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PATTERNS IN ASH-COVERED MELTING SNOW

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ABSTRACT. The formation of regular patterns in snow covered by a layer of dark material is investigated as an instability phenomenon. A detailed analysis of the energy balance of the system highlights the prevalent role of solar radiation in such phenomena, and puts the base for further investigations.

1. Introduction

When snow is covered by a layer of inert dark materials, such as volcanic ashes, the melting of the snow layer that occurs naturally in the warmer season is in many cases characterized by the formation of peculiar *patterns*. Most of the times these patterns are in the form of conic piles, but on a sloping ground the pattern is often composed of steps with an undulating profile (see Betterton (2001) and citations therein). This phenomenon has been observed and scientifically studied since at least a century (Wilson 1953), and many hypotheses on the acting mechanism have been proposed.

The volcanic origin of the cover layer is underlined just because there are no other mechanisms, apart from a volcanic eruption, that could lead to the formation a thick cover over a snow pack. High winds, bringing dust or sand, or even air pollution, can create layers of the order of millimeters, but we are considering here layers of the order of several centimeters.

Despite its apparent simplicity, the phenomenon has not yet been fully understood, and several possible explanations have been proposed, both for the elementary thermodynamics involved and, more importantly, for the mechanism of pattern formation (Swithinbank 1950; Drewry 1972). In the simpler case of an ice or snow layer, not covered by ashes or similar materials, the phenomenon of the "penitentes" is well studied (Nichols 1939; Bergeron *et al.* 2006; Claudin *et al.* 2015). They are very steep cones that appear on ice fields subject to intense solar radiation. The major role of solar radiation is in this case suggested by the orientation of the cones, which are not vertical but have axes directed along the prevalent direction of sun rays.

Usual mechanisms of pattern formation, as found in reaction-diffusion systems or thermal convection, can not apply here, since both the snow and covering layer do not behave as fluids.

The basic idea of this work is to study the phenomenon as some kind of instability, which could manifest itself, like the above phenomena, on a prevalent length scale. To this aim, it is necessary first to derive some model equations describing both the snow and lapilli layer motion to a sufficient degree of accuracy, and then perform an analysis of the stability of some basic motion. Both snow and the covering material are in reality not homogeneous and their physical properties present a high variability, but the basic behavior of the system should be independent of this details.

The principal, and probably unique, cause of macroscopic motion is the fusion of the snow layer. So most of this work is devoted to an analysis of the energy exchanges in the system, in particular the computation of the heat flux trough the layer covering the snow and its dependence on the external temperature and irradiation, but more importantly, on the thickness of the layer itself (Anderson 1968).

Section **2** is devoted to a general physical description of the system. In Section **3** we derive a model of the system and, in particular, the energy balance equation for the layer of ashes, which is the basis to the following study. In Section **4** we study the resulting system, its base solution, and stability.

2. General system description

We present the system in the most simplified form, since our main motivation is the determination of the basic dynamical and thermodynamical phenomena involved.

We assume that a uniform layer of snow of constant thickness, was, at some time, covered by a uniform layer of light "pyroclastic material" (which we will call generically "ashes" or "lapilli"), that is, small rock fragments expelled by a voulcanic eruption. Clearly, different sequences of events are possible, producing other configurations of alternating layers of snow and ashes, but we here consider just the simplest configuration of ashes-covered snow. Some more details and pictures are reported in Appendix A.

Notice first that "snow" does not denote some well defined material. A huge literature, motivated mainly by the study of avalanches or other phenomena, exists, which emphasizes the large differences which can be found in many physical parameters of snow such as: mass density, thermal conductivity and capacity, surface reflectivity, viscosity, etc. This large variability depends on the fact that snow is a mixture of ice crystals and air, with a wide range of possible proportions between the two components, which includes also a variable percentage of liquid water and water vapor. Moreover, even the average shape and size of the ice crystals can influence its thermal and mechanical properties.

Many of these properties change naturally over time under the influence of air temperature and humidity, wind and solar irradiation. Another naturally occurring phenomenon (very common, but not relevant to the present study) is the superposition, in successive times, of layers of snow of different composition.

The second component of the system, that is the ashes covering the snow layer, can be also, in general, of very different composition. They can present different average size, density, porosity, and also heat conductivity and heat capacity. According to their size (diameter) d, this fragments are generally called ashes or cinders for d less than 2 mm, lapilli for d in the range 2–64 mm and bombs for larger diameters.

