}{2}v^2 - \frac{a_0 m_0}{2k_0}\sigma^2 + (a_0x + c_0t)v + (m_0\sigma + k_0t + c_0x)\varepsilon + \\ &\quad + k_0\int \frac{dz}{\mu(z)} - \int \frac{h\left(x,\Omega(t,z,\sigma)\right)}{\mu(z)}dz \end{aligned} \tag{52}$$

$$G = -(m_0\sigma + k_0t + c_0x)v - (a_0x + c_0t)\sigma \tag{53}$$

while in the *II)* case we find

$$F = c_0\,(tv + x\varepsilon) - \int \nu_0(x,\xi)d\xi \tag{54}$$

$$G = -c_0\,(xv + t\sigma)\,. \tag{55}$$

As far as the *III)* case is concerned, in order to integrate explicitely (20)–(22), in (50) we choose

$$R = \frac{\varphi_0}{4}X^2, \qquad S = \frac{\varphi_0}{4}T^2 \tag{56}$$

so that $\tilde{m}(x,t)$ and $a(x,t)$ assume, respectively, the form

$$\tilde{m} = \frac{1}{2}\left(t^2 + \varphi_0 x^2\right), \qquad a = \varphi_0 xt. \tag{57}$$

Therefore from (20)–(22) we obtain

$$F = a(x,t)v + \tilde{m}(x,t)\varepsilon + t\left(zM' - M\right) - \int M''\left(M - zM'\right)dz \tag{58}$$

$$G = -\tilde{m}(x,t)v - a(x,t)\sigma \tag{59}$$

where $a(x,t)$ and $\tilde{m}(x,t)$ are given by (57) while we set

$$\frac{1}{\mu(z)} = M''(z)$$

and prime means for ordinary differentiation.

Finally, in the *IV)* case we deduce

$$F = \frac{m_0}{2}v^2 + a(x,t)v + \tilde{m}(x)\varepsilon - \int \nu_0(x,\hat{z})d\hat{z} + m_0\int \sigma\varphi(\sigma)d\sigma \tag{60}$$

$$G = -\tilde{m}(x)v - m_0\sigma v - a(x,t)\sigma \tag{61}$$

where $a(x,t)$ and $\tilde{m}(x)$ are given, respectively, by $(38)_1$ and $(38)_2$.

3.2. Second case. Here we assume $\varphi\mu_\varepsilon - \mu\varphi_\varepsilon + \mu_\sigma \neq 0$ so that from (29) we get

$$\nu = \frac{1}{p(\varepsilon,\sigma)}\left(a_x - \tilde{m}_t\varphi - m_0\mu\right) \tag{62}$$

where for simplicity we set

$$p = \varphi\mu_\varepsilon - \mu\varphi_\varepsilon + \mu_\sigma. \tag{63}$$

By substituting (62) into (27) and (28) we obtain

$$\left(\frac{\partial}{\partial\sigma}\left(\frac{1}{p}\right)+\frac{\partial}{\partial\varepsilon}\left(\frac{\varphi}{p}\right)\right)a_x-\left(\frac{\partial}{\partial\sigma}\left(\frac{\varphi}{p}\right)+\frac{\partial}{\partial\varepsilon}\left(\frac{\varphi^2}{p}\right)\right)\tilde{m}_t+$$
$$-m_0\mu\left(\frac{\partial}{\partial\sigma}\left(\frac{1}{p}\right)+\frac{\partial}{\partial\varepsilon}\left(\frac{\varphi}{p}\right)+\frac{\varphi_\varepsilon}{p}\right)=2m_0 \tag{64}$$

$$a_{xt}-\varphi\tilde{m}_{tt}=\left(1-\frac{\partial}{\partial\varepsilon}\left(\frac{\mu\varphi}{p}\right)\right)p\tilde{m}_t+a_x\frac{\partial}{\partial\varepsilon}\left(\frac{\mu}{p}\right)p+$$
$$-m_0\frac{\partial}{\partial\varepsilon}\left(\frac{\mu^2}{p}\right)p. \tag{65}$$