It is also very reasonable to assume that, by the close contact with the underlying snow, these ashes will contain a certain amount of water. An important mechanical quantity relate to ashes, as a granular material, is it angle of repose, or the maximum inclination of its free surface which does not give rise to a downward flow of the material. When the material is wet, this angle in general increases.

3. The model

We introduce a simplified model of the system. A layer of lapilli/ashes of thickness S has covered a layer of snow of height H. We assume that both layers are uniform and horizontal. When the climate gets warmer, the snow starts to melt, but the effect of the ash cover is not what one would expect. That is, melting is not simply accelerated (because of the darker color of ashes) or slowed down (because of a sort of thermal insulation) by the ash cover, but a regular pattern of piles of snow covered by ashes appears.

The variation of snow height is clearly due to its *fusion*. The resulting water will mainly percolate through the snow layer, but a small fraction can also move inside the ashes layer by capillarity. This phenomena are not irrelevant, since, for example, they justify the wetting of the ashes layer. Nevertheless these details are not in the scope of this work.

3.1. Energy balance of the lapilli layer. Since the snow is covered by the layer of ashes, to derive the amount of thermal energy reaching the snow we have to analyze the conduction of thermal energy inside the ashes layer, taking into account the possible causes of energy input and output (Anderson 1968; Holman 1989).

Some simplifying assumption can be safely done. We can restrict our study, in a first approximation, to a *stationary state* of conduction and also to a *temperature of* $T_0 = 0°C = 273.15K$ at the lower interface of the lapilli layer.

To justify the first assumption, we observe that these fusion phenomena occur generally in times of the order of weeks. More importantly, even if there is certainly a daily variation of the thermal exchanges, due to different air temperature, sun elevation and so on, these variations are very slow, and a quasi-static approximation is certainly possible.

Note that we can restrict our analysis to the hottest hours of the day, or more exactly to *the time in which fusion is happening*, and ignore the temperature variations in other times of the day. Since we assume that snow is melting, the temperature at the ash/snow interface is then necessarily T_0. A lower temperature would imply that no fusion is occurring, while an higher temperature is clearly not possible.

These assumptions, and the horizontal homogeneity of the layer, simplify greatly the determination of the thermodynamical state of the system.

Assume that the lapilli layer is limited by the planes $z = 0$, $z = S$. Let $T(x,y,z,t)$ be the temperature inside the layer, then the heat equation for the system, in stationary conditions, is simply $\Delta T = 0$, where Δ denotes the laplacian. Because of the homogeneity in the x, y directions, this reduces to $T''(z) = 0$ where the prime denotes the derivative with respect to z. Then we can say $T(z) = az + b$, where the constants have yet to be determined, from the boundary conditions or other considerations. Since we know that $T(0) = 0$, we have $b = 0$. Consider then the (vertical) heat flux Q in the system, where we assume for convenience that Q is positive for an entering heat flux. If we denote by κ the heat conductivity of the layer, by the Fourier law we obtain

$$Q = \kappa T' = \kappa a = \kappa \frac{T(S) - T(0)}{S}.$$

Since $T(0) = 0$ and denoting the temperature at the upper boundary $T(S)$ by T_u, we have then $Q = \kappa T_u / S$. Note that Q is also the value of heat flux at the two boundaries. In this section we show that, by physical considerations, it is possible to express Q as a function of several parameters, such as intensity of the solar radiation, air temperature and so on, and, more importantly, on the unknown temperature T_u of the upper layer. If we denote this heat flux by $Q(T_u)$, we can write then

$$Q(T_u) = \kappa \frac{T_u}{S}, \tag{1}$$

as an implicit equation for T_u. Note that once T_u is determined, so is $Q = Q(T_u)$ and then it is determined the heat flux *exiting* the layer at the lower boundary. We note that thermal conductivity κ, in a realistic computation, is probably the most difficult quantity to estimate, since in the layer we may have lapilli, water and air in different proportions and configurations. In the computations it was assumed $\kappa = 0.5$ W/(m K).

3.2. Heat flux at the upper boundary. We try to take into account all the physical phenomena that contribute, in a significant way, to the heat exchanges at the upper boundary of the layer, to obtain an explicit form for $Q(T_u)$.

We will consider (as depicted in Figure 1):

(1) *Solar irradiation*, by far the largest contribution of incoming energy.
(2) *Entering thermal radiation*, from the surroundings.
(3) *Exiting thermal radiation*, from the exposed surface of the layer.
(4) *Conductive/convective* exchanges between the surface and air (generally the main outgoing contribution).
(5) Heat loss because of *evaporation* of water permeating the ash layer.