After cumbersome calculations, from (64) and (65) the following two cases arise

$$i)\qquad a=k_1t+c_1x,\quad \tilde{m}=k_1x+m_1t \tag{66}$$

along with

$$\frac{\partial}{\partial\sigma}\left(\frac{1}{p}\right)+\frac{\partial}{\partial\varepsilon}\left(\frac{\varphi}{p}\right)=0 \tag{67}$$

$$\frac{m_1}{p}\left(\varphi_\sigma+\varphi\varphi_\varepsilon\right)+m_0\left(2+\frac{\mu\varphi_\varepsilon}{p}\right)=0 \tag{68}$$

$$m_1\left(1-\frac{\partial}{\partial\varepsilon}\left(\frac{\mu\varphi}{p}\right)\right)+c_1\frac{\partial}{\partial\varepsilon}\left(\frac{\mu}{p}\right)-m_0\frac{\partial}{\partial\varepsilon}\left(\frac{\mu^2}{p}\right)=0 \tag{69}$$

where k_1, c_1, m_1 are constants.

$$ii)\quad \tilde{m}_t=\hat{k}_0\tilde{m}+\hat{k}_1t+k_2(x) \tag{70}$$
$$a_x=\hat{c}_0\tilde{m}_t+\hat{c}_1 \tag{71}$$

along with

$$\frac{\partial}{\partial\sigma}\left(\frac{\varphi-\hat{c}_0}{p}\right)+\frac{\partial}{\partial\varepsilon}\left(\frac{\varphi\left(\varphi-\hat{c}_0\right)}{p}\right)=0 \tag{72}$$

$$\left(\hat{c}_1-m_0\right)\left\{\frac{\partial}{\partial\sigma}\left(\frac{1}{p}\right)+\frac{\partial}{\partial\varepsilon}\left(\frac{\varphi}{p}\right)\right\}=m_0\left\{2+\frac{\mu\varphi_\varepsilon}{p}\right\} \tag{73}$$

$$\hat{k}_0\left(\hat{c}_0-\varphi\right)=p\left\{1-\frac{\partial}{\partial\varepsilon}\left(\frac{\mu\left(\varphi-\hat{c}_0\right)}{p}\right)\right\} \tag{74}$$

$$\hat{k}_1\left(\hat{c}_0-\varphi\right)=p\left\{\hat{c}_1\frac{\partial}{\partial\varepsilon}\left(\frac{\mu}{p}\right)+m_0\frac{\partial}{\partial\varepsilon}\left(\frac{\mu^2}{p}\right)\right\} \tag{75}$$

where $\hat{c}_0$, $\hat{c}_1$, $\hat{k}_0$ and $\hat{k}_1$ are constants. Furthermore from (70) and (71) if $\hat{k}_0\neq 0$ we have

$$\tilde{m}=-\frac{k_2(x)}{\hat{k}_0}+\tilde{m}_0(x)e^{\hat{k}_0t} \tag{76}$$

$$a=\frac{1}{\hat{k}_0}\left(-\hat{k}_2t+\tilde{m}_0'(x)e^{\hat{k_0}t}\right)+\hat{c}_1x \tag{77}$$

or, if $\hat{k}_0 = 0$, we get

$$\tilde{m} = k_2(x)t + \tilde{m}_1 x \tag{78}$$

$$a = \frac{\hat{k}_2}{2}\left(t^2 + \hat{c}_0 x^2\right) + \tilde{m}_1 t + \left(\hat{c}_0 \bar{k}_2 + \hat{c}_1\right) x \tag{79}$$

In (76)-(79) the functions $k_2(x)$ and $\tilde{m}_0(x)$ are given by

$$k_2(x) = \bar{k}_2 + \hat{k}_2 x, \qquad \tilde{m}_0'' - \hat{c}_0 \hat{k}_0^2 \tilde{m}_0 = 0 \tag{80}$$

while $\bar{k}_2$, $\hat{k}_2$ and $\tilde{m}_1$ are constants. Therefore, once $\varphi(\varepsilon,\sigma)$ and $\mu(\varepsilon,\sigma)$ are determined according to (67)-(69) or (72)-(75), then, taking (62) into account along with (66) or (76) and (77) or (78) and (79), the set of multipliers A, B and C corresponding to the present case can be determined.

In order to verify that in the above two cases the overdermined systems (67)-(69) or (72)-(75) admit both some solutions, next we consider two particular cases. In the *i)* case if we assume $m_0 = m_1 = 0$, then from (67)-(69) we have

$$\mu_\sigma + \varphi \mu_\varepsilon - \mu \varphi_\varepsilon = \frac{\mu}{\sigma} \tag{81}$$

and, in turn, from (20)–(22) we obtain

$$F = (k_1 t + c_1 x) v + k_1 x \varepsilon - c_1 \int \frac{\sigma}{\mu(\varepsilon,\sigma) d\varepsilon} \tag{82}$$

$$G = k_1 x v + (k_1 t + c_1 x)\sigma. \tag{83}$$

In the *ii)* case we require $\varphi(\sigma)$, $\mu(\sigma)$ and $\hat{k}_1 = 0$. Therefore integration of (72)-(75) leads to

$$\varphi = \hat{c}_0 - \frac{\mu'(\sigma)}{\hat{k}_0} \tag{84}$$

and

$$\mu = \frac{\hat{c}_1}{m_0} - \frac{1}{m_0^2(\mu_0 + \mu_1 \sigma)} \quad \text{if} \quad m_0 \neq 0 \tag{85}$$

or

$$\mu = \mu_0 + \mu_1 \sigma \qquad \text{if} \qquad m_0 = 0 \tag{86}$$

where μ_0 and μ_1 are constants. Finally when $m_0 \neq 0$ from (20)-(22) we obtain

$$F = \frac{m_0}{2} v^2 + a(x,t) v - \left(\frac{k_2(x)}{\hat{k}_0} + \frac{m_0 \mu_0}{\mu_1}\right)\varepsilon + \int \varphi v \, d\sigma + \hat{f}(x,t) \tag{87}$$

$$G = -\left(m_0 \sigma + \tilde{m}(x,t)\right) v - a(x,t) v + \hat{g}(x,t) \tag{88}$$

where $a(x,t)$, $k_2(x)$ and $\tilde{m}(x,t)$ are given by (76), (77) and (80). Moreover v assumes the form

$$v = \tilde{m}(x,t) + \frac{k_2(x)}{\hat{k}_0} + \frac{m_0 \mu_0}{\mu_1} + m_0 \sigma \tag{89}$$

while the functions $\hat{f}(x,t)$ and $\hat{g}(x,t)$ must satisfy the relation

$$\hat{f}_t + \hat{g}_x = \frac{1}{m_0\mu_1}\left(1 - m_0\hat{c}_1\mu_0\right). \tag{90}$$

In passing we notice that the material laws (86) and, in turn, (84) lead to a linear stress-strain law (7), so that we do not consider such a situation.

4. Total energy conservation

In the previous section a general analysis of the system (16)-(19) has been carried on and classes of local conservation laws of (5)-(7) corresponding to suitable functional forms of the material response functions $\Phi(\varepsilon,\sigma)$ and $\Psi(\varepsilon,\sigma)$ there involved have been obtained. Here we consider a particular case which could be of a certain interest in modelling viscoelastic processes.