Note that a derivation from first principles of mechanics and thermodynamics of all the contribution to the energy exchanges in the system is totally infeasible. The first three terms of the list, but especially the last two terms, are expressed as phenomenological laws, which are generally just a best fit of some simple law to experimental data. Moreover, the above list can not be considered an exhaustive description of the physics of the system.

The first three terms take into account the exchange of energy in the form of electromagnetic radiation. We separate roughly the radiation involved in the system into visible and thermal radiation. This is not a rigorous separation, but reflects the fact that we are dealing with two very different radiation spectra: the spectrum of solar radiation which corresponds roughly to that of a black body at $5772K$ and the thermal radiation of the surroundings, and of the layer itself, will correspond to a much lower temperature (close to $0C$).

Recalling the Stefan-Boltzmann law, we may write for a perfect absorber and emitter (a "black body") at a temperature T (in Kelvin) $W = \sigma T^4$, where σ is the Stefan-Boltzmann constant, and W is the power emitted from the body as radiation per unit of surface. To describe a real body, an emissivity $0 < \varepsilon < 1$ is usually introduced, depending in general on the wavelength λ, rewriting the above law as $W = \varepsilon(\lambda)\sigma T^4$.

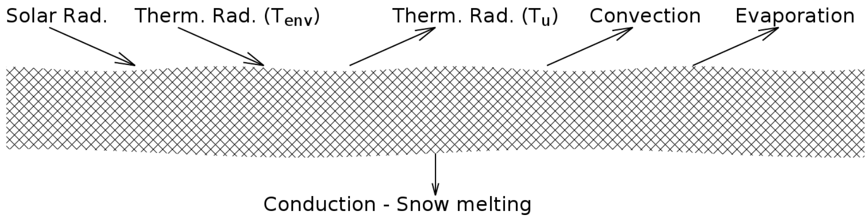

FIGURE 1. Representation of the thermal exchanges in the lapilli layer

We note here a very relevant point: volcanic materials act almost as ideal black body, for both visible and thermal radiation, with $\varepsilon \approx 0.95$, while snow reflects a good fraction of visible light ($\varepsilon \approx 0.25$) but acts also almost as a black body for thermal radiation.

We denote then by $\varepsilon_V(s)$ the emissivity of the lapilli layer in the visible region of the spectrum, where we include also a dependency on the thickness s of the layer.

This dependency on s is needed to provide a smooth transition of ε_V from the value typical of lapilli to that of snow as $s \to 0$. In this work we used

$$\varepsilon_V(s) = (1 - e^{-s/D_L})\varepsilon_L + e^{-s/D_L}\varepsilon_S,$$

where D_L represents an average size of the lapilli composing the layer, $\varepsilon_S, \varepsilon_L$ are the emissivity for visible radiation of snow and lapilli respectively. This function reflects the fact the light can penetrate the lapilli layer reaching directly the underlying snow if the lapilli layer has dimension of the order of D_L.

If we denote by ε_T the emissivity for thermal radiation of both snow and lapilli, we can write then these first three terms as

$$\varepsilon_V(s)W_S, \quad \varepsilon_T \sigma T_{env}^4, \quad -\varepsilon_T \sigma T_u^4,$$

where T_{env} denotes the temperature of the environment, T_u is the *unknown* temperature of the exposed surface of the lapilli layer, and W_S denotes the total flux of solar energy per unit surface. Note also that here a positive sign denotes energy entering the system at the upper boundary, with the same convention used for Q. The quantity W_S is a fraction of the "solar constant", the total power per unit of surface perpendicular to the sun rays reaching the upper atmosphere. This quantity is about $1361W/m^2$. Solar radiation at the ground level is obviously highly variable, depending first of all on the cloud cover, but also on the altitude of the location, on the relative angle between the sun rays and the surface receiving the rays. See e.g. Corripio (2018) and Appendix B for a sample estimate. For the fourth term, heat exchanges with the surrounding air, a simple approximation can be used, the Newton law, with the effective form $H_C = h_C(V)(T_u - T_{env})$ for the heat flux exiting the layer. This term can be in general positive or negative depending on the sign of $(T_u - T_{env})$. Note that the h_C coefficient depends also on wind speed V. For a moderate wind speed of $2m/s$ and an horizontal surface, some phenomenological models give a value of $h_C(V) \approx 22.5$. We used in particular the formula

$$h_C(V) = 10.45 - V + 10\sqrt{V}.$$

The fifth contribution, evaporation cooling, depends as the previous term on wind speed, but also on the actual humidity ratio of the surrounding air R and the saturation humidity ratio