We assume $A(v,\varepsilon,\sigma)$, $B(v,\varepsilon,\sigma)$ and $C(v,\varepsilon,\sigma)$ so that in (23)-(25), taking (30) into account, we require

$$a = 0, \quad \tilde{m} = 0, \quad \nu_x = \nu_t = 0. \tag{91}$$

Therefore integration of (27)-(29) gives

$$\nu = \frac{\alpha_0}{\mu}$$

along with

$$\alpha_0\left\{\frac{\partial}{\partial\varepsilon}\left(\frac{1}{\Psi}\right) + \frac{\partial}{\partial\sigma}\left(\frac{\Phi}{\Psi}\right)\right\} = m_0. \tag{92}$$

where α_0 is a constant and we used (26) and (31). Finally by solving (20)-(22) we obtain:

$$F = \frac{m_0}{2}v^2 + m_0\sigma\varepsilon - \alpha_0\int\frac{\Phi}{\Psi}d\varepsilon + \hat{f}_1(x,t) \tag{93}$$

$$G = -m_0\sigma v + \hat{g}_1(x,t) \tag{94}$$

with

$$\frac{\partial\hat{f}_1}{\partial t} + \frac{\partial\hat{g}_1}{\partial x} = -\alpha_0. \tag{95}$$

It should be of a certain interest to notice that from (92), by assuming without loss of generality $m_0 = 1$, we can introduce the function $e(\varepsilon,\sigma)$ such that

$$\frac{\partial e}{\partial\varepsilon} = \sigma - \alpha_0\frac{\Phi}{\Psi}, \qquad \frac{\partial e}{\partial\sigma} = \frac{\alpha_0}{\Psi}. \tag{96}$$

In such a case, owing (93)-(95) the corresponding local conservation law assumes the form

$$\frac{\partial}{\partial t}\left(\frac{v^2}{2} + e(\varepsilon,\sigma)\right) - \frac{\partial(\sigma v)}{\partial x} = \alpha_0. \tag{97}$$

If the function $e(\varepsilon,\sigma)$ denotes the internal energy and α_0 an energy production term, the equation (97) represents the total energy balance law admitted by the viscoelastic model (5)-(7). It is known that in the pure elastic case where $\sigma = \sigma(\varepsilon)$ (*i.e.*, the memory effects are neglected) it can be proved that the source $\alpha_0 = 0$ and the total energy is conserved. On the contrary, in the present case, by assuming $\alpha_0 \leq 0$, in absence of thermal effects and external forces, the equation (97) characterizes the dissipation of the total energy due to the memory effects described by the rate-type system (5)-(7). Furthermore from (96) we have

$$\frac{\partial e}{\partial \varepsilon} + \Phi(\varepsilon, \sigma) \frac{\partial e}{\partial \sigma} = \sigma \tag{98}$$

$$\Psi(\varepsilon, \sigma) \frac{\partial e}{\partial \sigma} = \alpha_0 \leq 0 \tag{99}$$

Therefore all the viscoelastic processes described by (5)-(7) must satisfy (98) and (99). Relations (98) and (99) were considered by Gurtin *et al.* (1980) and Podio-Guidugli and Suliciu (1984), who studied the existence of free energies admitted by the viscoelastic model (5)-(7) as well as stability problems.

5. Conclusions and final remarks

In this paper we considered the system of equations (5)-(7) which can describe one-dimensional viscoelastic processes where memory effects are taken into account. A general analysis concerning the full classification of first order conservation laws admitted by such a rate-type model has been carried on. Several functional forms of the material response functions $\varphi(\varepsilon, \sigma)$ and $\mu(\varepsilon, \sigma)$ (or equivalently $\Phi(\varepsilon, \sigma)$ and $\Psi(\varepsilon, \sigma)$) there involved allowing the existence of the corresponding conservation laws were deduced. Such constitutive laws can be useful for approximating real material behaviour in suitable range of variation of the concerned variables.

It could be of a certain interest to notice that some of the response functions Φ and Ψ here deduced contain, as particular cases, celebrated model laws known in litterature. For example, if in relation (45) we choose

$$\mu = \frac{1}{\tau} y$$

where τ indicates a relaxation time, owing (26) and (31) and setting $\varphi_0 = \frac{1}{E}$ with E denoting the Young modulus, we easily recover the Malvern's model (9). In such a case the equilibrium stress-strain law specializes to the Hooke's law.

Next in the *II)* case of section 3, taking (42) into account along with (26) and (31), we get

$$\Psi \frac{\partial \Phi}{\partial \sigma} - \Phi \frac{\partial \Psi}{\partial \sigma} - \frac{\partial \Psi}{\partial \varepsilon} = 0. \tag{100}$$

Therefore if we choose

$$\Phi = -\frac{1}{\tau} (\sigma - \sigma_0(\varepsilon)) \tag{101}$$

integration of (100) leads to

$$\Phi = \frac{d\sigma_0}{d\varepsilon} + \Phi_0(\varepsilon) (\sigma - \sigma_0(\varepsilon)) \tag{102}$$

so that the celebrated model (11), (12) deduced by Colemann and Noll is recovered. In such a case $\Phi_0(\varepsilon)$ is a constitutive function while the equilibrium and the instantaneous stress-strain laws coincide. Finally if in (81) we assume

$$\varphi = \frac{1}{E}, \qquad \mu = -\frac{1}{\tau E}\sigma$$

then the Maxwell equation (8) is found.

The conservation laws deduced by means of the analysis developed hitherto can be used for determining potential systems (Bluman and Cheviakov 2005) associated to (5)-(7) through the introduction of potential variables $w(x,t)$ defined by

$$w_x = F(x,t,v,\varepsilon,\sigma), \qquad w_t = -G(x,t,v,\varepsilon,\sigma).$$

Of course couplets, triplets, etc. of potential systems can be constructed depending on the number of conservation laws we consider. Therefore for such a systems non-local conservation laws and non-local symmetries of (5)-(7) can be determined.

Acknowledgments

This work was partially supported by the Italian National Group of Mathematical Physics (GNFM-INdAM) and partially supported by the research project PRIN2017 titled "Multi-scale phenomena in Continuum Mechanics: singular limits, off-equilibrium and transitions".

References

Anco, S. and Bluman, G. (1997). "Direct construction of conservation laws from field equations". *Physical Review Letters* **78**(15), 2869–2873. DOI: 10.1103/PhysRevLett.78.2869.

Bluman, G. and Cheviakov, A. F. (2005). "Framework for potential systems and nonlocal symmetries: Algorithmic approach". *Journal of Mathematical Physics* **46**, 123506. DOI: 10.1063/1.2142834.

Bluman, G., Cheviakov, A. F., and Anco, S. (2010). *Applications of Symmetry Methods to Partial Differential Equations*. Vol. 168. Applied Mathematical Sciences. Springer-Verlag, New York. URL: https://www.springer.com/us/book/9780387986128.

Boillat, G. and Ruggeri, T. (1979). "Reflection and transmission of discontinuity waves through a shock wave. General theory including also the case of characteristic shocks". *Proceedings of the Royal Society of Edinburgh* **83-A**, 17–24. DOI: 10.1017/S0308210500011331.

Cheviakov, A. F. (2010). "Computation of fluxes of conservation laws". *Journal of Engineering Mathematics* **66**(1-3), 153–173. DOI: 10.1007/s10665-009-9307-x.

Colemann, B. D. and Noll, W. (1961). "Foundations of linear viscoelasticity". *Reviews of Modern Physics* **33**, 239–249. DOI: 10.1103/RevModPhys.33.239.

Crampin, M. and Saunders, D. (2005). "On null Lagrangians". *Differential Geometry and its Applications* **22**(2), 131–146. DOI: 10.1016/j.difgeo.2004.10.002.

Cristescu, N. (1972). "A procedure for determining the constitutive equations for materials exhibiting both time–dependent and time–independent plasticity". *International Journal of Solids and Structures* **8**(4), 511–531. DOI: 10.1016/0020-7683(72)90020-0.