R_{Sat}. In turn, R_{Sat} depends on air pressure and temperature. Even this term can be evaluated only by some phenomenological formula as an evaporation rate E, measuring the mass of water evaporating in the unit surface in unit time. For E, the following phenomenological formula was used

$$E = (R_{Sat} - R)(25 + 19V)/3600$$

A very important circumstance should be noted now, ashes/lapilli are almost *saturated by water*. This could be observed directly, bus is also evident since the walls of many ash piles are much more inclined with respect to the normal "angle of repose" of the dry material. The contribution of this effect is then $H_E = EL_E$ where L_E is the latent heat of evaporation of water, which is approximately $2500J/kg$ at temperature T_0 (L_E depends also slightly on temperature, but this dependence can be neglected).

Considering all the above terms, the resulting heat flux Q entering the layer is then

$$Q(T_u) = \varepsilon_V(S)W_S - \varepsilon_T\sigma(T_u^4 - T_{env}^4) - H_C(T_u) - H_E, \qquad (2)$$

where only the dependence on T_u and S is shown. An example of energy balance is reported in Appendix B.

3.3. Rate of snow fusion. Once all the physical parameter entering in expression (2) are fixed (W_S, T_{env}, V, D_L, etc), and for any given thickness S of the layer, Eq.(1) can be solved numerically, obtaining the upper temperature T_u, and then the heat flux Q. As said before, Q is also equal to the heat flux at the lower face of the layer.

We assume now, as it is very reasonable in our conditions, and also commonly found in the literature, that snow has an almost uniform temperature of T_0, so no conduction or other heat exchanges are happening inside the snow pack. This implies that all of Q will be used in the process of snow melting. Note that Q and all quantities in the equation are power densities W/m^2.

Fusion of ice (and then snow) happens if it has reached its fusion temperature T_0, and then further thermal energy (latent heat of fusion) is supplied to obtain the phase transition from the solid to the liquid state. If U_f is the latent heat of fusion of ice, then Q/U_f is the mass of ice melting per unit time in the unit surface, and finally, we obtain the *melting rate* of snow as

$$F(S) = \frac{Q}{U_f\rho} \quad \text{[m/s]}$$

where ρ is the mass density of snow (in the computations it was assumed $\rho = 300\text{kg/m}^3$). Figure 2 shows two sample curves of the melting rate $F(s)$. Note also that after a maximum the melting rate decreases regularly, and below some thickness the layer actually produces an higher melting rate.

Let examine some properties of the $F(S)$. It will depend on many quantities or parameters, first of all the mean size of the ashes/lapilli, but also external temperature, intensity of solar radiation, and many other. The curve $F(S)$ presents a maximum for a thickness S of the order of the size D_L of the lapilli of the layer, then it is constantly decreasing. At some value S^* of S the fusion rate becomes smaller than the rate of fusion of exposed snow, and the layer can then be considered an "insulator". Note that the rate of solar and thermal radiation influx is clearly independent from S, at least for $S \gg D_L$, so the decreasing value of total heat flux Q (and then $F(s)$) can only be explained by an increase of the surface temperature

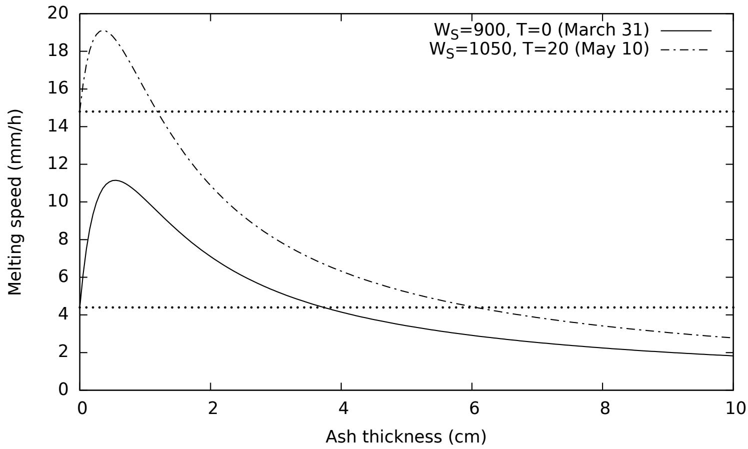

FIGURE 2. Typical curves for high solar radiation, relative to the two pictures in Appendix A. An average dimension $D_L = 5$mm of the lapilli is assumed. Horizontal lines denote the melting speed of the exposed snow (without ash cover)

T_u, and the consequent increment of both the emitted thermal radiation and the convective exchanges.