Cristescu, N. (2007). *Dynamic Plasticity*. World Scientific Pub., Singapore. DOI: 10.1142/6083.

Currò, C., Fusco, D., and Manganaro, N. (2011). "A reduction procedure for generalized Riemann problems with application to nonlinear transmission lines". *Journal of Physics A: Mathematical and Theoretical* **44**(33), 335205. DOI: 10.1088/1751-8113/44/33/335205.

Currò, C., Fusco, D., and Manganaro, N. (2012a). "Differential constraints and exact solution to Riemann problems for a traffic flow model". *Acta Applicandae Mathematicae* **1**(1), 167–178. DOI: 10.1007/s10440-012-9735-x.

Currò, C., Fusco, D., and Manganaro, N. (2012b). "Hodograph transformation and differential constraints for wave solutions to 2×2 quasilinear hyperbolic nonhomogeneous systems". *Journal of Physics A: Mathematical and Theoretical* **45**(19), 195207. DOI: 10.1088/1751-8113/45/19/195207.

Currò, C., Fusco, D., and Manganaro, N. (2013). "An exact description of nonlinear wave interaction processes ruled by 2×2 hyperbolic systems". *Zeitschrift für angewandte Mathematik und Physik (ZAMP)* **64**(4), 1227–1248. DOI: 10.1007/s00033-012-0282-0.

Currò, C., Fusco, D., and Manganaro, N. (2015a). "Exact description of simple wave interactions in multicomponent chromatography". *Journal of Physics A: Mathematical and Theoretical* **48**, 015201. DOI: 10.1088/1751-8113/48/1/015201.

Currò, C., Fusco, D., and Manganaro, N. (2015b). "Exact solutions in ideal chromatography via differential constraints method". *Atti della Accademia Peloritana dei Pericolanti. Classe di Scienze Fisiche, Matematiche e Naturali* **93**(1), A2 [14 pages]. DOI: 10.1478/AAPP.931A2.

Currò, C. and Manganaro, N. (2013). "Riemann problems and exact solutions to a traffic flow model". *Journal of Mathematical Physics* **54**(17), 071503. DOI: 10.1063/1.4813473.

Currò, C. and Manganaro, N. (2016). "Generalized Riemann problems and exact solutions for p-systems with relaxation". *Ricerche di Matematica* **65**(2), 549–562. DOI: 10.1007/s11587-016-0274-z.

Currò, C. and Manganaro, N. (2017). "Double–wave solutions to quasilinear hyperbolic systems of first–order PDEs". *Zeitschrift für angewandte Mathematik und Physik (ZAMP)* **68**, 103. DOI: 10.1007/s00033-017-0850-4.

Currò, C. and Manganaro, N. (2018). "Exact solutions and wave interactions for a viscoelastic medium". *Atti della Accademia Peloritana dei Pericolanti. Classe di Scienze Fisiche, Matematiche e Naturali* **96**(1), A1 [19 pages]. DOI: 10.1478/AAPP.961A1.

Currò, C. and Manganaro, N. (2019a). "Differential constraints and exact solutions for the ET6 model". *Ricerche di Matematica* **68**(1), 179–193. DOI: 10.1007/s11587-018-0396-6.

Currò, C. and Manganaro, N. (2019b). "Nonlinear wave interactions for a model of Extended Thermodynamics with six fields". *Ricerche di Matematica* **68**(1), 131–143. DOI: 10.1007/s11587-018-0391-y.

Currò, C., Manganaro, N., and Pavlov, M. (2017). "Nonlinear wave interaction problems in the three-dimensional case". *Nonlinearity* **30**, 207–224. DOI: 10.1088/1361-6544/30/1/207.

Dafermos C., M. (2010). *Hyperbolic Conservation Laws in Continuum Physics*. Vol. 35. A Series of Comprehensive Studies in Mathematics. Springer-Verlag, Berlin. URL: https://www.springer.com/it/book/9783642242427.

Faciu, C. (1991a). "Numerical aspects in modelling phase transitions by rate-type constitutive equations". *International Journal of Engineering Science* **29**(9), 1103–1119. DOI: 10.1016/0020-7225(91)90115-J.

Faciu, C. (1991b). "Stress-induced phase transformations in rate-type viscoelasticity". *Journal de Physique IV France* **1**(C4), C4-101–C4-106. DOI: 10.1051/jp4:1991415.

Faciu, C. (1996a). "Initiation and growth of strain bands in rate-type viscoelastic materials. Part I: Discontinuous strain solutions". *European Journal of Mechanics – A/Solids* **15**(6), 969–988.

Faciu, C. (1996b). "Initiation and growth of strain bands in rate-type viscoelastic materials. Part II: The energetics of the banding mechanism". *European Journal of Mechanics – A/Solids* **15**(6), 989–1011.

Faciu, C. and Suliciu, I. (1994). "A Maxwellian model for pseudoelastic materials". *Scipta Metallurgica et Materialia* **31**(10), 1399–1404. DOI: 10.1016/0956-716X(94)90125-2.

Fazio, R. (1992). "A moving boundary hyperbolic problem for a stress impact in a bar of rate–type material". *Wave Motion* **16**(4), 299–305. DOI: 10.1016/0165-2125(92)90019-X.

Frydrychowicz, W. and Singh, M. C. (1985). “Group theoretic technique for the similarity solution of a non–linear elastic rod subjected to velocity impact”. *Journal of Theoretical and Applied Mechanics* **23**(1), 19–37. URL: http://www.ptmts.org.pl/jtam/index.php/jtam/article/view/v23n1p19.

Fusco, D. and Manganaro, N. (2008). “A reduction approach for determining generalized simple waves”. *Zeitschrift für angewandte Mathematik und Physik (ZAMP)* **59**(19), 63–75. DOI: 10.1007/s00033-006-5128-1.

Giaquinta, M. and Hildebrandt, S. (1996). *Calculus of Variations. I. The Lagrangian Formalism*. Vol. 310. Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin. URL: https://www.springer.com/us/book/9783540506256.

Gurtin, M. E., Williams, W. O., and Suliciu, I. (1980). “On rate–type constitutive equations and the energy of viscoelastic and viscoplastic materials”. *International Journal of Solid and Structures* **16**(7), 607–617. DOI: 10.1016/0020-7683(80)90020-7.

Herrmann, W. and Nunziato, J. W. (1972). “Nonlinear Constitutive Equations”. In: *Dynamic Response of Materials to Intense Impulsive Loading*. Ed. by P. C. Chou and A. K. Hopkins. AD-768 416. U.S.A.: Air Force Material Laboratory. Chap. 5, pp. 123–281. URL: http://www.dtic.mil/dtic/tr/fulltext/u2/768416.pdf.

Ibragimov, N. H. (2007). “A new conservation theorem”. *Journal of Mathematical Analisis and Applications* **333**(4), 311–328. DOI: 10.1016/j.jmaa.2006.10.078.

Jeffrey, A. (1976). *Quasilinear Hyperbolic Systems and Waves*. Vol. 5. Research Notes in Mathematics. London: Pitman Publishing.

Lax, P. D. (1957). “Hyperbolic systems of conservation laws II”. *Communications on Pure and Applied Mathematics* **10**(4), 537–566. DOI: 10.1002/cpa.3160100406.

Malvern, L. E. (1951a). “Plastic wave propagation in a bar of metal exhibiting a strain rate effect”. *Quarterly of Applied Mathematics* **8**(4), 405–411. DOI: 10.1090/qam/48292.

Malvern, L. E. (1951b). “The propagation of longitudinal waves of plastic deformation in a bar of material exhibiting a strain rate effect”. *Journal of Applied Mechanics* **18**(2), 203–208.