4. Equations for snow and ashes

If we denote by $H(t,x,y)$ and $S(t,x,y)$ the height of the snowpack and the thickness of the ash layer as functions of time and position, we have then

$$\dot{H}(t,x,y) = -F(S(t,x,y)), \tag{3}$$

where the function F, as shown in the previous section, is determined implicitly through the computation of the upper temperature T_u.

A possible model of hash motion, as found (but just in 1D) in the literature (Betterton 2001), assumes that a layer of lapilli should be adherent to the snow, and then follow the motion of the snow surface by moving orthogonally to the surface. Assuming this model, in a two-dimensional spatial framework, by geometrical considerations (see Betterton (2001)), we can derive the horizontal component of the velocity of the lapilli layer

$$\mathbf{v} = -\dot{H}\,\nabla H$$

where the dot denotes the time derivative, and ∇H is the gradient of H, that is $\nabla H = [\partial_x H, \partial_y H]$. Note that an horizontal snow surface, that is $H =$ const, implies $\mathbf{v} = \mathbf{0}$. Assuming the conservation of lapilli, as an incompressible material, we derive also

$$\dot{S} = -\nabla \cdot (\mathbf{v}S)$$

and finally, the following equation for the height of the lapilli layer

$$\dot{S}(t,x,y) = \nabla \cdot (\dot{H}(t,x,y)\,\nabla H(t,x,y)\,S(t,x,y)). \tag{4}$$

Considering the two equations (3) and (4) we have then the following system:

$$\begin{cases} \dot{H}(t,x,y) = -F(S(t,x,y)) \\ \dot{S}(t,x,y) = \nabla\cdot(\dot{H}(t,x,y)\,\nabla H(t,x,y)\,S(t,x,y)). \end{cases}$$

We want to study this system to derive the initial emergence of the observed pattern (the snow piles) as an instability of an initial homogeneous solution.

4.1. Base solution. We look first for a homogeneous and stationary solution. Note that the base solution can not be constant in all fields, or static, since we assume that snow is melting. S will be simply given by a constant $S_0 = s_0$, to which will correspond a value $m = F(s_0)$ for the melting rate, and the solution for H is simply $H_0(t) = -mt + h_0$, which represents a constant melting. So our *base motion* is an homogeneous layer of snow, covered by ashes, melting at a constant rate.

4.2. Linear instability. We assume then that the solution is perturbed by a small term, as

$$H(t,x,y) = H_0(t) + h(t,x,y); \qquad S(t,x,y) = S_0 + s(t,x,y),$$

with $h \ll H$, $s \ll S$, and substitute into the system, obtaining

$$\begin{cases} -m + \dot{h}(t,x,y) = -F(S_0 + s(t,x,y)) \\ \dot{s}(t,x,y) = \nabla\cdot((-m + \dot{h}(t,x,y))\,\nabla h(t,x,y)\,(S_0 + s(t,x,y))), \end{cases} \tag{5}$$

and linearize the system, retaining only the first order terms in the perturbation fields.

We expand also F to first order as $F(S) = F(s_0) + F'(s_0)(S - s_0)$, with $F(s_0) = m$ and $F'(s_0) = -\beta$. We assume to have an s_0 such that F is decreasing, so it is $\beta > 0$.

If we consider only first order terms, the right hand side of the second equation reduces to $-ms_0\Delta h$, where Δ denotes the laplacian (in x, y).

By substituting and simplifying we obtain the following system.

$$\begin{cases} \dot{h}(t,x,y) = \beta s(t,x,y) \\ \dot{s}(t,x,y) = -ms_0\Delta h(t,x,y). \end{cases}$$

Suppose now that h, s have a periodic spatial dependency, and then the form

$$s = \delta_s e^{\sigma t} e^{i(ax+by)}, \quad h = \delta_h e^{\sigma t} e^{i(ax+by)},$$

with unknown constant δ_s, δ_h, a, b wave numbers, and σ a time evolution coefficient.

We get then the algebraic homogeneous system

$$\begin{cases} \sigma\delta_h = \beta\delta_s \\ \sigma\delta_s = -ms_0k^2\delta_h, \end{cases}$$

where $k^2 = a^2 + b^2$.

As usual in these analyses, we search for values of σ such that there are nontrivial solutions, so we consider the determinant

$$D = \sigma^2 - m\beta\, s_0\, k^2$$

which is null for $\sigma = \pm k\sqrt{m\beta\, s_0}$.