Manganaro, N. (2017). “Riemann problems for viscoelastic media”. *Rendiconti Lincei - Matematica e Applicazioni* **28**, 479–494. DOI: 10.4171/RLM/772.

Manganaro, N. and Pavlov, M. V. (2014). “The constant astigmatism equation. New exact solution”. *Journal of Physics A: Mathematical and Theoretical* **47**(7), 075203. DOI: 10.1088/1751-8113/47/7/075203.

Noether, E. (1918). “Invariante Variationsprobleme”. *Nachr. d. K önig. Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse*, 235–257. [For an English translation see M. A. Tavel (1971), “Milestones in mathematical physics. Noether’s theorem”. *Transport Theory and Statistical Physics*, **1**(3), 183-207. DOI: 10.1080/00411457108231445].

Olver, P. (1983). “Conservation laws and null divergences”. *Mathematical Proceedings of the Cambridge Philosophical Society* **94**(3), 529–540. DOI: 10.1017/S030500410000092X.

Olver, P. (2000). *Applications of Lie Groups to Differential Equations*. New York: Springer–Verlag. DOI: 10.1007/978-1-4684-0274-2.

Olver, P. and Sivaloganathan, J. (1988). “The structure of null Lagrangians”. *Nonlinearity* **1**(2), 389–398. DOI: 10.1088/0951-7715/1/2/005.

Podio-Guidugli, P. and Suliciu, I. (1984). “On rate-type viscoelasticity and the second law of thermodynamics”. *International Journal of Non-Linear Mechanics* **19**(6), 545–564. DOI: 10.1016/0020-7462(84)90050-7.

Rozhdestvenskii, B. L. and Janenko, N. N. (1983). *Systems of Quasilinear Equations and Their Applications to Gas Dynamics*. Vol. 55. Translation of Mathematical Monographs. Providence Rhode Island: Americal Mathematical Society.

Schuler, K. W. and Nunziato, J. W. (1974). “The dynamic mechanical behavior of polymethyl methacrylate”. *Rheologica Acta* **13**(2), 265–273. DOI: 10.1007/BF01520887.

Smoeller, J. (1983). *Shock Waves and Reaction-Diffusion Equation*. Vol. 258. A Series of Comprehensive Studies in Mathematics. Berlin: Springer – Verlag. DOI: 10.1007/978-1-4612-0873-0.

Suliciu, I. (1989). "On the Description of the Dynamics of Phase Transitions by Means of Rate-Type Constitutive Equations. A Model Problem". In: *Advances in Plasticity 1989. Proceedings of Plasticity '89*. Ed. by A. S. Khan and T. Masataka. New York: Pergamon Press, pp. 417–420. DOI: 10.1016/B978-0-08-040182-9.50104-1.

Suliciu, I. (1992). "Some stability-instability problems in phase transitions modelled by piecewise linear elastic or viscoelastic constitutive equations". *International Journal of Engineering Science* **30**(4), 483–494. DOI: 10.1016/0020-7225(92)90039-J.

Suliciu, I., Lee, S. I., and Ames, W. F. (1973). "Nonlinear traveling waves for a class of rate-type materials". *Journal of Mathemaical Analysis and Applications* **42**(2), 313–322. DOI: 10.1016/0022-247X(73)90140-6.

Tang, S., Qian, J., and Xiao, J. (2006). "Wave interactions in Suliciu model for dynamic phase transitions". *Applied Mathematics and Mechanics* **27**(1), 91–98. DOI: 10.1007/s10483-006-0112-z.

Whitham, G. B. (1974). *Linear and Nonlinear Waves*. New York: Wiley – Interscience. DOI: 10.1002/9781118032954.

* Università degli Studi di Messina
Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra
Viale F. Stagno D'Alcontres 31, 98166 Messina, Italy

Email: nmanganaro@unime.it

Communicated 6 December 2018, manuscript received 23 February 2019; published online 12 July 2019