Since σ can be positive or negative, we obtain both decaying and growing solutions. It is easy to verify $\sigma > 0$ implies $\delta_h\delta_s > 0$ so the growing solutions are those for which a spatial maximum of s corresponds to a maximum oh h, as expected.

What we *do not get* by this simple model, is a selection of the *preferred spatial wave number k*, which would simply give the average distance between piles as k^{-1}.

In Betterton (2001) a wave number is derived, but this is obtained by introducing an "ad hoc" diffusive term into the first equation, without a clear physical justification.

5. Conclusions

In this work we derived a model of the system, even if at the present state it can not be considered complete. The energy balance of the lapilli layer, and the consequent derivation of the dependency of the rate of snow fusion on the layer thickness, is based on simple physical principles, and seems to reproduce correctly the phenomena, as described in the little existing literature, see e.g. Driedger (1981).

In this work we used the model of lapilli motion similar to the one derived in Betterton (2001) even if it is not entirely satisfying. The total disappearance of lapilli between the isolated heaps of snow can not simply be explained by adherence of lapilli to the snow layer. Moreover, it appears that after snow is completely melted, the lapilli cover on the ground is uniform, and so it is difficult to assume that there was a real horizontal component of motion of the lapilli. A possible alternative, which requires essentially just a vertical motion of lapilli, could be the sinking of lapilli inside the snow layer. This motion should be investigated experimentally. Possible alternative form of both equations of system (5) should probably take into account other phenomena, such as: the influence of the inclination of the layer on solar irradiation and water evaporation, and, as said previously, sinking of some of the lapilli in the snow. This further analysis of the system will be the subject of future works.

Appendix A. The observed phenomenon

This work was motivated mainly by the observation, in several occasions, of striking regular patterns of snow fusion on mount Etna, when the snow is covered by a layer of ashes. This appears very strange since we expect that both snow and ashes should cover the ground almost uniformly during their fall. So, what we observe here must be an *evolution* of an initial uniform state.

It is useful to first recall briefly some basic facts about Etna pertinent to this work.

Mt. Etna is the highest active volcano in Europe (currently 3350m), with frequent strombolian, or explosive, episodes, which release pyroclastic materials of different shape, composition, and size. The smallest pyroclastic materials (ashes and lapilli) are generally very porous. They are easily transported by high altitude winds and can fall in a large area around the summit craters. Mt. Etna is also usually covered by snow in the winter starting from an altitude of about 1000m, and a significant snow cover may persist at higher quotes until the end of April. It is located at the relatively low latitude of $37.8°N$, which implies a high insolation (solar radiation flux) in spring and summer. Figure 3 shows clearly that it was not just some local phenomenon, since it appears to be the norm in the large area shown in the picture. We observe the formation of steep piles, composed by snow and a cover of hash, and note that there are almost no ashes in-between piles. Figure 4 is even more striking then the previous. The snow occupies here a limited strip of land, and the upper side of the picture shows a region in which snow is already totally melted. There are many relevant details to observe: most of the snow has already melted, and we can see in a single picture the evolution of piles, and their extreme steepness. The darker part of the picture includes the higher piles, which are covered by lapilli almost up to their bases. The smallest piles, close to the boundary of the snow pack, appear as isolated peaks. The inclination of the walls of the piles goes from about 45 degrees to almost 90 in the smallest piles.

Appendix B. An example of energy balance

We will assume the conditions corresponding approximately to Figure 4. We assume $T_{env} = 20C$, and a value of $W_S = 1050W/m^2$. This is a very large value, but it was estimated on the base of existing models of solar radiation (the utility in Corripio (2018) was used), for an altitude of $1700m$, a latitude of $37.8°N$, and around midday at the beginning of May, when the phenomena of Figure 4 where observed.

Also, we consider a $10cm$ layer of lapilli with a average size of $5mm$, relative umidity of 50% and a wind speed of $2m/s$.

Solar	Thermal	Total
+997.5	+397.8	+1395.3

Thermal	Convection	Evaporation	Snow fusion	Total
−510.0	−424.4	−383.4	−77.5	−1395.3

In this condition the upper temperature of ashes is $38.7C$, and the snow melts at a rate of $2.8mm/h$. Without an ash layer ($S = 0$) this rate would be $14.8mm/h$ (see also Figure 2).

FIGURE 3. A picture from March 31st, 2012

FIGURE 4. A picture from May 5th, 2013

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